INFOLINK UNIVERSITY COLLEGE

MATHEMATICS REMEDIAL COURSE

CHAPTER ONE:

SOLVING EQUATIONS AND INEQUALITIES

1.1 Solving Equations involving exponents and radicals

Definition: Equations with radicals that have variables in their radicands are called radical equations. An example of a radical equation is $2\sqrt{x+1} = 4$.

Solving Radical Equations

To solve a radical equation, follow these steps:

Step 1 Isolate the radical on one side of the equation, if necessary.

Step 2 Raise each side of the equation to the same exponent to eliminate the radical and obtain a linear, quadratic, or other polynomial equation.

Step 3 Solve the resulting equation using techniques you learned previously. Check your solution.

Examples

Solve the following equations

a. $2\sqrt{x+1} = 4$

b. $\sqrt[3]{2x-9} - 1 = 2$

c. $\sqrt{7x+15} = x+1$

d. $\sqrt{x+2} + 1 = \sqrt{3-x}$

Solutions

a.

$2\sqrt{x+1} = 4$	Write the original equation
$\sqrt{x+1} = 2$	Divide each side by 2
$\left(\sqrt{x+1}\right)^2 = 4$	Square each side to eliminate the radical
$x+1 = 4$	Simplification
$x = 3$	Subtract 1 from both sides

Therefore, the solution is $x = 3$.

b.

$\sqrt[3]{2x-9} - 1 = 2$	Write the original equation.
$\sqrt[3]{2x-9} = 3$	Add 1 to each side.
$\left(\sqrt[3]{2x-9}\right)^3 = 3^3$	Cube each side to eliminate the radical.
$2x - 9 = 27$	Simplify.
$2x = 36$	Add 9 to each side.
$x = 18$	Divide each side by 2.

▶ The solution is $x = 18$.

c. Solution

$x + 1 = \sqrt{7x + 15}$	Write the original equation.
$(x + 1)^2 = \left(\sqrt{7x + 15}\right)^2$	Square each side.
$x^2 + 2x + 1 = 7x + 15$	Expand left side and simplify right side.
$x^2 - 5x - 14 = 0$	Write in standard form.
$(x - 7)(x + 2) = 0$	Factor.
$x - 7 = 0 \quad \text{or} \quad x + 2 = 0$	Zero-Product Property
$x = 7 \quad \text{or} \quad x = -2$	Solve for x.

Check $\quad 7 + 1 \stackrel{?}{=} \sqrt{7(7) + 15}$ $\qquad -2 + 1 \stackrel{?}{=} \sqrt{7(-2) + 15}$

$8 \stackrel{?}{=} \sqrt{64}$ $\qquad -1 \stackrel{?}{=} \sqrt{1}$

$8 = 8$ ✓ $\qquad -1 \neq 1$ ✗

▶ The apparent solution $x = -2$ is extraneous. So, the only solution is $x = 7$.

d. Solution

$\sqrt{x + 2} + 1 = \sqrt{3 - x}$	Write the original equation.
$\left(\sqrt{x + 2} + 1\right)^2 = \left(\sqrt{3 - x}\right)^2$	Square each side.
$x + 2 + 2\sqrt{x + 2} + 1 = 3 - x$	Expand left side and simplify right side.
$2\sqrt{x + 2} = -2x$	Isolate radical expression.
$\sqrt{x + 2} = -x$	Divide each side by 2.
$\left(\sqrt{x + 2}\right)^2 = (-x)^2$	Square each side.
$x + 2 = x^2$	Simplify.
$0 = x^2 - x - 2$	Write in standard form.
$0 = (x - 2)(x + 1)$	Factor.
$x - 2 = 0 \quad \text{or} \quad x + 1 = 0$	Zero-Product Property
$x = 2 \quad \text{or} \quad x = -1$	Solve for x.

Check $\quad \sqrt{2 + 2} + 1 \stackrel{?}{=} \sqrt{3 - 2}$ $\qquad \sqrt{-1 + 2} + 1 \stackrel{?}{=} \sqrt{3 - (-1)}$

$\sqrt{4} + 1 \stackrel{?}{=} \sqrt{1}$ $\qquad \sqrt{1} + 1 \stackrel{?}{=} \sqrt{4}$

$3 \neq 1$ ✗ $\qquad 2 = 2$ ✓

▶ The apparent solution $x = 2$ is extraneous. So, the only solution is $x = -1$.

Solving Equations Involving Exponents

When an equation contains a power with a rational exponent, you can solve the equation using a procedure similar to the one for solving radical equations. In this case, you first isolate the power and then raise each side of the equation to the reciprocal of the rational exponent.

Examples

Solve the following equations

a. $(2x)^{\frac{3}{4}} + 2 = 10$ b. $(x + 30)^{\frac{1}{2}} = x$

Solutions

a. Solution

$(2x)^{3/4} + 2 = 10$	Write the original equation.
$(2x)^{3/4} = 8$	Subtract 2 from each side.
$[(2x)^{3/4}]^{4/3} = 8^{4/3}$	Raise each side to the four-thirds.
$2x = 16$	Simplify.
$x = 8$	Divide each side by 2.

▶ The solution is $x = 8$.

Check

$(2 \cdot 8)^{3/4} + 2 \stackrel{?}{=} 10$

$16^{3/4} + 2 \stackrel{?}{=} 10$

$10 = 10$ ✓

b. Solution

$(x + 30)^{1/2} = x$	Write the original equation.
$[(x + 30)^{1/2}]^2 = x^2$	Square each side.
$x + 30 = x^2$	Simplify.
$0 = x^2 - x - 30$	Write in standard form.
$0 = (x - 6)(x + 5)$	Factor.
$x - 6 = 0$ or $x + 5 = 0$	Zero-Product Property
$x = 6$ or $x = -5$	Solve for x.

▶ The apparent solution $x = -5$ is extraneous. So, the only solution is $x = 6$.

1.2 Solving system of linear equations in two variables

Definition:

The system $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$ where a_1, a_2, b_1, b_2, c_1 *and* c_2 are constants is system of linear equations in two variables.

Types of Systems

- There are three possible outcomes when graphing two linear equations in a plane.

- One point of intersection, so one solution
- Parallel lines, so no solution
- Coincident lines, so infinite number of solutions

- If there is at least one solution, the system is considered to be *consistent*.
- If the system defines distinct lines, the equations are *independent*.
- Change both linear equations into slope-intercept form. We can then easily determine if the lines intersect, are parallel, or are the same line.

Example

How many solutions does the following system have?

$$3x + y = 1 \quad and \quad 3x + 2y = 6$$

Write each equation in slope-intercept form.

First equation,

$3x + y = 1$

$y = -3x + 1$ (Subtract $3x$ from both sides)

Second equation,

$3x + 2y = 6$

$2y = -3x + 6$ (Subtract $3x$ from both sides)

$y = -\frac{3}{2}x + 3$ (Divide both sides by 2)

The lines are intersecting lines (since they have different slopes), so there is one solution.

Example

How many solutions does the following system have?

$$3x + y = 0 \quad and \quad 2y = -6x$$

Write each equation in slope-intercept form,

First equation,

$3x + y = 0$

$y = -3x$ (Subtract $3x$ from both sides)

Second equation,

$2y = -6x$

$y = -3x$ (Divide both sides by 2)

The two lines are identical, so there are infinitely many solutions.

Example

How many solutions does the following system have?

$$2x + y = 0 \quad and \quad y = -2x + 1$$

Write each equation in slope-intercept form.

First equation,

$2x + y = 0$

$$y = -2x \quad \text{(subtract 2x from both sides)}$$

Second equation,

$$y = -2x + 1 \quad \text{(already in slope-intercept form)}$$

The two lines are parallel lines (same slope, but different y-intercepts), so there are no solutions.

The Substitution Method

You solve one equation for one of the variables, then substitute the new form of the equation into the other equation for the solved variable.

Example

Solve the following system using the substitution method.

$$3x - y = 6 \ \ and - 4x + 2y = -8$$

Solving the first equation for y,

$$3x - y = 6$$
$$-y = -3x + 6 \quad \text{(subtract } 3x \text{ from both sides)}$$
$$y = 3x - 6 \quad \text{(divide both sides by } -1)$$

Substitute this value for y in the second equation.

$$-4x + 2y = -8$$
$$-4x + 2(3x - 6) = -8 \quad \text{(replace } y \text{ with result from first equation)}$$
$$-4x + 6x - 12 = -8 \quad \text{(use the distributive property)}$$
$$2x - 12 = -8 \quad \text{(simplify the left side)}$$
$$2x = 4 \quad \text{(add 12 to both sides)}$$
$$x = 2 \quad \text{(divide both sides by 2)}$$

Substitute $x = 2$ into the first equation solved for y.

$$y = 3x - 6 = 3(2) - 6 = 6 - 6 = 0$$

Our computations have produced the point (2, 0). Check the point in the original equations.

First equation,

$$3x - y = 6$$
$$3(2) - 0 = 6 \quad \text{true}$$

Second equation,

$$-4x + 2y = -8$$
$$-4(2) + 2(0) = -8 \quad \text{true}$$

The solution of the system is $\{(2,\ 0)\}$

The Substitution Method

Steps in solving a System of Linear Equations by the Substitution Method

1) Solve one of the equations for a variable.
2) Substitute the expression from step 1 into the other equation.
3) Solve the new equation.
4) Substitute the value found in step 3 into either equation containing both variables.
5) Check the proposed solution in the original equations.

Example

Solve the following system of equations using the substitution method.

$$y = 2x - 5 \ \ and \ \ 8x - 4y = 20$$

Since the first equation is already solved for y, substitute this value into the second equation.

$$\begin{aligned} 8x - 4y &= 20 & \\ 8x - 4(2x - 5) &= 20 & \text{(replace } y \text{ with result from first equation)} \\ 8x - 8x + 20 &= 20 & \text{(use distributive property)} \\ 20 &= 20 & \text{(simplify left side)} \end{aligned}$$

There are an infinite number of solutions for this system. Any solution of one equation would automatically be a solution of the other equation. This represents a consistent system and the linear equations are dependent equations.

Example

Solve the following system of equations using the substitution method.

$$3x - y = 4 \ \ and \ \ 6x - 2y = 4$$

Solve the first equation for y.

$$\begin{aligned} 3x - y &= 4 & \\ -y &= -3x + 4 & \text{(subtract 3x from both sides)} \\ y &= 3x - 4 & \text{(multiply both sides by } -1) \end{aligned}$$

Substitute this value for y into the second equation.

$$\begin{aligned} 6x - 2y &= 4 & \\ 6x - 2(3x - 4) &= 4 & \text{(replace y with the result from the first equation)} \\ 6x - 6x + 8 &= 4 & \text{(use distributive property)} \\ 8 &= 4 & \text{(simplify the left side)} \end{aligned}$$

There is no solution to this system.

This represents an inconsistent system, even though the linear equations are independent.

The Elimination Method

Another method that can be used to solve systems of equations is called the ***addition, arithmetic*** or ***elimination method***.

You multiply both equations by numbers that will allow you to combine the two equations and eliminate one of the variables.

Example

Solve the following system of equations using the elimination method.

$$6x - 3y = -3 \quad and \quad 4x + 5y = -9$$

Multiply both sides of the first equation by 5 and the second equation by 3.

First equation,

$$5(6x - 3y) = 5(-3)$$
$$30x - 15y = -15 \quad \text{(use the distributive property)}$$

Second equation,

$$3(4x + 5y) = 3(-9)$$
$$12x + 15y = -27 \quad \text{(use the distributive property)}$$

Combine the two resulting equations (eliminating the variable y).

$$\begin{array}{rcl} 30x - 15y & = & -15 \\ \underline{12x + 15y} & \underline{=} & \underline{-27} \\ 42x & = & -42 \\ x & = & -1 \quad \text{(divide both sides by 42)} \end{array}$$

Substitute the value for x into one of the original equations.

$$6x - 3y = -3$$
$$6(-1) - 3y = -3 \quad \text{(replace the } x \text{ value in the first equation)}$$
$$-6 - 3y = -3 \quad \text{(simplify the left side)}$$
$$-3y = -3 + 6 = 3 \quad \text{(add 6 to both sides and simplify)}$$
$$y = -1 \quad \text{(divide both sides by } -3\text{)}$$

Our computations have produced the point (–1, –1). Check the point in the original equations.

First equation,

$$6x - 3y = -3$$
$$6(-1) - 3(-1) = -3 \quad \textbf{\textit{true}}$$

Second equation,

$$4x + 5y = -9$$
$$4(-1) + 5(-1) = -9 \quad \textbf{\textit{true}}$$

The solution of the system is (–1, –1).

Steps in Solving a System of Linear Equations by Elimination Method

1) Rewrite each equation in standard form, eliminating fraction coefficients.
2) If necessary, multiply one or both equations by a number so that the coefficients of a chosen variable are opposites.
3) Add the equations.
4) Find the value of one variable by solving equation from step 3.
5) Find the value of the second variable by substituting the value found in step 4 into either original equation.
6) Check the proposed solution in the original equations.

Example

Solve the following system of equations using the elimination method.

$$\frac{2}{3}x+\frac{1}{4}y=-\frac{3}{2}$$
$$\frac{1}{2}x-\frac{1}{4}y=-2$$

First multiply both sides of the equations by a number that will clear the fractions out of the equations.

Multiply both sides of each equation by 12. (Note: you don't have to multiply each equation by the same number, but in this case it will be convenient to do so.)

First equation,

$$\frac{2}{3}x+\frac{1}{4}y=-\frac{3}{2}$$
$$12\left(\frac{2}{3}x+\frac{1}{4}y\right)=12\left(-\frac{3}{2}\right)$$
$$8x+3y=-18$$

Second equation,

$$\frac{1}{2}x-\frac{1}{4}y=-2$$
$$12\left(\frac{1}{2}x-\frac{1}{4}y\right)=12(-2)$$
$$6x-3y=-24$$

Combine the two equations.

$$\begin{array}{l} 8x + 3y = -18 \\ \underline{6x - 3y = -24} \\ 14x \quad\quad = -42 \\ x = -3 \quad\quad \text{(divide both sides by 14)} \end{array}$$

Substitute the value for x into one of the original equations.

$$8x + 3y = -18$$
$$8(-3) + 3y = -18$$
$$-24 + 3y = -18$$
$$3y = -18 + 24 = 6$$
$$y = 2$$

Our computations have produced the point $(-3, 2)$.

1.3 Solving equations involving absolute value

Properties of absolute value.

1. For any real number a, $|a| = |-a|$.
2. For any real number a, $|a| \geq 0$.

3. $|x| = \begin{cases} x, & for\ x \geq 0 \\ -x, & for\ x < 0 \end{cases}$

4. For any non- negative number a, $|x| = a$ means $x = a\ or\ x = -a$.

Example

Solve the equation, $|x + 7| = 14$.

Solution:

For any non- negative number a, $|x| = a$ means $x = a\ or\ x = -a$.

So, you begin by making it into two separate equations and then solve them separately.

$|x + 7| = 14$ means:

$x + 7 = 14$	or	$x + 7 = -14$
$x = 14 - 7$	or	$x = -14 - 7$
$x = 7$	or	$x = -21$

Hence, $x = -21$ *and* 7 are solutions of the equation.

Remark: An absolute value equation has no solution if the absolute value expression equals a negative number since an absolute value can never be negative. For example: $|x| = -2$ has no solution. (Why?)

Example

Solve the equation, $|x + 3| + |2x - 4| = 5$

Solution

Use case method.

The zeros of $x + 3 = 0$ and $2x - 4 = 0$ are -3 and 2

Then construct sign chart

$x < -3$ (-3)	$-3 \leq x \leq 2$ (2)	$x > 2$
$\|x + 3\|$ $-x - 3$	$x + 3$	$x + 3$
$\|2x - 4\|$ $4 - 2x$	$4 - 2x$	$2x - 4$

Case 1: when $x < -3$

$$|x + 3| + |2x - 4| = 5$$

$$-x - 3 + 4 - 2x = 5$$

$$x = -\frac{4}{3} \notin (-\infty, -3)$$

The value of $x = -\frac{4}{3}$ is not in the interval $\boldsymbol{x < -3}$. Therefore, $S_1 = \emptyset$

Case 2: when $-3 \leq x \leq 2$

$$|x + 3| + |2x - 4| = 5$$

$$x + 3 + 4 - 2x = 5$$

$$x = 2 \in [-3,\ 2]$$

Hence, $S_2 = \{2\}$

Case 3: when $x > 2$

$$|x\ +\ 3| + |2x - 4| = 5$$

$$x + 3 + 2x - 4 = 5$$

$$x = 2 \notin (2,\ \infty)$$

Hence, $S_3 = \emptyset$

Therefore, the solution set is $S.S = S_1\ \cup\ S_2\ \cup\ S_3\ = \{2\}$

1.4 Solving inequality involving absolute value

Definition

1. For $a \geq 0$. Then
   a. $|x| \geq a \quad iff \quad x \geq a \quad or \quad x \leq -a$
   b. $|x| > a \quad iff \quad x > a \quad or \quad x < -a$
   c. $|x| \leq a \quad iff \quad -a \leq x \leq a$
   d. $|x| < a \quad iff \quad -a < x < a$
2. For $a < 0$. Then
   a. $|x| > 0$, then the solution set is the set of real numbers
   b. $|x| < a$, then the solution set is empty set

Example

Solve the following inequality

a. $|x - 3| > 2$ b. $|x + 2| - 3 \leq 1$ c. $|x + 2| + |2x - 1| \geq 4$

Solution

a. $|x - 3| > 2$
   $x - 3 > 2 \quad or \quad x - 3 < -2$
   $$x > 5 \quad or \quad x < 1$$
   The solution is $\{x: x < 1\ \ or\ x > 5\} = (-\infty,\ 1) \cup (5,\ \infty)$
b. $|x + 2| - 3 \leq 1$
   $$|x + 2| \leq 4$$
   $$-4 \leq x + 2 \leq 4$$
   $$-6 \leq x \leq 2$$
   Therefore, the solution is $\{x: -6 \leq x \leq 2\} = [-6,\ 2]$
c. $|x + 2| + |2x - 1| \geq 4$
   The zeros of $x\ +\ 2 = 0$ and $2x - 1 = 0$ are -2 and $\frac{1}{2}$
   Then construct sign chart

	$x < -2$	$-2 \le x \le \frac{1}{2}$	$x > \frac{1}{2}$
$\lvert x+3 \rvert$	$-x-2$	$x+2$	$x+2$
$\lvert 2x-4 \rvert$	$1-2x$	$1-2x$	$2x-1$

Case 1: when $x < -2$

$$|x+2| + |2x-1| \ge 4$$
$$-x - 2 + 1 - 2x \ge 4$$
$$x \le -\frac{5}{3} = \left(-\infty, -\frac{5}{3}\right]$$

The solution set is the intersection of $(-\infty, -2) \cap \left(-\infty, -\frac{5}{3}\right]$

Therefore, $S_1 = (-\infty, -2)$

Case 2: when $-2 \le x \le \frac{1}{2}$

$$|x+2| + |2x-1| \ge 4$$
$$x + 2 + 1 - 2x \ge 4$$
$$-x \ge -1$$
$$x \le 1 = (-\infty, 1]$$

The solution set is the intersection of $(-\infty, 1] \cap \left[-2, \frac{1}{2}\right]$

Therefore, $S_2 = [-2, 1]$

Case 3: when $x > \frac{1}{2}$

$$|x+2| + |2x-1| \ge 4$$
$$x + 2 + 2x - 1 \ge 4$$
$$x \ge 1 = [1, \infty)$$

The solution set is the intersection of $[1, \infty) \cap \left(\frac{1}{2}, \infty\right)$

Therefore, $S_3 = [1, \infty)$

Therefore, the solution set is $S.S = S_1 \cup S_2 \cup S_3 = (-\infty, \infty) = \mathbb{R}$

1.5 Solving system of linear inequality in two variables

Important procedures in solving inequalities in two variables graphically

- To solve an inequality in two variables graphically, first sketch the boundary lines on the same xy − plane.
- This graph is the boundary line (or curve); since all points that make the inequality true lie on one side or the other of the line. Before graphing the equations, decide whether the line or curve is part of the solution or not, that is, whether it is solid or dashed. If the inequality symbol is either $\le$ *or* $\ge$, then the points on the boundary line are solutions to the inequality and the line must be solid. If the symbol is either $<$ *or* $>$, then the points on the boundary line are not solutions to the inequality and the line is dashed.
- Next, decide which side of the boundary line must be shaded to show the part of the graph that represents all (x, y) coordinate pairs that make the inequality true. To do this, choose a point not on the boundary line. Substitute the x- and y-values of this point into the original inequalities. If the inequality is true for the test point, then shade the region on the side of the boundary line that contains the test point.
- If the inequality is false for the test point, then shade the opposite region. The shaded portion represents all of the (x, y) coordinate pairs that are solutions to the original inequality. The solution to the system of inequalities is the overlap of the shading from the individual inequalities. When two boundary lines are graphed, there are often four

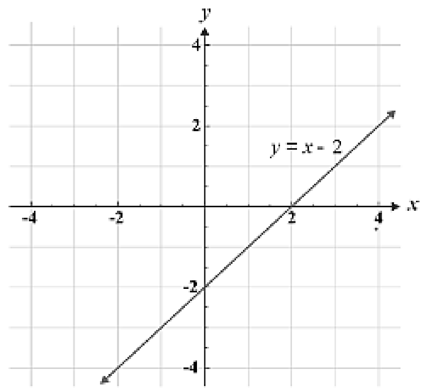
regions. The region containing the coordinate pairs that make both of the inequalities true is the solution region.

Example

Solve the system of linear inequalities $y \le x - 2$ and $y > -3x + 5$ using graphical method.

Solution:

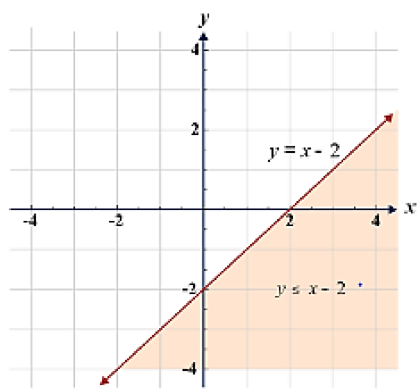
First, graph the inequality $y \le x - 2$. The related equation is $y = x - 2$ Since the inequality is ≤ , not a strict one, the border line is solid. Graph the straight line.



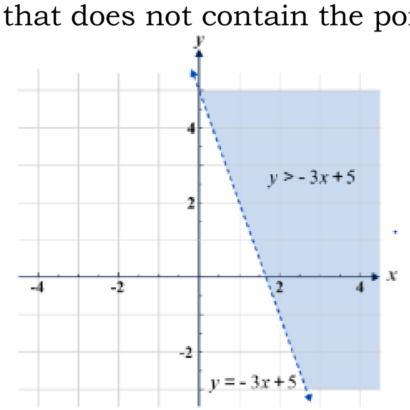
Consider a point that is not on the line - say, (0, 0) - and substitute in the inequality

$$y \le x - 2$$
$$0 \le 0 - 2$$
$$0 \le -2$$

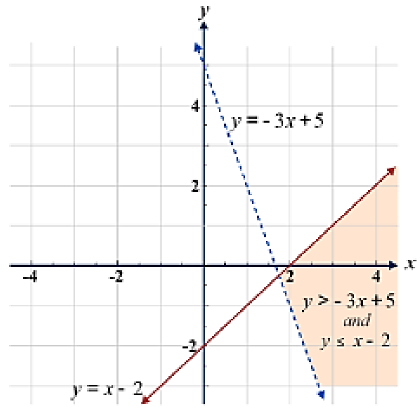
This is false. So, the solution does not contain the point(0, 0). Shade the lower half of the line



Similarly, draw a dashed line for the related equation of the second inequality $y > -3x + 5$ which has a strict inequality. The point (0, 0) does not satisfy the inequality, so shade the half that does not contain the point (0, 0).



The solution of the system of inequalities is the intersection region of the solutions of the two inequalities



1.6 Solving quadratic equations and inequality

Definition

An equation of the form $ax^2 + bx + c = 0$ where a, b, c, ∈ R and $a \neq 0$ is a quadratic equation. Here, a is called the leading coefficient, b is the middle term and c is the constant term

Solving quadratic equations

There are three basic methods for solving quadratic equations: factorization (if possible), completing the square and using the quadratic formula method.

How to solve a quadratic equation by factorization method?

1. Put all terms on one side of the equal sign, leaving zero on the other side.
2. Factorize the equation.
3. Set each factor equal to zero.
4. Solve each of these equations.
5. Check by inserting your answer in the original equation.

Example

Solve the quadratic equation $x^2 - 6x - 16 = 0$.

Solution:

Factorizing this we have, (Find two numbers whose sum is -6 *and* product is -16).
They are -8 and 2. Hence, $(x-8)(x+2) = 0$.

$$x - 8 = 0 \text{ or } x + 2 = 0.$$

Recall that ab $=$ 0 if and only if a $=$ 0 or b $=$ 0

$$x = 8, x = -2.$$

Hence, the solution of the quadratic equation is $x = 8, x = -2$.

Example

Solve the quadratic equation $4x^2 + 4x + 1 = 0$.

Solution:

writing $4x^2 + 2x + 2x + 1 = 0$.

$$2x(2x+1) + 1(2x + 1) = 0.$$

$$(2x+1)(2x+1) = 0. \Rightarrow (2x+1)^2 = 0. \Rightarrow x = -\frac{1}{2}$$

Hence, the solution set of the quadratic equation is $\left\{-\frac{1}{2}\right\}$

Completing the square method

Example

Is it possible to solve $x^2 + 6x + 4 = 0$ using factorization method?

Solution:

Since there are no two integers whose sum is equal to 6 and product is equal to 4, this quadratic equation may not be solved using factorization method. Hence, we need another method to solve the equation.

Completing the square is where we take the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ and convert it into $\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}$ as follows

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad (since, \qquad a \neq 0)$$

$\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + \frac{c}{a} - \frac{b^2}{4a^2} = 0$. Taking half of the coefficient of the middle term and squaring it and adding its opposite). The expression in the bracket is a perfect square. Hence, after simplifying, we have

$$\left(\mathrm{x} + \frac{\mathrm{b}}{2\mathrm{a}}\right)^2 + \frac{4\mathrm{ac} - \mathrm{b}^2}{4\mathrm{a}^2} = 0 - - - - (*)$$

Equivalentely, $ax^2 + bx + c = 0$ if and only if $\left(\mathrm{x} + \frac{\mathrm{b}}{2\mathrm{a}}\right)^2 + \frac{4\mathrm{ac}-\mathrm{b}^2}{4\mathrm{a}^2} = 0$

Example

Solve $x^2 + 6x + 7 = 0$ using completing the square method.

Solution:

$x^2 + 6x + 9 + 7 - 9 = 0$ (Adding the square of half the coefficient of the middle term 6 and its opposite)

$(x^2 + 6x + 9) + 7 - 9 = 0$ (Collecting those terms which sum up as a perfect square)

$$(x + 3)^2 = 2$$

$$x + 3 = \pm\sqrt{2} \text{ (taking the square root)}$$

$$x = -3 \pm \sqrt{2}$$

Therefore, $x = -3 + \sqrt{2}$ and $x = -3 - \sqrt{2}$ are solutions

Example

Solve $3x^2 + 10x + 7 = 0$ using completing the square method.

Solution:

$x^2 + \frac{10}{3}x + \frac{7}{3} = 0$ (Dividing both sides by 3)

$(x^2 + \frac{10}{3}x + \frac{100}{36}) + \frac{7}{3} - \frac{100}{36} = 0$(Colleting those terms which sum up as a perfect square)

$\left(x + \frac{10}{6}\right)^2 - \frac{16}{36} = 0$ (Writing as a perfect square)

$\left(x + \frac{10}{6}\right)^2 = \frac{4}{9}$. Therefore, $x = -\frac{10}{6} \pm \frac{2}{3}$ (Extracting from the square root)

Therefore, $x = -1$ *and* $x = -\frac{7}{3}$ are the solutions.

The quadratic formula method

There are quadratic equations that cannot be solved by factorization method. This is generally true when the roots are not rational numbers. Consider the quadratic equation $ax^2 + bx + c = 0, \ \ where\ a \neq 0.$

From the completing the square method, we have $\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2} = 0$

Solving for x, $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Quadratic formula

For quadratic equation $ax^2 + bx + c = 0, a \neq 0,$

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Example

Solve $x^2 - 6x + 3 = 0$ using the quadratic formula.

Solution:

$1, -6$ and 3 are the values for a, b, and c, respectively in the quadratic formula. Since $b^2 - 4ac = 24 > 0$, we have two distinct real roots. That is,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6)\pm\sqrt{(-6)^2-4(1)(3)}}{2(1)}$$

$$x = \frac{6 \pm \sqrt{36-12}}{2} = \frac{6 \pm \sqrt{24}}{2} = 3 \pm \sqrt{6}.$$

Hence, $3+\sqrt{6}$ and $3-\sqrt{6}$ are the solutions. Can you solve this using other methods?

Discriminant

The discriminant is the value under the radical sign which is b^2-4ac. A quadratic equation with real numbers as coefficients can have the following: The roots of the quadratic equation are: $x = \frac{-b \pm \sqrt{D}}{2a}$, where $D = b^2 - 4ac$

- If $b^2 - 4ac < 0$, it has no real root
- If $b^2 - 4ac = 0$, it has only one real root
- If $b^2 - 4ac > 0$, it has two distinct real roots

Example

Check whether $x^2 + 2x + 2 = 0$ has distinct real roots, one real root or no real roots. If root exists, find it.

Solution

$a = 1,\ b = 2\ and\ c = 2.$

Since $b^2 - 4ac = 24 < 0$, we have no real roots; the roots are imaginary.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm \sqrt{-4}}{2}.$$

Since the discriminant $D = b^2 - 4ac$ is negative, this quadratic equation has no real root.

Example

Solve $x^2 + 4x + 3 = 0$ using the quadratic formula.

Solution:

$a = 1, b = 4\ and\ c = 3$. Since $b^2 - 4ac > 0$, the equation has two distinct real roots. The roots are $x = -1\ and\ x = -3$.

Relationships between roots and coefficients of a quadratic equation.

$$If\ r_1 = \frac{-b + \sqrt{b^2-4ac}}{2a}\ and\ r_2 = \frac{-b - \sqrt{b^2-4ac}}{2a}, then$$

$$r_1 + r_2 = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{-b}{a} \text{ and}$$

$$r_1 . r_2 = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right) = \frac{c}{a}$$

Example

Let $ax^2 + 6x + c = 0$.

If the sum of the roots is $\frac{-6}{5}$ and the product is $\frac{1}{5}$, then find the values of a and c.

Solution:

Let r_1 and r_2 be the roots

Hence, we have

$r_1 + r_2 = \frac{-6}{5} = \frac{-6}{a}$,

Which implies $a = 5$ and

$r_1 r_2 = \frac{1}{5} = \frac{c}{a} = \frac{c}{5}$ which also implies $c = 1$. Therefore, the quadratic equation is

$5x^2 + 6x + 1 = 0$.

Example

If the difference between the roots of the equation $x^2 - 13x + k = 0$ is 7, find the value of k.

Solution:

Let r_1 and r_2 be the roots.

$r_1 + r_2 = \frac{-b}{a} = \frac{-(-13)}{1} = 13$ *and* $r_1 - r_2 = 7$

$$\begin{cases} r_1 + r_2 = 13 \\ r_1 - r_2 = 7 \end{cases}$$

$r_1 = 10$ *and* $r_2 = 3$

$r_1 r_2 = (10)(3) = 30 = \frac{c}{a} = \frac{k}{1}$

Therefore, $k = 30$

Quadratic inequality

Definition

An inequality that can be reduced to any one of the following forms

$$ax^2 + bx + c \leq 0 \text{ or } ax^2 + bx + c < 0$$

$$ax^2 + bx + c \geq 0 \text{ or } ax^2 + bx + c > 0$$

where a, b and c are constants and $a \neq 0$ is called **quadratic inequality**

Note

1. When the discriminant $b^2 - 4ac < 0$, then

i. The solution set of $ax^2 + bx + c > 0$ is the set of real numbers if $a > 0$ and empty set if $a < 0$.

ii. The solution set of $ax^2 + bx + c < 0$ is the set of real numbers if $a < 0$ and empty set if $a > 0$.

2. Product property

i. $mn < 0$ if and only if m < 0 and n > 0 or m > 0 and n < 0

ii. $mn > 0$ if and only if m < 0 and n < 0 or m > 0 and n > 0

Example

Solve each of the following inequalities:

a. $\boldsymbol{x^2 - 2x - 3 > 0}$ **b.** $\boldsymbol{x^2 - x - 2 \leq 0}$

Solution

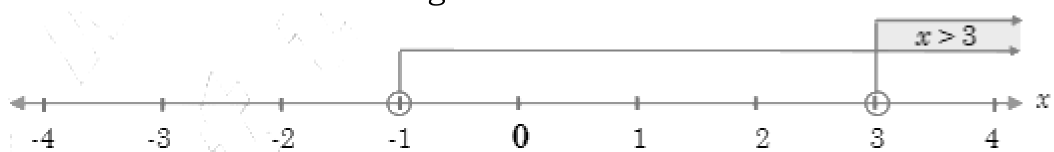
a. By product property 1: $\boldsymbol{x^2 - 2x - 3 = (x-3)(x+1)}$ is positive if either both factors are positive or both are negative.

Now, consider case by case as follows

Case i When both the factors are positive

$$x - 3 > 0 \text{ and } x + 1 > 0$$
$$x > 3 \text{ and } x > -1$$

The intersection of $x > 3$ and $x > -1$ is $x > 3$. This can be illustrated on the number line as shown in figure below.

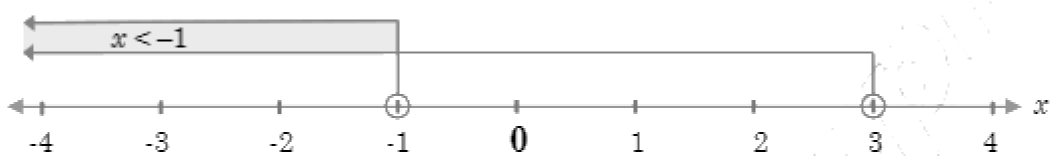


The solution set for this first case is $S_1 = \{x: x > 3\} = (3,\ \infty)$.

Case ii When both the factors are negative

$$x - 3 < 0 \text{ and } x + 1 < 0$$
$$x < 3 \text{ and } x < -1$$

The intersection of $x < 3$ and $x < -1$ is $x < -1$. This can be illustrated on the number line as shown in figure below.



The solution set for this second case is $S_2 = \{x: x < -1\} = (-\infty, -1)$.

Therefore, the solution set of $\boldsymbol{x^2 - 2x - 3 > 0}$ is:

$$S_1 \cup S_2 = \{x: x < -1 \ or \ x > 3\} = (-\infty, -1) \cup (3,\ \infty).$$

b. By product property 2: $\boldsymbol{x^2 - x - 2 = (x-2)(x+1)}$ is negative if one of the factors is negative and the other is positive. To solve $\boldsymbol{(x-2)(x+1) \leq 0}$, consider case by case as follows:

Case i When $x - 2 \geq 0$ and $x + 1 \leq 0$

$$x \geq 2 \text{ and } x \leq -1$$

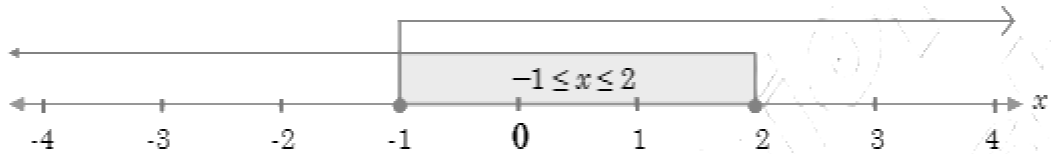
There is no intersection of $x \geq 2$ and $x \leq -1$. Graphically

So, $S_1 = \emptyset$.

Case ii When $x - 2 \le 0$ and $x + 1 \ge 0$

$x \le 2$ and $x \ge -1$

The intersection of $x \le 2$ and $x \ge -1$ is $-1 \le x \le 2$. This can be illustrated on the number line as shown in figure below.



The solution set for this second case is $S_2 = \{x: -1 \le x \le 2\} = [-1,\ 2]$.

Therefore, the solution set of $(\boldsymbol{x-2})(\boldsymbol{x+1}) \le \mathbf{0}$ is:

$$S_1 \cup S_2 = \{x: -1 \le x \le 2\} = [-1,\ 2].$$

Solving Quadratic Inequalities Using the Sign Chart Method

Steps to solve quadratic inequality using sign chart method

Step 1 Factorize the expression

Step 2 Find the zeros of each factors

Step 3 Draw a sign chart, noting the sign of each factors

Step 4 Read the solution from the last line of the sign chart

Example

Solve the inequality $2x^2 + 3x - 2 \ge 0$ using the sign chart method:

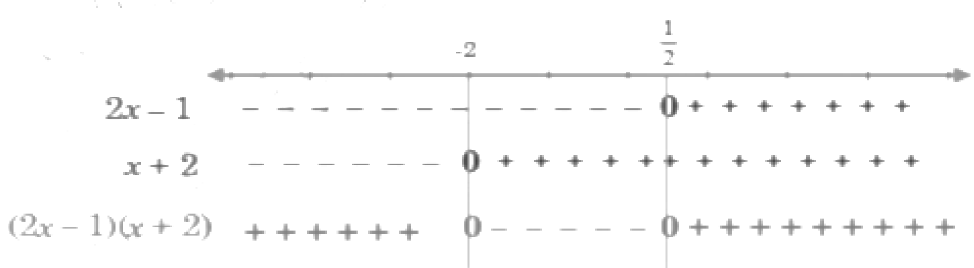
Solution

$2x^2 + 3x - 2 = (2x-1)(x+2) \ge 0.$

$2x - 1 < 0$ for each $x < \frac{1}{2}$, $2x - 1 = 0$ at $x = \frac{1}{2}$, and $2x - 1 > 0$ for each $x > \frac{1}{2}$.

Similarly, $x + 2 < 0$ for each $x < -2$, $x + 2 = 0$ at $x = -2$ and $x + 2 > 0$ for each $x > -2$.

The above results are shown in the sign chart given below:



From the sign chart, you can conclude that

$(2x-1)(x+2) \ge 0$ for each $x \in (-\infty, -2] \cup \left[\frac{1}{2}, \infty\right)$ and

$(2x-1)(x+2) < 0$ for each $x \in \left(-2,\ \frac{1}{2}\right)$.

Therefore, the solution set of $2x^2 + 3x - 2 \ge 0$ is $(-\infty, -2] \cup \left[\frac{1}{2}, \infty\right)$

1.7 Rational Expression and Rational Equation

A. Rational Expressions

Definition

A rational expression is the quotient $\frac{P(x)}{Q(x)}$ of two polynomials $P(x)$ *and* $Q(x)$, where $Q(x) \neq 0$. $P(x)$ is called the numerator and $Q(x)$ is called the denominator.

To simplify a rational expression:

1 Find the domain.
2 Factorize the numerator and denominator completely.
3 Divide the numerator and denominator by any common factor (i.e. cancel like terms).

Example

Find the domain of the rational expression $\frac{x+4}{2x^2-11x+5}$.

Solution

$2x^2 - 11x + 5 = 0$ Set the denominator equal to zero and solve for x.

$(2x - 1)(x - 5) = 0$ This is a factorable quadratic equation.

$2x - 1 = 0$ or $x - 5 = 0$

$x = \frac{1}{2}$ or $x = 5$

Domain: $\{x \mid x \text{ is a real number and } x \neq \frac{1}{2}, x \neq 5\}$

Simplify the expression $\frac{2x^3+12x^2+16x}{6x+24}$. to the lowest term

a. Factor the numerator and denominator.
b. Determine the domain and write the domain in set-builder notation.
c. Simplify the expression.

Solution:

a. $\frac{2x^3 + 12x^2 + 16x}{6x + 24}$

$= \frac{2x(x^2 + 6x + 8)}{6(x + 4)}$ Factor the numerator and denominator.

$= \frac{2x(x + 4)(x + 2)}{6(x + 4)}$

b. To determine the domain, set the denominator equal to zero and solve for x.

$6(x + 4) = 0$ Set $x + 4 = 0$. Note that the factor of 6 will never equal 0.

$x = -4$

The domain of the function is $\{x \mid x \text{ is a real number and } x \neq -4\}$.

c. $\frac{\overset{1}{\cancel{2}}x\overset{1}{\cancel{(x + 4)}}(x + 2)}{\cancel{2} \cdot 3\cancel{(x + 4)}}$ Simplify the ratio of common factors to 1.

$= \frac{x(x + 2)}{3}$ provided $x \neq -4$

Simplify

$$\frac{x^{-1}-x^{-2}}{1+2x^{-1}-3x^{-2}}$$

Solution

$$\frac{x^{-1}-x^{-2}}{1+2x^{-1}-3x^{-2}}$$

$$=\frac{\frac{1}{x}-\frac{1}{x^2}}{1+\frac{2}{x}-\frac{3}{x^2}}$$

Rewrite the expression with positive exponents. The LCD of all individual terms is x^2.

$$=\frac{x^2\cdot\left(\frac{1}{x}-\frac{1}{x^2}\right)}{x^2\cdot\left(1+\frac{2}{x}-\frac{3}{x^2}\right)}$$

Step 1: Multiply the numerator and denominator of the complex fraction by the LCD x^2.

$$=\frac{x^2\left(\frac{1}{x}\right)-x^2\left(\frac{1}{x^2}\right)}{x^2(1)+x^2\left(\frac{2}{x}\right)-x^2\left(\frac{3}{x^2}\right)}$$

Step 2: Apply the distributive property.

$$=\frac{x-1}{x^2+2x-3}$$

$$=\frac{x-1}{(x+3)(x-1)}$$

Step 3: Factor and simplify to lowest terms.

$$=\frac{\overset{1}{\cancel{x-1}}}{(x+3)\cancel{(x-1)}}$$

$$=\frac{1}{x+3}$$

B. Rational Equation

Definition

Equation that can be reduced to the form of $\frac{P(x)}{Q(x)}=0$, where $P(x)$ *and* $Q(x)$ are polynomials and $Q(x)\neq 0$ is called rational equation.

Steps for Solving a Rational Equation

1. Factor the denominators of all rational expressions. Identify any values of the variable for which any expression is undefined.
2. Identify the LCD of all terms in the equation.
3. Multiply both sides of the equation by the LCD.
4. Solve the resulting equation.
5. Check the potential solutions in the original equation. Note that any value from step 1 for which the equation is undefined cannot be a solution to the equation.

Examples

Solve the equation. $3 - \dfrac{6w}{w+1} = \dfrac{6}{w+1}$

Solution:

$$3 - \frac{6w}{w+1} = \frac{6}{w+1}$$

The LCD of all terms in the equation is $w + 1$.

$$(w+1)(3) - (w+1)\left(\frac{6w}{w+1}\right) = (w+1)\left(\frac{6}{w+1}\right)$$

Multiply by $(w + 1)$ on both sides to clear fractions.

$$(w+1)(3) - \cancel{(w+1)}\left(\frac{6w}{\cancel{w+1}}\right) = \cancel{(w+1)}\left(\frac{6}{\cancel{w+1}}\right)$$

Apply the distributive property.

$$3w + 3 - 6w = 6$$
$$-3w = 3$$
$$w = -1$$

Solve the resulting equation.

Check: $3 - \dfrac{6w}{w+1} = \dfrac{6}{w+1}$

$$3 - \frac{6(-1)}{(-1)+1} \stackrel{?}{=} \frac{6}{(-1)+1}$$

The denominator is 0 for the value of $w = -1$.

Because the value $w = -1$ makes the denominator zero in one (or more) of the rational expressions within the equation, the equation is *undefined* for $w = -1$. No other potential solutions exist, so the equation $3 - \dfrac{6w}{w+1} = \dfrac{6}{w+1}$ has no solution.

Example

Solve the equation $\frac{36}{p^2-9} + 1 = \frac{2p}{p+3}$.

Solution

$$\frac{36}{(p+3)(p-3)} = \frac{2p}{p+3} - 1$$

The LCD is $(p + 3)(p - 3)$. Expressions will be undefined for $p = 3$ and $p = -3$.

Multiply both sides by the LCD to clear fractions.

$$(p+3)(p-3)\left[\frac{36}{(p+3)(p-3)}\right] = (p+3)(p-3)\left(\frac{2p}{p+3}\right) - (p+3)(p-3)(1)$$

$$\cancel{(p+3)(p-3)}\left[\frac{36}{\cancel{(p+3)(p-3)}}\right] = \cancel{(p+3)}(p-3)\left(\frac{2p}{\cancel{p+3}}\right) - (p+3)(p-3)(1)$$

$36 = 2p(p-3) - (p+3)(p-3)$ Solve the resulting equation.

$36 = 2p^2 - 6p - (p^2 - 9)$ The equation is quadratic.

$36 = 2p^2 - 6p - p^2 + 9$

$36 = p^2 - 6p + 9$

$0 = p^2 - 6p - 27$ Set the equation equal to zero and factor.

$0 = (p-9)(p+3)$

$p = 9 \quad \text{or} \quad p = -3$ $p = -3$ is not in the domain of the original expressions. Therefore, it cannot be a solution. This can be verified when we check.

<u>Check</u>: $p = 9$

$$\frac{36}{p^2 - 9} = \frac{2p}{p+3} - 1$$

$$\frac{36}{(9)^2 - 9} \stackrel{?}{=} \frac{2(9)}{(9)+3} - 1$$

$$\frac{36}{72} \stackrel{?}{=} \frac{18}{12} - 1$$

$$\frac{1}{2} \stackrel{?}{=} \frac{3}{2} - 1$$

$$\frac{1}{2} = \frac{1}{2} \checkmark$$

The solution is $p = 9$.

<u>Check</u>: $p = -3$

$$\frac{36}{p^2 - 9} = \frac{2p}{p+3} - 1$$

$$\frac{36}{(-3)^2 - 9} \stackrel{?}{=} \frac{2(-3)}{(-3)+3} - 1$$

$$\frac{36}{0} \stackrel{?}{=} \frac{-6}{0} - 1$$

Denominator is zero.

Application problems

Example

An athlete's average speed on her bike is 14 mph faster than her average speed running. She can bike 31.5 mi in the same time that it takes her to run 10.5 mi. Find her speed running and her speed biking.

Solution

Because the speed biking is given in terms of the speed running, let x represent the running speed.

Let x represents the speed running. Then $x + 14$ represents the speed biking. Organize the given information in a chart given below.

	Distance	Rate	Time
Running	10.5	x	$\frac{10.5}{x}$
Biking	31.5	$x + 14$	$\frac{31.5}{x+14}$

$$\text{Because } d = rt \Rightarrow t = \frac{d}{r}$$

The time required to run 10.5 mi is the same as the time required to bike 31.5 mi, so we can equate the two expressions for time:

$$\frac{10.5}{x} = \frac{31.5}{x+14} \qquad \text{The LCD is } x(x+14).$$

$$x(x+14)\left(\frac{10.5}{x}\right) = x\cancel{(x+14)}\left(\frac{31.5}{\cancel{x+14}}\right) \qquad \text{Multiply by } x(x+14) \text{ to clear fractions.}$$

$$10.5(x+14) = 31.5x \qquad \text{The resulting equation is linear.}$$

$$10.5x + 147 = 31.5x \qquad \text{Solve for } x.$$

$$-21x = -147$$

$$x = 7$$

The athlete's speed running is 7 mph.

The speed biking is $x + 14$ or $7 + 14 = 21$ mph.

Example

The ratio of male to female police officers in a certain town is 11:3. If the total number of officers is 112, how many are men and how many are women?

Solution:
Let x represent the number of male police officers. Then 112 – x represents the number of female police officers.

$$\boxed{\frac{\text{Male}}{\text{Female}}} \rightarrow \frac{11}{3} = \frac{x}{112-x} \leftarrow \boxed{\frac{\text{Number of males}}{\text{Number of females}}}$$

$$3(112-x)\left(\frac{11}{3}\right) = 3\cancel{(112-x)}\left(\frac{x}{\cancel{112-x}}\right) \qquad \text{Multiply both sides by } 3(112-x).$$

$$11(112-x) = 3x \qquad \text{The resulting equation is linear.}$$

$$1232 - 11x = 3x$$

$$1232 = 14x$$

$$\frac{1232}{14} = \frac{14x}{14}$$

$$x = 88$$

The number of male police officers is $x = 88$.

The number of female officers is $112 - x = 112 - 88 = 24$.

Review Exercise

1. Solving equations involving exponents and radicals

Solve the following equations

a. $2\sqrt{x+1} = 3x$ c. $\sqrt[3]{2x-9} - 1 = 2$ e. $2\sqrt[3]{x-3} = 4$ g. $(x+30)^{\frac{1}{2}} = x$

b. $\sqrt{x+2} + 1 = \sqrt{3-x}$ d. $\sqrt{7x+15} = x+1$ *f.* $\sqrt{2x+5} = \sqrt{x+7}$ h. $(5x^2-4)^{\frac{1}{4}} = x$

2. Solving equations involving system of linear equations

Solve the following system of linear equations

a. $\begin{cases} 2x - y = 4 \\ 4x + 2 = 2y \end{cases}$ b. $\begin{cases} -x + 3y = 6 \\ 6y = 2x + 12 \end{cases}$ C. $\begin{cases} 2x = 3y + 3 \\ y = \frac{2}{3}x - 1 \end{cases}$ d. $\begin{cases} 2x + y = 4 \\ x + 2y = -1 \end{cases}$

3. Solving equations involving absolute value

Solve the following absolute value equations

a. $|2x-3| = 7$ c. $|3x+1| + |x-3| = 5$ e. $|2x+1| = |x-3|$

b. $|3x+1| = 2 - x$ d. $|3x+1| - x = 5$ *f.* $2|x+1| - |x-3| = 1$

4. Solving inequality involving absolute value

Solve the following absolute value inequality

a. $|2x-3| \le 7$ c. $|3x+1| + |x-3| > 5$ e. $|2x+1| < |x-3|$

b. $|3x+1| \ge 2 - x$ d. $|3x+1| - x < 5$ *f.* $2|x+1| - |x-3| \ge 1$

5. Solving system of linear inequality in two variables

a. $\begin{cases} 2x - y \le 4 \\ 4x + 2 > 2y \end{cases}$ b. $\begin{cases} -x + 3y < 6 \\ 6y > 2x + 12 \end{cases}$ C. $\begin{cases} 2x \ge 3y + 3 \\ y \ge \frac{2}{3}x - 1 \end{cases}$ d. $\begin{cases} 2x + y \le 4 \\ x + 2y \le -1 \end{cases}$

6. Solving quadratic equations and inequality

Solve the following quadratic equations

a. Find the roots of i) $t^2 - 12t + 35 = 0$ ii) $5x^2 - 2 = 3x$

b. Solve for x: i) $x^2 + 11x + 24 = 0$ ii) $2x^2 - 4x = 1$

Solve the following quadratic inequalities

a. $x^2 - 11x + 28 \le 0$ c. $x^2 + x - 12 \le 0$ e. $x^2 - 3x + 4 < 0$

b. $x^2 + x + 4 \ge 0$ d. $x^2 - 11x + 28 \ge 0$ *f.* $-2x^2 + x - 3 > 0$

7. Solving equations involving rational expressions

Solve the rational equations

a. $\frac{t}{t-1} - \frac{1}{t-2} = \frac{11}{t^2-3t+2}$ d. $\frac{2x}{x+1} + \frac{3x}{x+1} = \frac{x^2}{x^2+2t+1}$ g. $\frac{x-3}{x+6} + \frac{x-2}{x-3} = \frac{x^2}{x^2+3x-18}$

b. $\frac{2x}{x+1} - \frac{3}{x+5} = -\frac{8x^2}{x^2+6x+5}$ e. $\frac{x}{x-8} - \frac{2}{x-4} = \frac{x^2}{x^2-12x+32}$ h. $\frac{x}{x-2} + \frac{2}{x-4} = \frac{4x-12}{x^2-6x+8}$

c. $\frac{6x+5}{2x^2-2x} + \frac{2}{x^2-1} = \frac{3x}{x^2-1}$ *f.* $\frac{x-5}{x-9} + \frac{x+3}{x-3} = -\frac{4x^2}{x^2-12x+27}$

Word Problem

1. Find two consecutive odd integers whose product is 99.
2. A certain number added to its square is 30. Find the number.
3. A plot of land for sale has a width of x cm., and a length that is 8 cm less than its width. A farmer will only purchase the land if it measures 240 square cm. What value of x will cause the farmer to purchase the land?

4. The length of a rectangle is 5 inches more than twice a number. The width is 4 inches less than the same number. If the area of the rectangle is 15, find the number.
5. The square of a number exceeds the number by 72. Find the number.
6. The ages of three family children can be expressed as consecutive integers. The square of the age of the youngest child is 4 more than eight times the age of the oldest child. Find the ages of the three children.
7. The altitude of a triangle is 5 less than its base. The area of the triangle is 42 square inches. Find its base and altitude
8. The length of a rectangle exceeds its width by 4 inches. Find the dimensions of the rectangle if its area is 96 square inches.
9. If the measure of one side of a square is increased by 2 centimeters and the measure of the adjacent side is decreased by 2 centimeters, the area of the resulting rectangle is 32 square centimeters. Find the measure of one side of the square (the original figure).
10. Kassa is building a deck on the back of his house. He has enough lumber for the deck to be 144 square feet. The length should be 10 feet more than its width. What should the dimensions of the deck be?
11. The product of two consecutive negative odd integers is 483. Find the integers.
    Ans:
12. Suppose the solution set of $2x^2 + tx + 1 > 0$ consists of the set of all real numbers. Find all possible values of t.
    Ans: $\boldsymbol{t \in \left[-2\sqrt{2},\ 2\sqrt{2}\right]}$
13. Michael buys two bags of chips and three boxes of pretzels for \$5.13. He then buys another bag of chips and two more boxes of pretzels for \$3.09. Find the cost of each bag of chips and each box of pretzels.

    Ans \$ 0.99 for a bag of chips; \$ 1.05 for a box of pretzels

14. At a restaurant four people order fried crab claws and four people order a cup of gumbo, with a total bill of \$ 31. If only two people had ordered the crab claws and one person ordered the gumbo, the bill would have been \$ 12.25. How much is each order of fried crab claws and each cup of gumbo?
    Ans: \$4.50 for an order of crab claws; \$3.25 for a cup of gumbo
15. If one of the roots of a quadratic equation $x^2 + tx + 6 = 0$ exceeds the other by 1, then find the value of t.
    Ans: t = 5
16. If r_1 and r_2 are the roots of the quadratic equation $2x^2 - 4x + 1 = 0$, then find the value of $\frac{1}{r_1} + \frac{1}{r_2}$.
17. The product of the ages of Abebe and Belay is 130 less than the product of their ages in 5 years. If Belay is 3 years older than Abebe, what are their current ages?

CHAPTER 2

RELATION AND FUNCTION

2.1 Definition and Examples of Relation

A relation R from a non-empty set A to a non-empty set B is a subset of the Cartesian product A × B. The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R. The set of all second elements in a relation R from a set A to a set B is called the range of the relation R. The whole set B is called the codomain of the relation R. Note that range is always a subset of the codomain

Types of Relations

A relation R in a set A is subset of A × A. Thus empty set φ and A × A are two extreme relations.
(i) A relation R in a set A is called empty relation, if no element of A is related to any element of A, i.e., $R = \varphi \subset A \times A$.

(ii) A relation R in a set A is called universal relation, if each element of A is related to every element of A, i.e., $R = A \times A$.

(iii) A relation R in A is said to be reflexive if aRa for all a∈A, that is for all a∈A, $(a, a) \in R$. R is symmetric if aRb ⇒ bRa, ∀ a, b ∈ A and it is said to be transitive if aRb and bRc ⇒ aRc ∀ a, b, c ∈ A.

Any relation which is reflexive, symmetric and transitive is called an equivalence relation.

Example

Example 1 Let A = {0, 1, 2, 3} and define a relation R on A as follows:

R = {(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)}.

Is R reflexive? symmetric? transitive?

Solution R is reflexive and symmetric, but not transitive since for (1, 0) ∈ R and (0, 3) ∈ R whereas (1, 3) ∉ R.

Example

Let A = {0, 1, 2, 3}. R = {(0, 0), (0, 2), (0, 3), (2, 3)}. Is R reflexive, symmetric, and transitive?

Solution

Not reflexive. e.g.: (1, 1) 6 ∈ R. ∃x ∈ A, (x, x) 6 ∈ R.

Not symmetric. e.g.: (0, 3) ∈ R but (3, 0) 6 ∈ R. ∃x, y ∈ A, if (x, y) ∈ R, then (y, x) 6 ∈ R.

Transitive. ∀x, y, z ∈ A, if (x, y) ∈ R and (y, z) ∈ R, then (x, z) ∈ R.

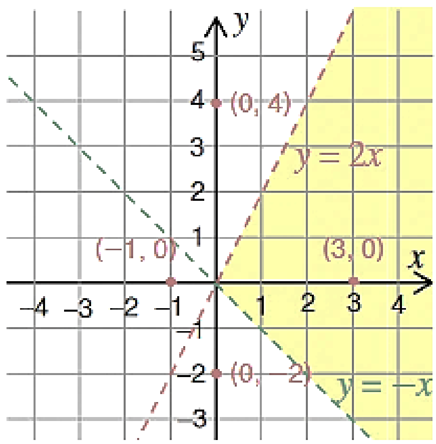
Example

Find the domain range the following relations

a. $R_1 = \{(1,\ 2), (3,\ 4), (5,\ 6), (7,\ 8), (9,\ 10)\}$ b. $R_2 = \{(x,\ y): y < 2x$ and $y > -x\}$

Solution

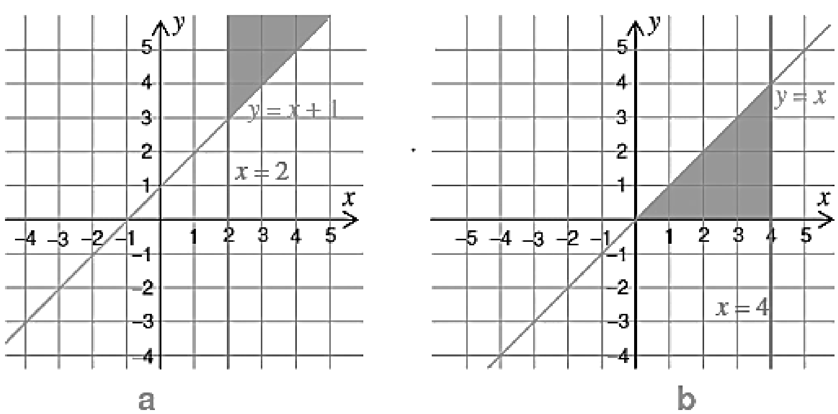
a. Domain of $R_1 = \{1,\ 3,\ 5,\ 7,\ 9\}$ and Range of $R_1 = \{2,\ 4,\ 6,\ 8,\ 10\}$
b. First sketch the graph of R_2. Sketch the graphs of y < 2x and y > –x on same coordinate system. Note that these two lines divide the coordinate system into four regions. Take any points one from each region and check if they satisfy the relation. Say, (3, 0), (0, 4), (–1, 0) and (0,–2). (3, 0) satisfies both inequalities of the relation. So the graph of the relation is the region that contains (3, 0).



From the graph above,
Domain of $R_2 = \{x: x > 0\}$ Range of $R_2 = \{y: y \in \mathbb{R}\}$

Example

From the graph of each of the following relations, represented by the shaded region, specify the relation and determine the domain and the range:



Solution

a. R = {(x, y): y ≥ x+1 and x ≥ 2}; Domain = {x: x ≥ 2} and Range = {y: y ≥ 3}
b. R = $\{(x, y): y \leq x, x \leq 4$ and $y \geq 0\}$; Domain = $\{x: 0 \leq x \leq 4\}$ and Range = $\{y: 0 \leq y \leq 4\}\}$

Inverse of a relation

Definition:

- Let R be a relation from A to B. Then inverse relation R^{-1} from B to A is:

$$R^{-1} = \{(y,x) \in B \times A \mid (x,y) \in R\}.$$

- For all x ∈ A and y ∈ B, (x, y) ∈ R ⇔ (y, x) ∈ R^{-1}

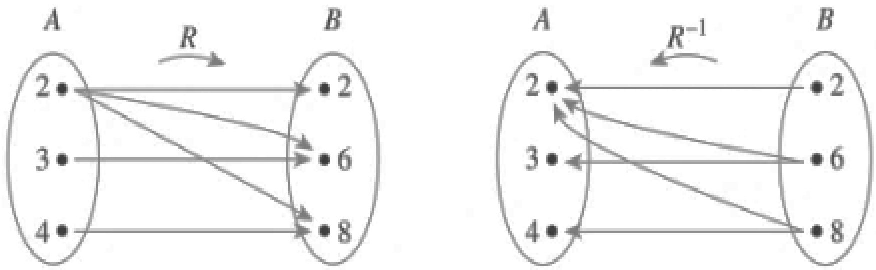
Example

Let A = {2, 3, 4} and B = {2, 6, 8}. Let R: A to B. For all (a, b) ∈ A × B, a R b ⇔ a | b

Determine R and R^{-1}. Draw arrow diagrams for both. Describe R^{-1} in words.

Solution

R = {(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)} R^{-1} = {(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)} For all (b, a) ∈ B × A, (b, a) ∈ R^{-1} ⇔ b is a multiple of a



Example

Let R = $\{(x,y): y \leq x + 3 \; and \; y > -2x + 6\}$. Find R^{-1}

Solution

$$\begin{aligned} R^{-1} &= \{(y, x): y \leq x + 3 \text{ and } y > -2x + 6\} \\ &= \{(x, y): x \leq y + 3 \text{ and } x > -2y + 6\} \\ &= \left\{(x, y): y \geq x - 3 \text{ and } y > -\frac{1}{2}x + 3\right\} \end{aligned}$$

Remark!

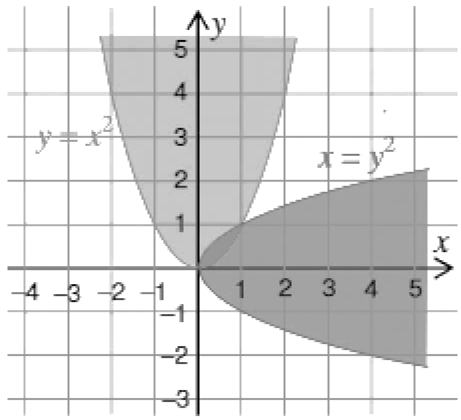
1. Domain of R^{-1} = Range of R
2. Range of R^{-1} = Domain of R
3. $(R^{-1})^{-1}$ = **R**
4. If the boundary curve in the graph of a relation is not part of the relation, it is shown using a broken line otherwise it is shown by solid line
5. The graph of R^{-1} is the mirror image of the graph of **R** about the line **y** = **x**.

Example

Let $\boldsymbol{R} = \{(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{y} \geq \boldsymbol{x}^2\}$. Draw the graph of R and $\mathbf{R}^{-1}$ using the same coordinate axes.

Solution

$R^{-1} = \{(\mathrm{x}, \mathrm{y}) : \mathrm{x} \geq \mathrm{y}^2\}$



Notice that $y = x^2$ and $x = y^2$ meet at (0, 0) and (1, 1). The equation of the line through the two points is y = x.

2.2 Definition and examples on Function

Definition: A function is a relation in which each element of the domain is paired with EXACTLY one element of the range. That is. A function is a relation such that no two ordered pairs have the same first-coordinates and different second-coordinates.

Examples

1. The relation $R_1 = \{(1, 3), (2, 5), (1, 5)\}$ is not function. Because 1 element of the domain related with 3 and 5.
2. The relation $R_2 = \{(1, 3), (2, 5), (1, 3)\}$ is a function. Because 1 element of the domain related with 3 and 3.
3. The relation $R = \{(x, y) : y \leq x\}$ is not function.

Example : If A = {1, 2, 3} and f, g are relations corresponding to the subset of A × A indicated against them, which of f, g is a function? Why?

$f = \{(1, 3), (2, 3), (3, 2)\}$

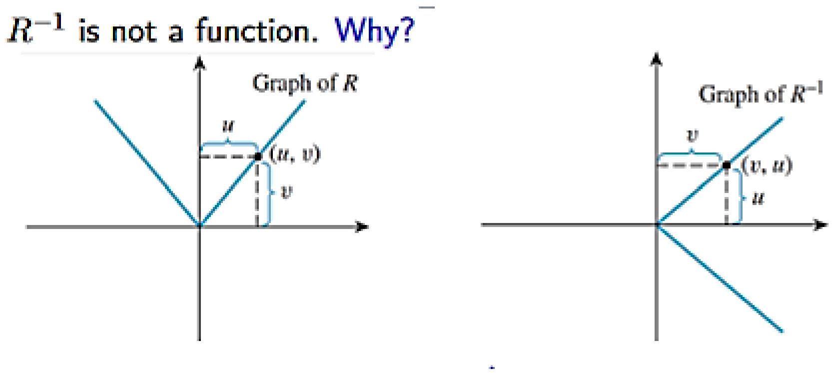
$g = \{(1, 2), (1, 3), (3, 1)\}$

Solution f is a function since each element of A in the first place in the ordered pairs is related to only one element of A in the second place while g is not a function because 1 is related to more than one element of A, namely, 2 and 3.

If $f = \{(5, 2), (6, 3)\}$ and g = {(2, 5), (3, 6)}, write the range of f *and* g.

Solution: The range of $f = \{2, 3\}$ and g = {5, 6}.

Define a relation R from R to R as follows: For all (u, v) ∈ R × R, u R v ⇔ v = 2|u|. Draw the graphs of R and R^{-1} in the Cartesian plane. Is R^{-1} a function?



2.3 Classification of Function

Definition

A function f : A → B is said to be

i. Odd, if and only if, for any x ∈A, $f(-x) = -f(x)$.

ii. Even, if and only if, for any x∈A, $f(-x) = f(x)$. The evenness or oddness of a function is called its **parity.**

Example

a. $f(x) = 2x^3$ is odd function, since $f(-x) = 2(-x)^3 = -2x^3 = -f(x)$.

b. $f(x) = 3x^2$ is even function, since $f(-x) = 3(-x)^2 = 3x^2 = f(x)$.

c. $f(x) = x^3 + 5$ is neither odd nor even, since $f(-x) = (-x)^3 + 5 = -x^3 + 5 \neq -(x^3 + 5) = -f(x)$ and $f(-x) = 2(-x)^3 + 5 = -2x^3 + 5 \neq 2x^3 + 5 = f(x)$.

One to one function

Definition

A function f: X → Y is defined to be one-one (or injective), if the images of distinct elements of X under f are distinct. i.e.

$$x_1, x_2 \in X,\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

Example

Show that, $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x$ is one-to-one function.

Solution

Let $x_1, x_2 \in \mathbb{R}$ be any two elements such that $f(x_1) = f(x_2)$.

Then, $3x_1 = 3x_2 \Rightarrow \frac{1}{3}(3x_1) = \frac{1}{3}(3x_2) \Rightarrow x_1 = x_2$. Thus, f is one-to-one.

The horizontal line test:

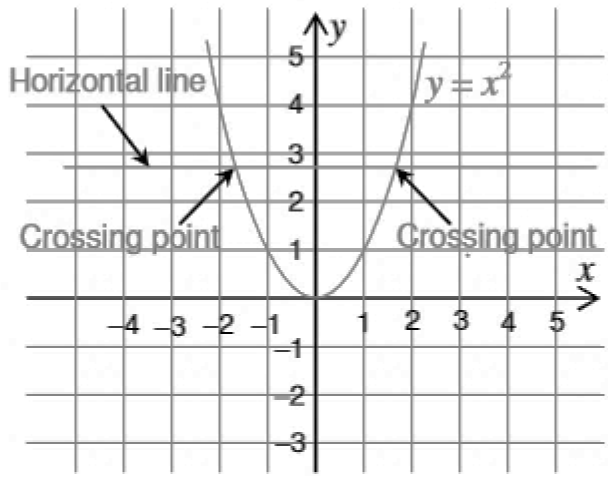
A function f : A → B is one-to-one, if and only if any horizontal line crosses its graph at most once.

Example

Using the horizontal line test show that, $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not one-to-one.

Solution

A horizontal line crosses the graph of $y = x^2$ at two points (Figure below). Thus, f is not one- to- one.



Onto function

Definition

A function $f : A \to B$ is onto (a surjection), if and only if, Range of $f = B$. That is.

A function $f: X \to Y$ is defined to be onto (or surjective), if every element of Y is the image of some element of X under f. i.e. for every $y \in Y$ there exists an element $x \in X$ such that $f(x) = y$.

Example Show that the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \dfrac{x}{x^2+1}, \forall x \in \mathbf{R}$, is neither one-one nor onto.

Solution For $x_1, x_2 \in \mathbf{R}$, consider

$$f(x_1) = f(x_2)$$

$$\Rightarrow \frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1}$$

$$\Rightarrow x_1 x_2^2 + x_1 = x_2 x_1^2 + x_2$$

$$\Rightarrow x_1 x_2 (x_2 - x_1) = x_2 - x_1$$

$$\Rightarrow x_1 = x_2 \text{ or } x_1 x_2 = 1$$

We note that there are point, x_1 and x_2 with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$, for instance, if we take $x_1 = 2$ and $x_2 = \frac{1}{2}$, then we have $f(x_1) = \frac{2}{5}$ and $f(x_2) = \frac{2}{5}$ but $2 \neq \frac{1}{2}$. Hence f is not one-one. Also, f is not onto for if so then for $1 \in \mathbf{R}$ $\exists$ $x \in \mathbf{R}$ such that $f(x) = 1$

which gives $\frac{x}{x^2+1} = 1$. But there is no such x in the domain $\mathbf{R}$, since the equation $x^2 - x + 1 = 0$ does not give any real value of x.

One-to-one Correspondence

Definition

A function: A Bf → is a one-to-one correspondence (a bijection), if and only if, f is one-to-one and onto.

Example

Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 5x - 7$. Show that f is a one-to- one correspondence.

Solution

Let $x_1, x_2 \in \mathbb{R}$, such that $f(x_1) = f(x_2)$

$$\Rightarrow 5x_1 - 7 = 5x_2 - 7 \Rightarrow 5x_1 - 7 + 7 = 5x_2 - 7 + 7$$
$$\Rightarrow 5x_1 = 5x_2 \Rightarrow x_1 = x_2$$

So, f is one-to-one.
Let $y \in \mathbb{R}$. Is there $x \in \mathbb{R}$ such that $y = f(x)$?
If there is, it can be found by solving $y = f(x) = 5x - 7$

$$\Rightarrow y + 5 = 7x \Rightarrow x = \frac{y+5}{7}$$

So for any $y \in \mathbb{R}$, take $x = \frac{y+5}{7} \in \mathbb{R}$.

Then, $f(x) = f\left(\frac{y+5}{7}\right) = 7\left(\frac{y+5}{7}\right) - 5 = y$

So, f is onto. Therefore, f is one-to-one correspondence.

Example

Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3^x$. Check whether or not f is a one-to-one correspondence.

Solution

For any $x_1, x_2 \in \mathbb{R}$,

$$f(x_1) = f(x_2) \Rightarrow 3^{x_1} = 3^{x_2} \Rightarrow \frac{3^{x_1}}{3^{x_2}} = 1 \Rightarrow 3^{x_1 - x_2} = 1 = 3^0$$

$$\Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$$

Thus, f is **one-to-one.** But, **f is not onto**, because negative numbers cannot be images. For instance, take $\boldsymbol{y} = -\mathbf{4}$.
Since $3^x > 0$, for every $x \in \mathbb{R}$, it is not possible to have $x \in \mathbb{R}$, for which $3^x = -4$ Thus, f is not onto. Therefore, f is not a one-to-one correspondence.

Invertible Functions

Definitions

(i) A function $f : X \to Y$ is defined to be invertible, if there exists a function $g : Y \to X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}.

(ii) A function $f : X \to Y$ is invertible if and only if f is a bijective function.

(iii) If $f : X \to Y$, $g : Y \to Z$ and $h : Z \to S$ are functions, then $h \circ (g \circ f) = (h \circ g) \circ f$.

(iv) Let $f : X \to Y$ and $g : Y \to Z$ be two invertible functions. Then $g \circ f$ is also invertible with $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Steps to find the inverse of a function f.

1. Interchange x and y in the formula of f.
2. Solve for y in terms of x.
3. Write $y = f^{-1}(x)$.

Example : Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = 4x - 3 \ \forall \ x \in \mathbf{R}$. Then write f^{-1}.

Solution Given that $f(x) = 4x - 3 = y$ (say), then

$4x = y + 3$

$\Rightarrow \quad x = \dfrac{y+3}{4}$

Hence $\quad f^{-1}(y) = \dfrac{y+3}{4} \quad \Rightarrow \quad f^{-1}(x) = \dfrac{x+3}{4}$

Example : If $A = \{a, b, c, d\}$ and $f = \{a, b), (b, d), (c, a), (d, c)\}$, show that f is one-one from A onto A. Find f^{-1}.

Solution f is one-one since each element of A is assigned to distinct element of the set A. Also, f is onto since $f(A) = A$. Moreover, $f^{-1} = \{(b, a), (d, b), (a, c), (c, d)\}$.

Example: Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 3x - 4$. Then $f^{-1}(x)$ is given by

(A) $\frac{x+4}{3}$ (B) $\frac{x}{3} - 4$

(C) $3x + 4$ (D) None of these

Solution (A) is the correct choice.

Example: Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2 + 1$. Then, pre-images of 17 and -3, respectively, are

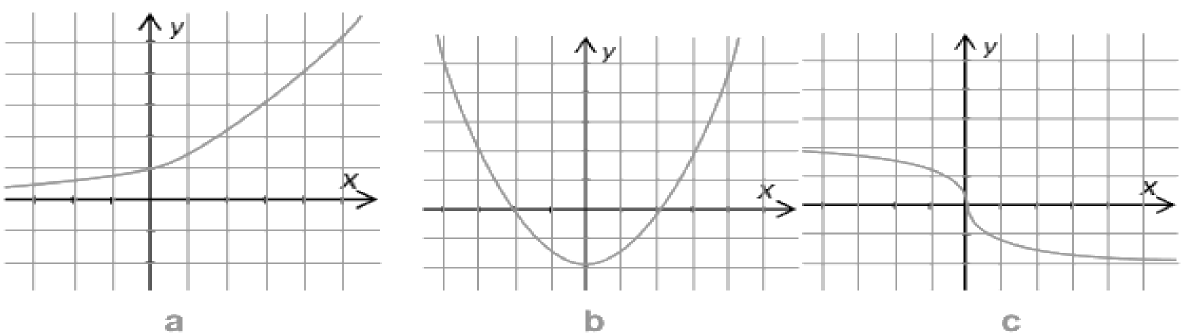
(A) $\phi, \{4, -4\}$ (B) $\{3, -3\}, \phi$

(C) $\{4, -4\}, \phi$ (D) $\{4, -4, \{2, -2\}$

Solution (C) is the correct choice since for $f^{-1}(17) = x \Rightarrow f(x) = 17$ or $x^2 + 1 = 17$ $\Rightarrow x = \pm 4$ or $f^{-1}(17) = \{4, -4\}$ and for $f^{-1}(-3) = x \Rightarrow f(x) = -3 \Rightarrow x^2 + 1 = -3 \Rightarrow x^2 = -4$ and hence $f^{-1}(-3) = \phi$.

Example

Which of the following functions are invertible?

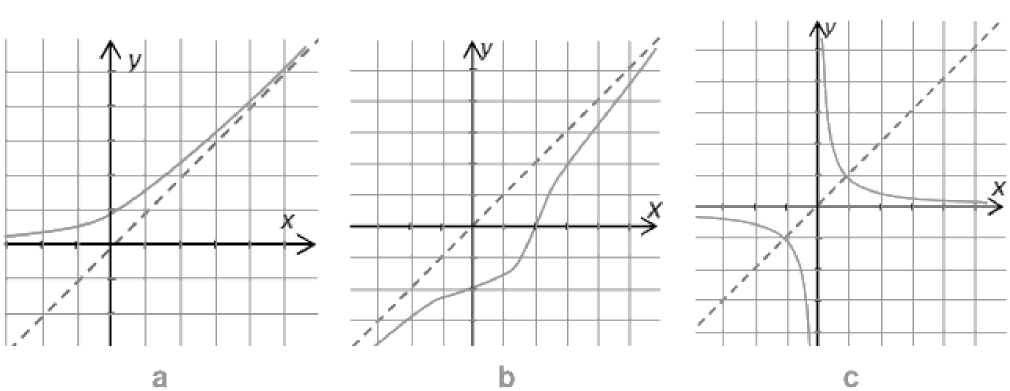


Solution

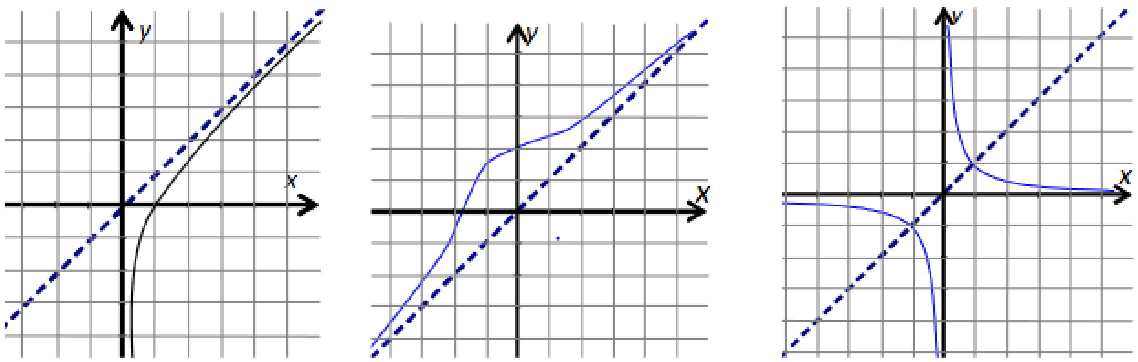
Functions given under a and c are invertible, but choice be is not invertible because it is not one-to-one function.

Example

Sketch $f^{-1}$for each of the following functions.



Solution



a. b. c.

2.4 Composition of Functions

Combination of functions

Note

Recall the following

- **Let f and g** be any two functions, then
  - $(f+\mathrm{g})(x) = f(x) + \mathrm{g}(\mathrm{x})$
  - $(f-\mathrm{g})(x) = f(x) - \mathrm{g}(\mathrm{x})$
  - $(f\mathrm{g})(x) = f(x)\mathrm{g}(\mathrm{x})$
  - $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{\mathrm{g}(\mathrm{x})}$
- Domain of $(f+\mathrm{g})$ = Domain of $(f-\mathrm{g})$ = Domain of $(f\mathrm{g})$

  =Domain of $f \cap$ Domain of g
- Domain of $\left(\frac{f}{\mathrm{g}}\right)$ = Domain of $f \cap$ Domain of g $\backslash\{x : \mathrm{g}(x) = 0\}$.

Example

Let $f = \{(2,\ 3),\ (3,\ 1),\ (4,\ 2),\ (5,\ 6)\}$ and $\mathrm{g} = \{(2,\ 1),\ (3,\ 4),\ (4,\ 0),\ (6,\ 3)\}$ be two functions. The find

a. $f+\mathrm{g}$
b. $f-\mathrm{g}$
c. $f.\mathrm{g}$
d. $f \div \mathrm{g}$
e. $3f - \mathrm{g}^2$
f. $4(f \div \mathrm{g})$

Solution

First find their domains as

Domain of $f = \{2, 3, 4, 5\}$ Domain of $\mathrm{g} = \{2, 3, 4, 6\}$

Domain of $f \cap$ Domain of $\mathrm{g} = \{2, 3, 4\}$

a. $f+\mathrm{g}=\{(2,f(2)+\mathrm{g}(2)),\ (3,\ f(3)+\mathrm{g}(3)),\ (4,f(4)+\mathrm{g}(4))\}=\{(2,4),\ (3,\ 5),\ (4,2)\}$
b. $f-\mathrm{g}=\{(2,f(2)-\mathrm{g}(2)),\ (3,\ f(3)-\mathrm{g}(3)),\ (4,f(4)-\mathrm{g}(4))\}=\{(2,2),\ (3,-3),\ (4,2)\}$
c. $f.\mathrm{g}=\{(2,f(2).\mathrm{g}(2)),\ (3,\ f(3).\mathrm{g}(3)),\ (4,f(4).\mathrm{g}(4))\}=\{(2,3),\ (3,\ 4),\ (4,0)\}$

Domain of f ∩ Domain of g $\backslash\{x:\mathrm{g}(x)=0\}=\{2,\ 3\}$

d. $\frac{f}{\mathrm{g}}=\{(2,f(2)\div\mathrm{g}(2)),\ (3,\ f(3)\div\mathrm{g}(3))\}=\left\{(2,3),\ \left(3,\ \frac{1}{4}\right)\right\}$
e. $3f-\mathrm{g}^2=\{(2,\ 9),\ (3,\ 3),\ (4,\ 6)\}-\{(2,\ 1),\ (3,\ 16),\ (4,\ 0)\}=\{(2,\ 8),\ (3,-13),\ (4,\ 6)\}$
f. $4\left(\frac{f}{\mathrm{g}}\right)=\left\{\left(2,4(f(2)\div\mathrm{g}(2))\right),\ \left(3,\ 4(f(3)\div\mathrm{g}(3))\right)\right\}=\{(2,12),\ (3,\ 1)\}$

Definition of composite of functions

i. Let $f:\mathrm{A}\to\mathrm{B}$ and $\mathrm{g}:\mathrm{B}\to\mathrm{C}$ be two functions. Then the composition of f and g denoted by $\mathrm{g}of$, is defined as the function $\mathrm{g}of:\mathrm{A}\to\mathrm{C}$ given by $(\mathrm{g}of)(x)=\mathrm{g}(f(x))\ \forall x\in A$
ii. If $f:\mathrm{A}\to\mathrm{B}$ and $\mathrm{g}:\mathrm{B}\to\mathrm{C}$ are one to one, then $\mathrm{g}of:\mathrm{A}\to\mathrm{C}$ is also one to one.
iii. If $f:\mathrm{A}\to\mathrm{B}$ and $\mathrm{g}:\mathrm{B}\to\mathrm{C}$ are onto, then $\mathrm{g}of:\mathrm{A}\to\mathrm{C}$ is also onto.
However, converse of above stated results (ii) and (iii) need not be true. Moreover, we have the following results in this direction.
iv. Let $f:\mathrm{A}\to\mathrm{B}$ and $\mathrm{g}:\mathrm{B}\to\mathrm{C}$ be given functions such that, $\mathrm{g}of$ is one-one. Then f is one-one.
v. If $f:\mathrm{A}\to\mathrm{B}$ and $\mathrm{g}:\mathrm{B}\to\mathrm{C}$ be given functions such that, $\mathrm{g}of$ is onto. Then g is onto.

Examples

Example: If $f=\{(5,2),(6,3)\}$, $g=\{(2,5),(3,6)\}$, write $f\,o\,g$.

Solution $f\,o\,g=\{(2,2),(3,3)\}$

Example: Let f, $g:\mathbf{R}\to\mathbf{R}$ be two functions defined as $f(x)=|x|+x$ and $g(x)=|x|-x\ \ \forall\ x\in\mathbf{R}$. Then, find $f\,o\,g$ and $g\,o\,f$.

Solution Here $f(x)=|x|+x$ which can be redefined as

$$f(x)=\begin{cases}2x \text{ if } x\geq 0\\ 0 \text{ if } x<0\end{cases}$$

Similarly, the function g defined by $g(x)=|x|-x$ may be redefined as

$$g(x)=\begin{cases}0 \text{ if } x\geq 0\\ -2x \text{ if } x<0\end{cases}$$

Therefore, $g\,o\,f$ gets defined as :

For $x \geq 0$, $(g \circ f)(x) = g(f(x) = g(2x) = 0$

and for $x < 0$, $(g \circ f)(x) = g(f(x) = g(0) = 0$.

Consequently, we have $(g \circ f)(x) = 0$, $\forall\ x \in \mathbf{R}$.

Similarly, $f \circ g$ gets defined as:

For $x \geq 0$, $(f \circ g)(x) = f(g(x) = f(0) = 0$,

and for $x < 0$, $(f \circ g)(x) = f(g(x)) = f(-2x) = -4x$.

i.e. $$(f \circ g)(x) = \begin{cases} 0, x > 0 \\ -4x, x < 0 \end{cases}$$

Example : Let $\mathbf{R}$ be the set of real numbers and $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = 4x + 5$. Show that f is invertible and find f^{-1}.

Solution Here the function $f : \mathbf{R} \to \mathbf{R}$ is defined as $f(x) = 4x + 5 = y$ (say). Then

$$4x = y - 5 \qquad \text{or} \qquad x = \frac{y-5}{4}.$$

This leads to a function $g : \mathbf{R} \to \mathbf{R}$ defined as

$$g(y) = \frac{y-5}{4}.$$

Therefore, $(g \circ f)(x) = g(f(x) = g(4x + 5)$

$$= \frac{4x+5-5}{4} = x$$

or $g \circ f = I_R$

Similarly $(f \circ g)(y) = f(g(y))$

$$= f\left(\frac{y-5}{4}\right)$$

$$= 4\left(\frac{y-5}{4}\right) + 5 = y$$

or $f \circ g = I_R$.

Hence f is invertible and $f^{-1} = g$ which is given by

$$f^{-1}(x) = \frac{x-5}{4}$$

Example

Let $f(x) = \frac{1}{x-3}$ **and** $\mathrm{g}(\mathrm{x}) = \frac{2}{x+1}$, Then find the formula for $(fo\mathrm{g})$ (x) and($\mathrm{g}of$) (x) and also find

Domain of $(fo\mathrm{g})(\mathrm{x})$ and $(\mathrm{g}of)$ (x)**.**

Solution

$$(fo\mathrm{g})\ (\mathrm{x}) = f(\mathrm{g}(\mathrm{x})) = \frac{1}{\mathrm{g}(\mathrm{x})-3} = \frac{1}{\frac{2}{x+1}-3} = \frac{1}{\frac{2-3x-3}{x+1}} = \frac{x+1}{-3x-1},\quad x \neq -1$$

$$(\mathrm{g}of)\ (\mathrm{x}) = \mathrm{g}(f(\mathrm{x})) = \frac{2}{f(\mathrm{x})+1} = \frac{2}{\frac{1}{x-3}+1} = \frac{2}{\frac{1+x-3}{x-3}} = \frac{2x-6}{x-2},\quad x \neq 3$$

Domain of $(fo\mathrm{g})(\mathrm{x}) = \mathbb{R} \setminus \left\{-1, -\frac{1}{3}\right\}$

Domain of $(\mathrm{g}of)\ (x) = \mathbb{R} \setminus \{2,\ 3\}$

Method 2

The two step method

To find Domain of $(\mathrm{g}of)$ (x)

Step 1: Find domain of the inner function f which is $\mathbb{R} \setminus \{3\}$

Step 2: Solve the equation $f(x) = -1, \Rightarrow \frac{1}{x-3} = -1 \Rightarrow 1 = -x + 3 \Rightarrow x = 2$ and exclude this value of x from the domain. That is.

Domain of $(\mathrm{g}of)\ (x) = \mathbb{R} \setminus \{2,\ 3\}$

Similarly, you can find Domain of $(fo\mathrm{g})(\mathrm{x})$**,** which is left for you as an exercise.

Example

Let $f\,(x) = 4x + 1$ and $\mathrm{g}(x) = 3x + k$, find the value of k for which $(fo\mathrm{g})(x) = (\mathrm{g}of)(x)$.

Solution

$(fo\mathrm{g})(x) = 4(3x + k) + 1 = 12x + 4k + 1$ and $(\mathrm{g}of)(x) = \mathrm{g}(4x + 1) = 3(4x + 1) + k = 12x + 3 + k$

Now, $(fo\mathrm{g})(x) = (\mathrm{g}of)(x) \Rightarrow 12x + 4k + 1 = 12x + 3 + k$

$$\Rightarrow 3k = 2$$

$$\Rightarrow k = \frac{2}{3}$$

2.5. Types of Functions

2.5.1 Polynomial function

Definition:

Let n be a non-negative integer and let $a_n, a_{n-1}, ---, a_1, a_0$, be real numbers with $a_n \neq 0$. The function $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is called a polynomial function in variable x of degree n.

Constant, linear and quadratic functions are all special cases of a wider class of functions called polynomial functions.

Note, in the equation: $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$

1. $a_n, a_{n-1}, ---, a_1, a_0$ are called coefficients
2. The number a_n is called the leading coefficient of the polynomial function and $a_n x^n$ is the leading term.
3. The number a_0 is called the constant term of the polynomial.
4. The number n (the exponent of the highest power of x), is the degree of the polynomial.
5. Note that the domain of a polynomial function is $\mathbb{R}$.

Example

Which of the following functions are polynomial functions? For those which are polynomials, find the degree, leading coefficient, and constant term.

a. $f(x) = \frac{3}{2}x^5 - 6x^2 + x + \frac{5}{6}$
b. $f(x) = \sqrt{x^2 + x + 4}$
c. $f(x) = \frac{x^2+3}{x^2+3}$
d. $f(x) = \left(1 - \sqrt{3}x\right)\left(1 + \sqrt{3}x\right)$
e. $f(x) = \frac{x+1}{x+1}$
f. $f(x) = \sqrt{x^2 + x - 4}$
g. $f(x) = 3x^{-5} + 2x^2 + x + 1$
h. $f(x) = \frac{3}{x}$

Solution

a. It is a polynomial function of degree 5 with leading coefficient $\frac{3}{2}$ and constant term $\frac{5}{6}$.
b. $f(x) = \sqrt{x^2 + x + 4} = x^2 + x + 4$, so it is a polynomial function because it can be written in the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$.
c. $f(x) = \frac{x^2+3}{x^2+3} = 1$, so it is a polynomial function of degree 0 with leading coefficient 1 and constant term 1.
d. It is a polynomial function of degree 2 with leading coefficient – 3 and constant term 1.
e. It is not a polynomial function because its domain is not $\mathbb{R}$.
f. $f(x) = \sqrt{x^2 + x - 4} \neq x^2 + x - 4$, so it is not a polynomial function because it cannot be written in the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$.

g. It is not a polynomial function because one of its terms has a negative exponent.
h. It is not a polynomial function because its domain is not $\mathbb{R}$.

We can restate the definitions of linear and quadratic functions using the terminology for polynomials. Linear functions are polynomial functions of degree 1. Nonzero constant functions are polynomial functions of degree 0. Similarly, quadratic functions are polynomial functions of degree 2. The zero function, $p(x) = 0$, is also considered to be a polynomial function but is not assigned a degree at this level. Note that in expressing a polynomial, we usually omit all terms which appear with zero coefficients and write others in decreasing order, or increasing order, of their exponents.

Example 2 For the polynomial function $p(x) = \dfrac{x^2 - 2x^5 + 8}{4} + \dfrac{7}{8}x - x^3$,

a what is its degree?

b find a_n, a_{n-1}, a_{n-2} and a_2.

c what is the constant term?

d what is the coefficient of x?

Solution: $$p(x) = \frac{x^2 - 2x^5 + 8}{4} + \frac{7}{8}x - x^3 = \frac{x^2}{4} - \frac{2}{4}x^5 + \frac{8}{4} + \frac{7}{8}x - x^3$$

$$= -\frac{1}{2}x^5 - x^3 + \frac{1}{4}x^2 + \frac{7}{8}x + 2$$

a The degree is 5.

b $a_n = a_5 = \dfrac{-1}{2}$, $a_{n-1} = a_4 = 0$, $a_{n-2} = a_3 = -1$ and $a_2 = \dfrac{1}{4}$.

c The constant term is 2.

d The coefficient of x is $\dfrac{7}{8}$.

Note: Based on the types of coefficients it has, a polynomial function, p is said to be:

- a polynomial function over integers, if the coefficients of p(x) are all integers.
- a polynomial function over rational numbers, if the coefficients of p (x) are all rational numbers.
- a polynomial function over real numbers, if the coefficients of p (x) are all real numbers.

Example

The function $f(x) = \frac{3}{2}x^5 - 6x^2 + x + \frac{5}{6}$ is a polynomial function over rational numbers.

The function $f(x) = 3x^{51} - 6x^2 + x$ is a polynomial function over integer.

OPERATIONS ON POLYNOMIAL FUNCTIONS

ADDITION AND SUBTRACTION POLYNOMIALS:

Adding and subtracting polynomials is the same as the procedure used in combining like terms.

- When adding polynomials, simply drop the parenthesis and combine like terms.
- When subtracting polynomials, distribute the negative first, then combine like terms.

Example

Addition:

$$(3x^2 + 4x + 5) + (4x^2 - 2x + 9) = 3x^2 + 4x^2 + 4x - 2x + 5 + 9 = 7x^2 + 2x + 14$$

Subtraction:

$$(6x^2 - 4x + 3) - (4x^2 + 3x - 5) = 6x^2 - 4x + 3 - 4x^2 - 3x + 5 = 2x^2 - 7x + 8$$

MULTIPLICATION POLYNOMIAL:

1. **Monomial times Monomial:** To multiply two monomial terms, just multiply the numbers then multiply the variables using the rules for exponents.

Example

$$3x(4x^3) = 12x^4$$

2. **Monomial times Polynomial:** Simply use the distributive property to multiply a monomial times a polynomial.

Example

$$3x(4x^2 + 3x - 5) = 3x(4x^2) + 3x(3x) + 3x(-5) = 12x^3 + 9x^2 - 15x$$

3. **Binomial times a Binomial:** To multiply two binomials, use the FOIL method (First times first, Outside times outside, Inside times inside, and Last times last).

Example

$$(4x + 3)(4x^2 - 5) = 4x(4x^2) + 4x(-5) + 3(4x^2) + 3(-5) = 16x^3 + 12x^2 - 20x - 15$$

Special Products: The following formulas may be used in these special cases as a short cut to the FOIL method.

Difference of square	**Example**
$(a - b)(a + b) = a^2 - b^2$	$(2x - 3)(2x + 3) = 4x^2 - 9$

Perfect squares | **Example**

$(a-b)^2 = a^2 - 2ab + b^2$ | $(3x-5)^2 = 9x^2 - 30x + 25$

$(a+b)^2 = a^2 + 2ab + b^2$ | $(3x+5)^2 = 9x^2 + 30x + 25$

4. **Polynomial times polynomial:** To multiply two polynomials where at least one has more than two terms, distribute each term in the first polynomial to each term in the second.
Examples:

i. $(4x+5)(4x^2-2x+9)$

$$= 4x(4x^2) + 4x(-2x) + 4x(9) + 5(4x^2) + 5(-2x) + 5(9)$$

$$= 16x^3 - 8x^2 + 36x + 20x^2 - 10x + 45$$

$$= 16x^3 + 12x^2 + 26x + 45$$

ii. $(3x^2+4x+5)(4x^2-2x+9)$

$$= 3x^2(4x^2) + 3x^2(-2x) + 3x^2(9) + 4x(4x^2) + 4x(-2x) + 4x(9) + 5(4x^2) + 5(-2x) + 5(9)$$

$$= 12x^4 - 6x^3 + 27x^2 + 16x^3 - 8x^2 + 36x + 20x^2 - 10x + 45$$

$$= 12x^4 + 10x^3 + 39x^2 + 26x + 45$$

DIVISION OF POLYNOMIALS:

1. **Division by Monomial:** Each term of the polynomial is divided by the monomial and it is simplified as individual fractions.

Examples

$$\frac{6x^2+12x-15}{3x} = \frac{6x^2}{3x} + \frac{12x}{3x} - \frac{15}{3x} = 2x^2 + 4 - \frac{5}{x}$$

2. **Division by Binomial or Larger Polynomial:**

Use the long division format as follows:

- Both the divisor and the dividend must be written in descending order.
- Any missing powers should be replaced by zero.
- All remainders are in fraction form (remainder/ divisor) and are added to the quotient.

Examples:

a. $(x^2 - 7x + 12) \div (x - 4) = x - 3$

b. $\frac{4x^3+15}{2x-4} = 2x^2 + 4x + 8 + \frac{47}{2x-4}$

Detail solution

a

$$\begin{array}{r}
x-3 \\
x-4\overline{\big)\,x^2-7x+12} \\
\underline{-\ (x^2-4x)} \\
-3x+12 \\
\underline{\underline{-(-3x+12)}} \\
0
\end{array}$$

b

$$\begin{array}{r}
2x^2+4x+8 \\
2x-4\overline{\big)\,4x^3-15} \\
\underline{-(4x^3-8x^2)} \\
8x^2-15 \\
\underline{\underline{-(8x^2-32)}} \\
17
\end{array}$$

Theorems on Polynomials

Theorem 1.1 Polynomial division theorem

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, $f(x)$ divides exactly into $d(x)$.

Theorem 1.2 Remainder theorem

Let $f(x)$ be a polynomial of degree greater than or equal to 1 and let c be any real number. If $f(x)$ is divided by the linear polynomial $(x - c)$, then the remainder is $f(c)$.

Theorem 1.3 Factor theorem

Let $f(x)$ be a polynomial of degree greater than or equal to one, and let c be any real number, then

i. $x - c$ is a factor of $f(x)$, if $f(c) = 0$, and
ii. $f(c) = 0$, if $x - c$ is a factor of $f(x)$.

Zeros of Function

Definition:

For a polynomial function f and a real number c, if $f(c) = 0$, then c is a zero of f.

If $(x - c)^k$ is a factor of $f(x)$, but $(x - c)^{k+1}$ is not, then c is said to be a zero of multiplicity k of f.

A polynomial function f has a real zero r if and only if $(x - r)$ is a factor of $f(x)$.

If r is a zero of even multiplicity, then the factor (x − r) occurs an even number of times. The graph then looks like the graph of an even power function at that zero. Hence the function "bounces" there.

If r is a zero of odd multiplicity, then the factor $(x - r)$ occurs an odd number of times. The graph then looks like the graph of an odd power function at that zero. Hence, if (x − r) occurs once, the function passes "straight through" at that zero and if (x − r) occurs any other odd number of time, the function "flattens" there.

Example

If $x + 2$ and $x - 1$ are the factors of a polynomial function P(x), then the following holds.

$x - 3$ and x are factors of $P(x - 2)$,

$x + 5$ and $x + 2$ are factors of $P(x + 3)$,

Example

If $f(x) = x^4 - 2x^3 + x - 3$ is divided by $x + 2$, then find the remainder.

Solution

Since -2 is the zero of $x + 2$, using remainder theorem, the remainder is $f(-2)$.

$f(-2) = (-2)^4 - 2(-2)^3 + (-2) - 3 = 16 + 16 - 5 = 27$

Theorem 1.4 Location Theorem

If P is a polynomial function and $P(a)$ and $P(b)$ have opposite signs, then there exists at least one value c between a and b for which $P(c) = 0$.

Theorem 1.5 Rational Root Test

If the rational number $\frac{p}{q}$, in its lowest terms, is a zero of the polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

With integer coefficients, then p must be a factor of a_0 and q must be a factor of a_n.

GRAPH OF POLYNOMIAL:

Graph of a polynomial functions are smooth curves with no breaks or corners. The end behavior of a polynomial is a description of what happens as x becomes large in the positive or negative direction.

Notation:

- $x \to \infty$ means "x becomes large in the positive direction"
- $x \to -\infty$ means "x becomes large in the negative direction"

For any polynomial, the end behavior is determined by the term that contains the highest power of x.

If P is a polynomial function, then c is called a zero of P if P(c) = 0. In other words, the zeros of P are the solutions or roots of the polynomial equation P(x) = 0.

If P is a polynomial and c is a real number, then the following are equivalent.

- c is a zero of P.
- $x = c$ is a solution of the equation $P(x) = 0$.
- $x - c$ is a factor of $P(x)$.
- $x = c$ is an x-intercept of the graph of P.

GRAPHING POLYNOMIAL FUNCTIONS

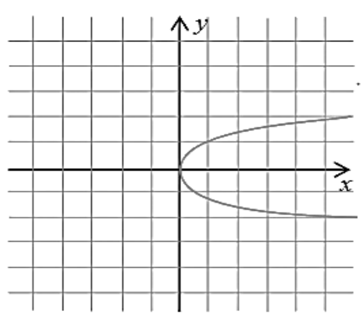
If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is a polynomials of degree n, then the graph of P has at most n – 1 turning points or local extreme.

The graph of every polynomial function has no sharp corners; it is a smooth and continuous curve.

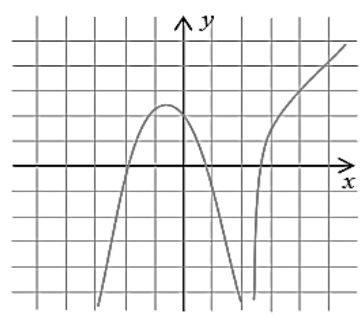
1. Find all the real zeros or x-intercepts. Break it down into linear factors by factoring methods and/or quadratic formula.
2. Plot the x-intercepts and determine the shape near the intercepts according to the multiplicity of the factor.
3. Determine the end behavior of the polynomial.
4. Make a table of values including test points to determine where the graph is above or below the x-axis and y-intercept.
5. Plot the test points and y-intercept and sketch a smooth curve passing through the points and having the required end behavior.

Example

Which of the following are graphs of a polynomial functions and which are not?

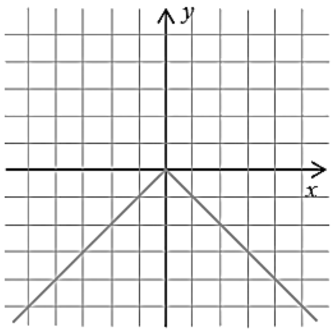


a

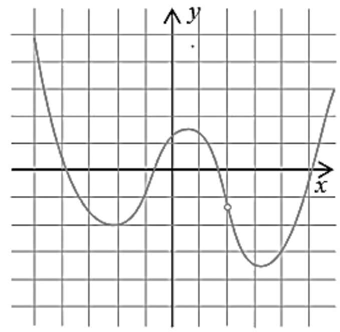


b

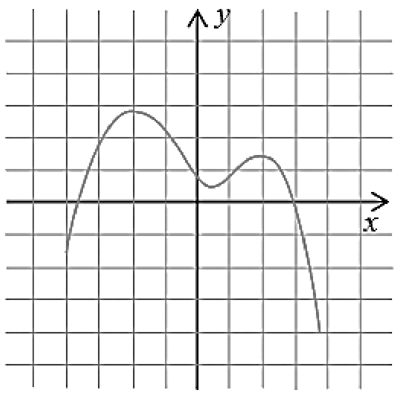
a. Not polynomial function because vertical line cross twice.
b. Not polynomial function because there is gap.



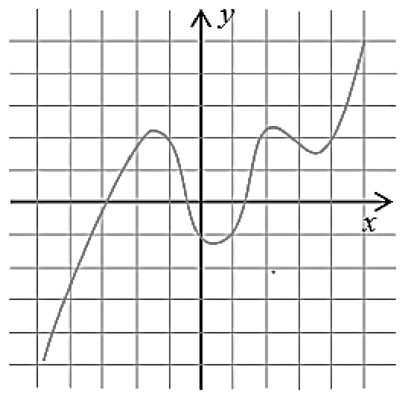
c



d



e

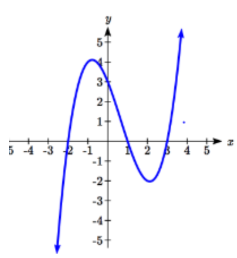
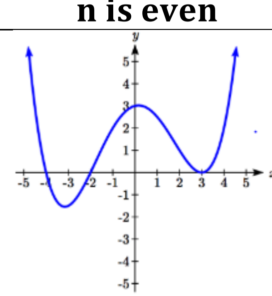
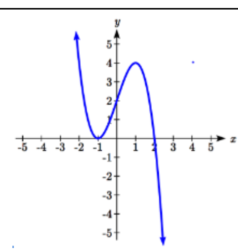
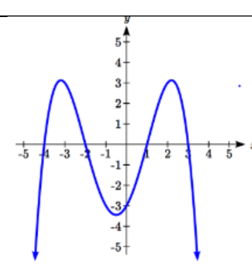


f

Answer

- **c.** Not polynomial function because there is sharp edge.
- **d.** Not polynomial function because there is hole.
- **e.** It is polynomial function.
- **f.** It is polynomial function.

General Shapes of graphs of a polynomial functions with leading term $a_n x^n$

	n is odd	**n is even**
$a_n > 0$		
$a_n < 0$		

The chart below summarizes the end behavior of a Polynomial Function.

Degree	Leading Coefficient	End behavior of graph
Even	Positive	Graph goes up to the far left and goes up to the far right.
Even	Negative	Graph goes down to the far left and down to the far right.
Odd	Positive	Graph goes down to the far left and up to the far right.
Odd	Negative	Graph goes up to the far left and down to the far right.

Example

Use the four-step process to sketch the graphs of the following polynomial functions:

$$f(x) = -2(x+2)^2(x-1)$$

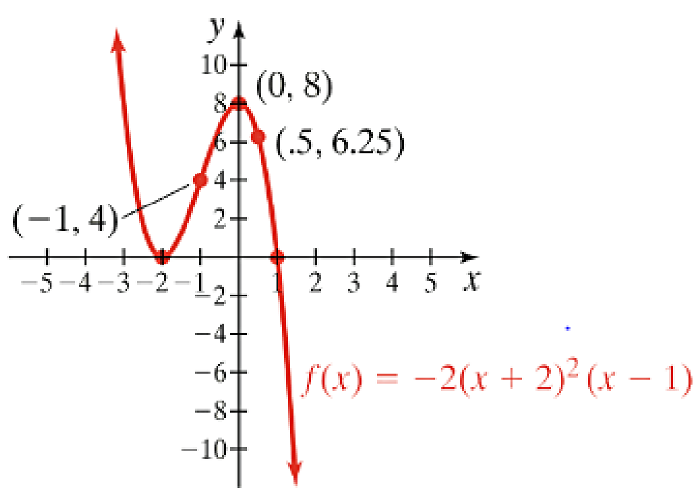
Step 1. Determine the end behavior. Odd with leading -, so left hand up, right hand down.

Step 2. Plot the y-intercept $f(0) = a_0$ $f(0) = -2(0+2)^2(0-1) = 8.$

Step 3. Completely factor f to find all real zeros and their multiplicities. - $x = -2$ with multiplicity 2 $x = 1$ with multiplicity 1.

Step 4. Choose a test value between each real zero and sketch the graph.

x	y
−1	4
0.5	6.25



2.5.2 Rational function and their graphs

Definition

A rational function is a function of the form $(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials in x and $q\ (x)\ \neq\ 0$.

Example

Which of the following are rational functions?

a. $f(x) = \frac{x+3}{x^3-3x+4}$ b. $g(x) = \frac{3x+4}{1}$ c. $k(x) = \sqrt{\frac{x+3}{x^3-3x+4}}$

Solution

f *and* g are rational functions, while k is not rational function. $g(x) = \frac{3x+4}{1}$ is the same as $y = 3x + 4$, so any polynomial function is a rational function.

Note:

- A rational function is said to be in lowest terms, if $p\,(x)$ and $q\,(x)$ have no common factor other than 1.
- The domain of a rational function f is the set of all real numbers except the values of x that make the denominator $q\,(x)$ zero.

Example

Give the domain of the function $(x) = \frac{x+3}{2x^3-3x-2}$.

Solution

The denominator $2x^3 - 3x - 2 = 0 \Rightarrow (2x+1)(x-2) = 0 \Rightarrow x = -\frac{1}{2}$ *or* $x = 2.$ Thus, our domain is the set of all real numbers except $-\frac{1}{2}$ and 2, because $-\frac{1}{2}$ and 2 both make the denominator equal to 0.

Definition

- The line $x = a$ is called a vertical asymptote of the graph of f if $f\,(x) \to \infty$ or $f\,(x) \to -\infty$ *as* $x \to a$, either from the left or from the right.
- The line $y = b$ is called a horizontal asymptote of the graph of f if $f\,(x) \to b$ as $x \to \infty$ or as $x \to -\infty$.

Rules for asymptotes and holes

Let $f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n1} x^{n-1} + --- a_1 x + a_0}{b_m x^m + b_{m1} x^{m-1} + --- b_1 x + b_0}$, be a rational function, where n is the largest exponent in the numerator and m is the largest exponent in the denominator.

- The graph will have a vertical asymptote at $x = a$ if $q\,(a) = 0$ and $p\,(a) \neq 0$. In case $p(a) = q(a) = 0$, the function has either a hole at $x = a$ or requires further simplification to decide.
- If n < m, then the x-axis is the horizontal asymptote.
- If $n = m$, then the line $y = \frac{a}{b}x$ is a horizontal asymptote.
- If $n = m + 1$, the graph has an oblique asymptote and we can find it by long division.
- If $n > m + 1$, the graph has neither an oblique nor a horizontal asymptote.

Example

Give asymptote to the function $f(x) = \frac{x^2-1}{x^2+3x+2}$.

Solution

Factorizing numerator and denominator gives:

$$f(x) = \frac{x^2 - 1}{x^2 + 3x + 2} = \frac{p(x)}{q(x)} = \frac{(x-1)(x+1)}{(x+2)(x+1)}$$

At $x = -1$, $p(-1) = q(-1) = 0$. Reducing to lowest terms we have:

$g(x) = \frac{x-1}{x+2}$, which gives $g(-1) = -2 \neq 0$. Thus, f has a hole at $(-1, -2)$.

At $x = -2$, $p(-2) = 3 \neq 0$ and $q(-2) = 0$. Thus $x = -2$ gives a vertical asymptote. Since the degree of the numerator is equal to the degree of the denominator, $y = 1$ is a horizontal asymptote.

Example

Find the oblique asymptote of the function $f(x) = \frac{x^2+3}{x+2}$.

Solution

Since the degree of the numerator is one more than that of the denominator, the graph of f has an oblique asymptote. Applying long division yields:

$$f(x) = \frac{x^2 + 3}{x + 2} = (x - 2) + \frac{7}{x + 2}$$

Thus, the equation of the oblique asymptote is the quotient part of the answer which would be $y = x - 2$.

The zeros of a rational function

Definition

Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function. An element a in the domain of f is called a zero of f, if and only if $p(a) = 0$.

Example

Find the zeros of the rational functions: $f(x) = \frac{x^2-2x-3}{x^2+3x+2}$.

Solution

We first factorize both numerator and denominator.

$f(x) = \frac{x^2-2x-3}{x^2+3x+2} = \frac{(x-3)(x+1)}{(x+2)(x+1)}$. The domain is $\mathbb{R} \setminus \{-1, 3\}$. Now for any x in the domain $f(x) = 0$ means the numerator, $x^2 - 2x - 3 = 0$, i.e. x = –1 or $x = 3$. But, since $x = -1$ is not in the domain of f, the only zero of f is $x = 3$.

Graphs of Rational Functions

Steps to sketch the graph of a rational function:

1. Reduce the rational function to lowest terms and check for any open holes in the graph.
2. Find x-intercept(s) by setting the numerator equal to zero.
3. Find the y-intercept (if there is one) by setting x = 0 in the function.
4. Find all its asymptotes (if any).
5. Determine the parity (i.e. whether it is even or odd or neither).
6. Use the x-intercepts and vertical asymptote(s) to divide the x-axis into intervals. Choose a test point in each interval to determine if the function is positive or negative there. This will tell you whether the graph approaches the vertical asymptote in an upward or downward direction.
7. Sketch the graph! Except for breaks at the vertical asymptotes or cusps, the graph should be a nice smooth curve with no sharp corners.

To draw the graph of $f(x) = \frac{p(x)}{q(x)}$

We need to find	Criteria
Domain	$\mathbb{R}\backslash\{x: q(x) = 0\}$
x - intercept	Zeros of f
y - intercept	$x = 0$ or $0 \in$ domain of f
Vertical asymptote	$p(x) \neq 0$ *and* $q(x) = 0$
Horizontal asymptote	Degree of $p(x) \leq$ Degree of $q(x)$
Oblique asymptote	Degree of $p(x) =$ Degree of $q(x) + 1$
Parity	f is odd or even or neither

Example

Sketch the graph of each of the functions: $\boldsymbol{f(x) = \frac{3x^2}{(x-2)(x+1)}}$.

Solution

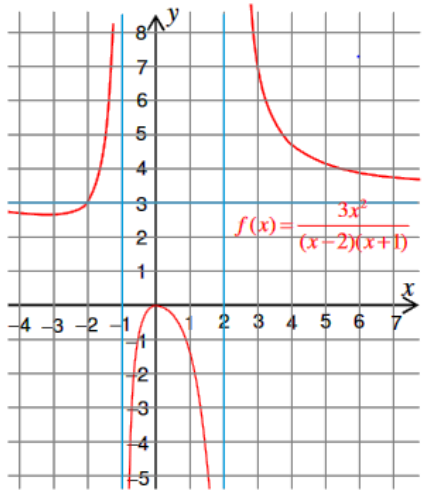
The function $f(x) = \frac{3x^2}{(x-2)(x+1)}$ cannot be reduced any further. This means that there will be no open holes on the graph of this function.

x - intercept	x = 0 or (0, 0)
y - intercept	y = 0 or (0, 0)
Vertical asymptote	x = −1 and x = 2
Horizontal asymptote	y = 3 Note that the graph crosses the horizontal asymptote at x = −2.
Oblique asymptote	None
Parity	f is neither even nor odd. You can check this by taking a test point. For instance, f (−4) ≠ f (4) and f (−4) ≠ −f (4).

Next, we find and plot several other points on the graph.

X	−3	1	4	5
y	2.7	− 1.5	4.8	4.17

Finally, we draw curves through the points, approaching the asymptotes.
Thus, the graph of f is:



2.5.3 The exponential functions and their graphs

Definition

A function $f : \mathbb{R} \to (0,\infty)$ given by $f(x) = a^x$, $a > 0, a \neq 1$ is called an exponential function.

Basic properties

The graph of $f(x) = a^x$, , $a > 1$ has the following basic properties:

1. The domain is the set of all real numbers.
2. The range is the set of all positive real numbers.
3. The graph includes the point (0, 1), i.e. the y-intercept is 1.
4. The function is increasing.
5. The values of the function are greater than 1 for x > 0 and between 0 and 1 for x < 0.
6. The graph approaches the x − axis as an asymptote on the left and increases without bound on the right.

Example

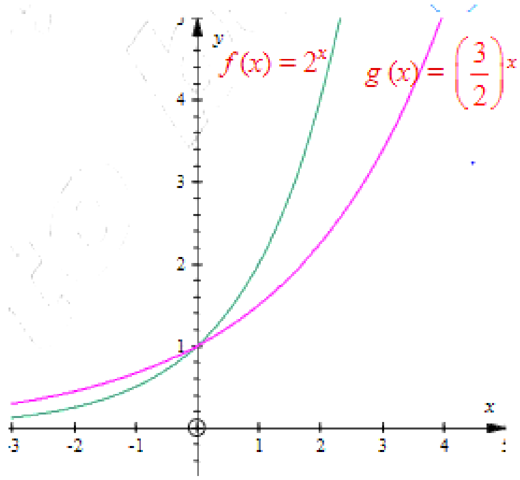
Draw the graph of $f(x) = 2^x$ and $g(x) = \left(\frac{3}{2}\right)^x$ on the same co-ordinate system.

Solution

Evaluate $y = 2^x$ and $y = \left(\frac{3}{2}\right)^x$ for some integral values of x and prepare a table of values.

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$g(x) = \left(\frac{3}{2}\right)^x$	$\frac{8}{27}$	$\frac{4}{9}$	$\frac{2}{3}$	1	$\frac{3}{2}$	$\frac{9}{4}$	$\frac{27}{8}$

Now plot these points on the co-ordinate system and join them by a smooth curve to obtain the graph of $f(x) = 2^x$ *and* $g(x) = \left(\frac{3}{2}\right)^x$.



Basic properties

The graph of $f(x) = a^x$, , $0 < a < 1$ has the following basic properties:

1. The domain is the set of all real numbers.
2. The range is the set of all positive real numbers.
3. The graph includes the point $(0, 1)$, i.e. the y - intercept is 1.
4. The function is decreasing.
5. The values of the function are greater than 1 for $x < 0$ and between 0 and 1 for $x > 0$.
6. The graph approaches the x-axis as an asymptote on the right and increases without bound on the left.

Example

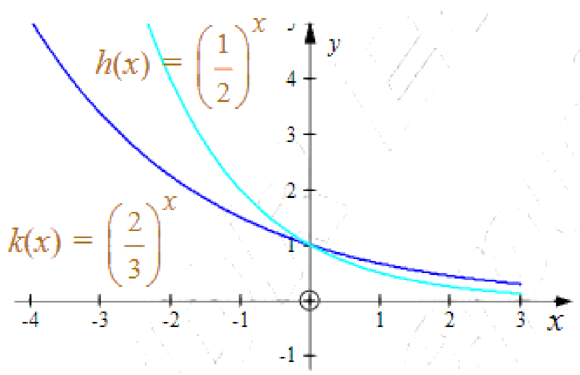
Draw the graph of $h(x) = \left(\frac{1}{2}\right)^x$ and $k(x) = \left(\frac{2}{3}\right)^x$ on the same co-ordinate system.

Solution

Evaluate $y = \left(\frac{1}{2}\right)^x$ and $y = \left(\frac{2}{3}\right)^x$ for some integral values of x and prepare a table of values.

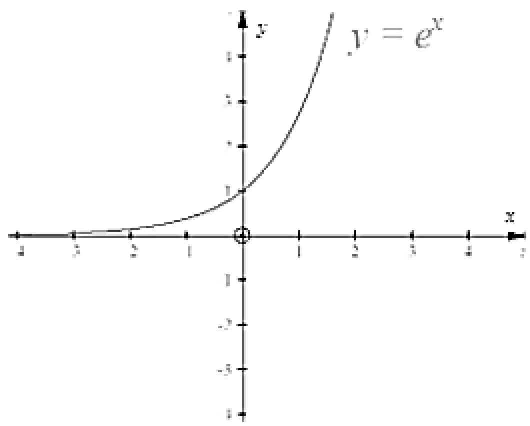
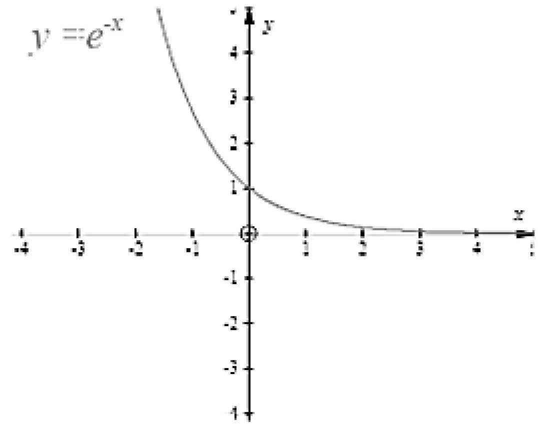
x	-3	-2	-1	0	1	2	3
$h(x) = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$k(x) = \left(\frac{2}{3}\right)^x$	$\frac{27}{8}$	$\frac{9}{4}$	$\frac{3}{2}$	1	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{8}{27}$

Now plot these points on the co-ordinate system and join them by a smooth curve to obtain the graph of $h(x) = \left(\frac{1}{2}\right)^x$ *and* $k(x) = \left(\frac{2}{3}\right)^x$.



The Natural Exponential Function

For any real number x, the function, $f\ (x)\ =\ e^x$ defines the exponential function with base e, usually called the natural exponential function.

	
The domain of $y = e^x$ *is* $\mathbb{R}$.	The domain of $y = e^{-x}$ is $\mathbb{R}$.
The range is $(0, \infty)$	The range is $(0, \infty)$
$y = e^x$ is an increasing function.	$y = e^{-x}$ is a decreasing function.
The graph of $y = e^x$ intersects the y-axis at $(0, 1)$.	The graph of $y = e^{-x}$ intersects the y-axis at $(0, 1)$.
$e^x > 1,$ *if* $x > 0$	$e^{-x} > 1,$ *if* $x < 0$
$0 < e^x < 1,$ *if* $x < 0$	$0 < e^{-x} < 1,$ *if* $x > 0$

2.5.4 The logarithmic functions and their graphs

Definition

A function $f : (0, \infty) \rightarrow \mathbb{R}$ given by $f\ (x)\ =\ \log_a^x,\ \ a\ >\ 0, a\ \neq\ 1$ is called a logarithmic function.

Basic properties
The graph of $y = \ \log_a^x$ for $a > 1$ has the following properties.
1 The domain is the set of all positive real numbers.
2 The range is the set of all real numbers.
3 The graph includes the point $(1, 0)$ i.e. the x-intercept of the graph is 1.
4 The function increases, as x increases.
5 The y-axis is a vertical asymptote of the graph.
6 The values of the function are negative for $0\ <\ x\ <\ 1$ and they are positive for $x\ >\ 1$.

Example

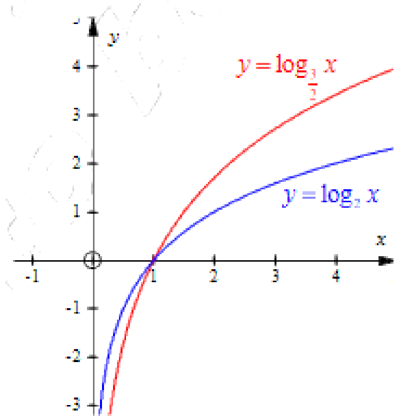
Draw the graph of $f(x) = \log 2^x$ and $\mathrm{g(x)} = \log \left(\frac{3}{2}\right)^x$ on the same co-ordinate system.

Solution

The tables below indicate some values for $f\,(x)$ and $g\,(x)$. Plot the corresponding points on the co-ordinate system. Join these points by smooth curves to get the required graphs as indicated in figure below.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4		x	$\frac{4}{9}$	$\frac{2}{3}$	1	$\frac{3}{2}$	$\frac{9}{4}$
$f(x) = \log 2^x$	-2	-1	0	1	2		$g(x) = \log\left(\frac{3}{2}\right)^x$	-2	-1	0	1	2

Now plot these points on the co-ordinate system and join them by a smooth curve to obtain the graph of $f(x) = \log 2^x$ *and* $g(x) = \log\left(\frac{3}{2}\right)^x$.



Basic properties

The graph of $y = \log_a^x$ for $0 < a < 1$ has the following properties.
1 The domain is the set of all positive real numbers.
2 The range is the set of all real numbers.
3 The graph has its x-intercept at $(1, 0)$ i.e. its x-intercept is 1.
4 The function decreases as x increases.
5 The y-axis is an asymptote of the graph.
6 The values of the function are positive when $0 < x < 1$ and they are negative when $x > 1$.

Example

Draw the graph of $h(x) = \log\left(\frac{1}{2}\right)^x$ and $k(x) = \log\left(\frac{2}{3}\right)^x$ on the same co-ordinate system.

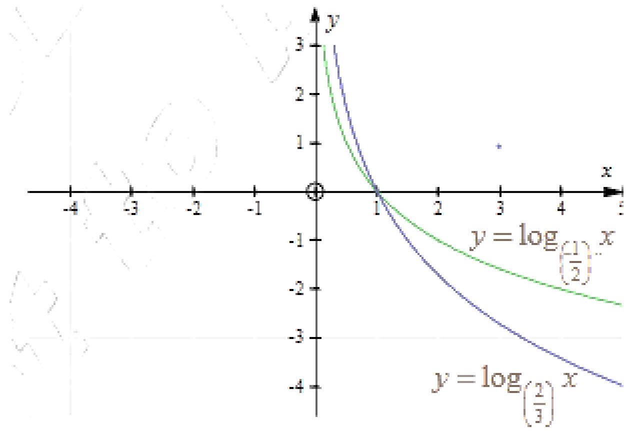
The tables below indicate some values for $f\,(x)$ and $g\,(x)$. Plot the corresponding points on the co-ordinate system. Join these points by smooth curves to get the required graphs as indicated in figure below.

Solution

Evaluate $y = \left(\frac{1}{2}\right)^x$ and $y = \left(\frac{2}{3}\right)^x$ for some integral values of x and prepare a table of values.

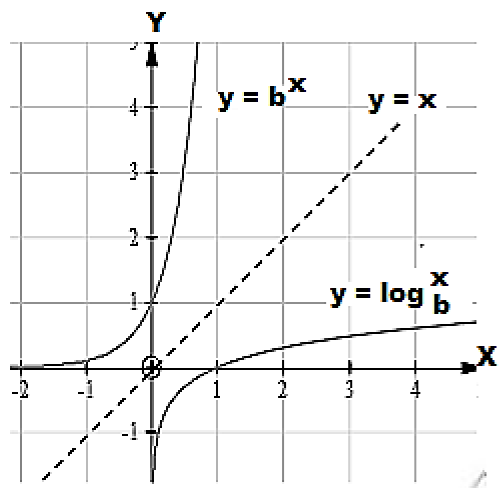
x	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$		x	$\frac{9}{4}$	$\frac{3}{2}$	1	$\frac{2}{3}$	$\frac{4}{9}$
$h(x) = \log\left(\frac{1}{2}\right)^x$	-2	-1	0	1	2		$k(x) = \log\left(\frac{2}{3}\right)^x$	-2	-1	0	1	2

Now plot these points on the co-ordinate system and join them by a smooth curve to obtain the graph of $h(x) = \log\left(\frac{1}{2}\right)^x$ *and* $k(x) = \log\left(\frac{2}{3}\right)^x$.



The relationship between exponential function and logarithmic functions

In general, the relation between the functions $y = b^x$ and $y = \log_b^x$ $(b > 1)$ is shown below:



From the graphs above, we observe the following relationships:

- The domain of $y = b^x$ is the set of all real numbers, which is the same as the range of $y = \log_{\mathrm{b}}^{\mathrm{x}}$.
- The range of $y = b^x$ is the set of all positive real numbers, which is the same as the domain of $y = \log_{\mathrm{b}}^{\mathrm{x}}$.
- The x-axis is the asymptote of $y = b^x$, whereas the y-axis is the asymptote of $y = \log_{\mathrm{b}}^{\mathrm{x}}$.
- $y = b^x$, has a y-intercept at $(0, 1)$ whereas $y = \log_{\mathrm{b}}^{\mathrm{x}}$ has an x-intercept at $(1, 0)$.

Domain of $y = b^x$ is equal to the range of $y = \log_{\mathrm{b}}^{\mathrm{x}}$.
Range of $y = b^x$is equal the domain of $y = \log_{\mathrm{b}}^{\mathrm{x}}$.
The functions $f(x) = b^x$ and $\mathrm{g(x)} = \log_{\mathrm{b}}^{\mathrm{x}}$ $(b > 1)$ are inverses of each other.

The Natural Logarithm

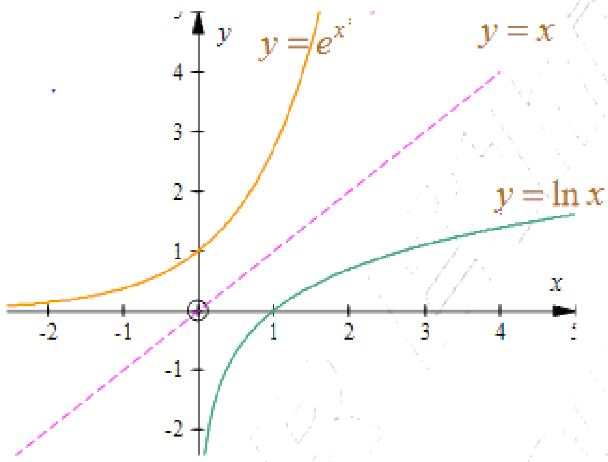
If we start with natural exponential function $y = e^x$and interchange x and y, we obtain $x = e^y$ which is the same as $y = \log_e^x$.

$y = \log_e^x$ is the mirror image of $y = e^x$ along the line y = x.

Notation

- lnx is used to represent $\log_e^x$.
- ln x is called the natural logarithm of x.

The graphs of $y = e^x$, $y = lnx$ and the line $y = x$ are shown below:



Solving Logarithmic Equations

If a, b, c, x and y are positive numbers and $a \neq 1, b \neq 1$, then

1. $\log_b^{xy} = \log_b^x + \log_b^y$
2. $\log_b^{\left(\frac{x}{y}\right)} = \log_b^x - \log_b^y$
3. For any real number k, $\log_b^{x^k} = k\log_b^x$
4. $\log_b^b = 1$
5. $\log_b^1 = 0$
6. $\log_a^x = \frac{\log_b^x}{\log_b^a}$ --- Change of base law.
7. $b^{\log_b^x} = x$

Example

Solve each of the following for x, checking that your solutions are valid.

a. $\log 8x + \log (x - 20) = 3$ b. $\log_3^{(x+1)} - \log_3^{(x+3)} = 1$

Solution

a. log 8x + log (x – 20) is valid for $8x > 0$ and $x - 20 > 0$; i.e. for $x > 0$ and $x > 20$. So $U = (20, \infty)$.

Now log 8x + log (x – 20) = 3

$\Rightarrow \log 8x(x - 20) = 3$ ---- $\log_b^{xy} = \log_b^x + \log_b^y$

$\Rightarrow 8x(x - 20) = 10^3 = 1000$

$\Rightarrow 8x^2 - 160x - 1000 = 0$

$\Rightarrow x^2 - 20x - 125 = 0$ --- dividing both side by 8.

$\Rightarrow (x + 5)(x - 25) = 0$ --- Factoring the equation.

So, $x = 25$ *or* x= -5. But $-5 \notin (20, \infty)$.

So the only solution is x = 25.

b. $\log_3^{(x+1)} - \log_3^{(x+3)}$ is valid for $x + 1 > 0$ and $x + 3 > 0$,
i.e. for x > −1 and x > −3.

Therefore the U = (–1, ∞).

$\log_3^{(x+1)} - \log_3^{(x+3)} = 1$

$\Rightarrow \log_3^{\left(\frac{x+1}{x+3}\right)} = 1$ --- since $\log_b^{\left(\frac{x}{y}\right)} = \log_b^{x} - \log_b^{y}$.

$\Rightarrow \frac{x+1}{x+3} = 3$

$$\Rightarrow 3x + 9 = x + 1$$

Therefore $-2x = 8$ and $x = -4$.

However, – 4 is NOT in the universe. Hence, there is no x satisfying the given equation and the solution set is the empty set.

2.5.5 Power function, Absolute value function, Signum function and Greatest integer function

1. Power function

Definition

A power function is a function which can be written in the form $f(x) = ax^r$, where r is a rational number and a in $\mathbb{R}$, is a fixed number.

Note

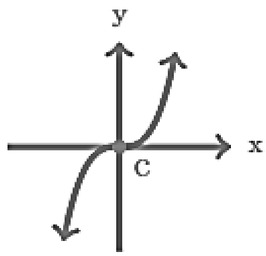
Don't confuse power functions with exponential functions.

- Exponential function: $y = a^x$ (a fixed base is raised to a variable exponent)
- Power function: $y = ax^r$ (variable base is raised to a fixed exponent)

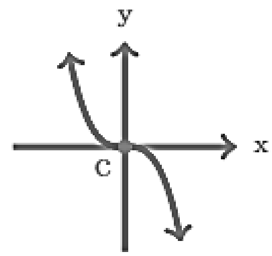
Example

The function $f(x) = x^3$ is a power function

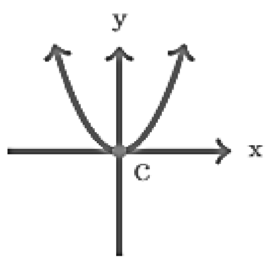
The end behavior is the behavior of the graph of a function as the input decreases without bound and increases without bound.
A power function is of the form: $f(x) = kx^p$ where k and p are constant. p determines the degree of the power function and both k and p determine the end behavior.



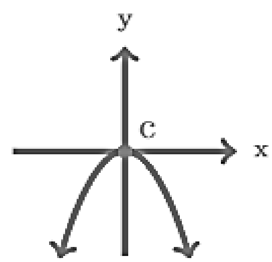
Power function, p: odd, $k > 0$
End behavior:
$y \to \infty$ as $x \to \infty$
$y \to -\infty$ as $x \to -\infty$



Power function, p: odd, $k < 0$
End behaviour:
$y \to -\infty$ as $x \to \infty$
$y \to \infty$ as $x \to -\infty$

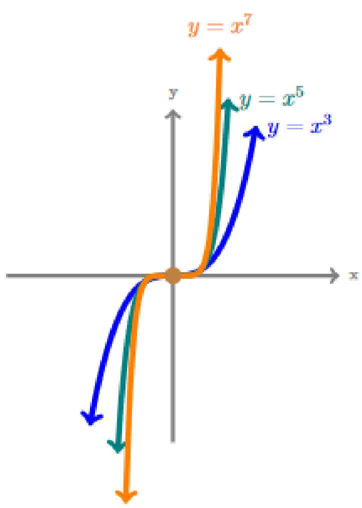


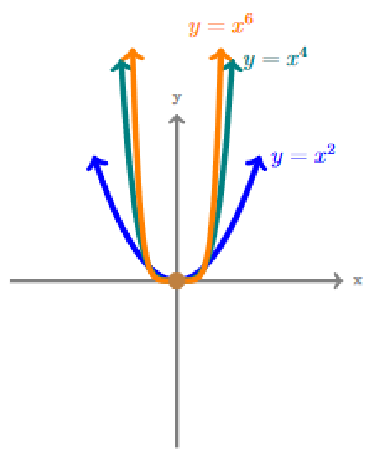
Power function, p: even, $k > 0$
End behaviour:
$y \to \infty$ as $x \to \infty$
$y \to \infty$ as $x \to -\infty$



Power function, p: even, $k < 0$
End behaviour:
$y \to -\infty$ as $x \to \infty$
$y \to -\infty$ as $x \to -\infty$

Comparing Power Functions:





2. Absolute value function

Definition

The modulus (Absolute value) function is the function given by $f(x) = |x|$.

Note:
Domain of $f = \mathbb{R}$. Since $f(x) = |x| \geq 0$, for each $x \in \mathbb{R}$, Range of $f = [0, \infty)$.

Example

Give the domain of each of the following functions.

a. $f(x) = |x| + 2$

b. $f(x) = |x - 3|$

Solution

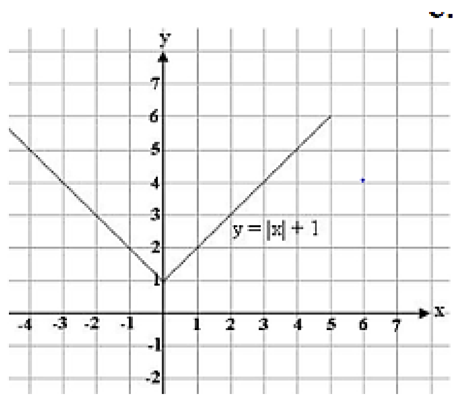
a. $\mathbb{R}$

b. $\mathbb{R}$

Example

Sketch the graph of $f(x) = |x| + 1$

Solution



3. **Signum function**

Definition

The signum function, read as signum x, is written as sgn x and is defined by

$$y = f(x) = \operatorname{sgn} x = \begin{cases} 1, & if \;\; x > 0 \\ 0, & if \;\; x = 0 \\ -1, & if \; x < 0 \end{cases}$$

Observe that, $\frac{|x|}{x} = \begin{cases} 1, & if \;\; x > 0 \\ \nexists, & if \;\; x = 0 \\ -1, & if \; x < 0 \end{cases}$, we have $\mathbf{sgn}\, \boldsymbol{x} = \begin{cases} \frac{|x|}{x}, & \boldsymbol{if \;\; x \neq 0} \\ \mathbf{0}, & \boldsymbol{if \; x = 0} \end{cases}$

Note

The domain of $\operatorname{sgn} x$ is the set of real number and its range is $\{-1,\ 0,\ 1\}$.

A signum function is an odd function.

Example

Sketch the graph of the following functions and also find their domain and range

a. $f(x) = \operatorname{sgn} x$

b. $f(x) = x^2 \operatorname{sgn} x$

c. $f(x) = x + \operatorname{sgn} x$

d. $f(x) = x^3 \operatorname{sgn} x$

Solution

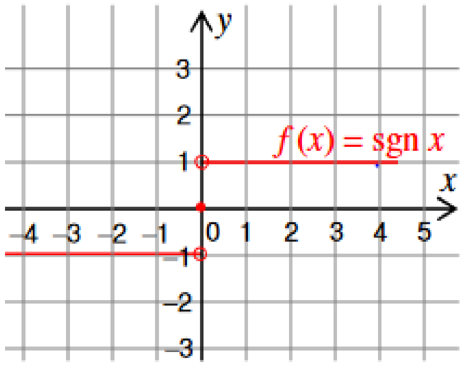
a. $\begin{cases}\text{Domain} = \mathbb{R} \\ \text{Range} = \{-1,\ 0,\ 1\}\end{cases}$

b. $\begin{cases}\text{Domain} = \mathbb{R} \\ \text{Range} = \mathbb{R}\end{cases}$

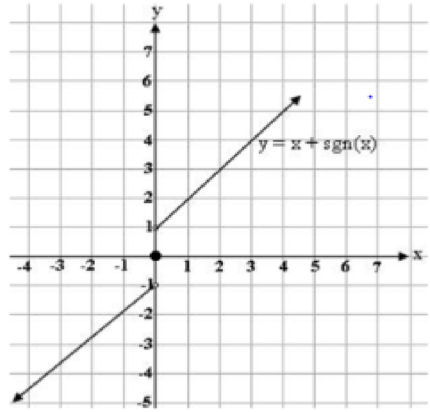
c. $\begin{cases}\text{Domain} = \mathbb{R} \\ \text{Range} = (-\infty, -1) \cup \{0,\} \cup (1, \infty)\end{cases}$

d. $\begin{cases}\text{Domain} = \mathbb{R} \\ \text{Range} = [0, \infty)\end{cases}$

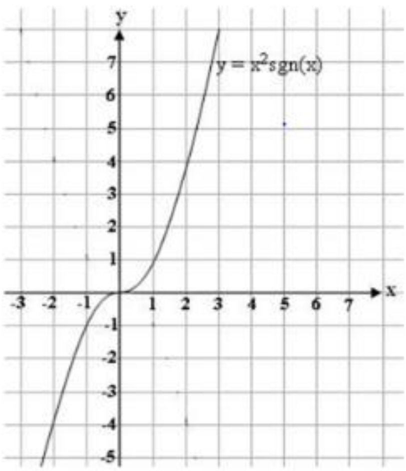
Their graphs



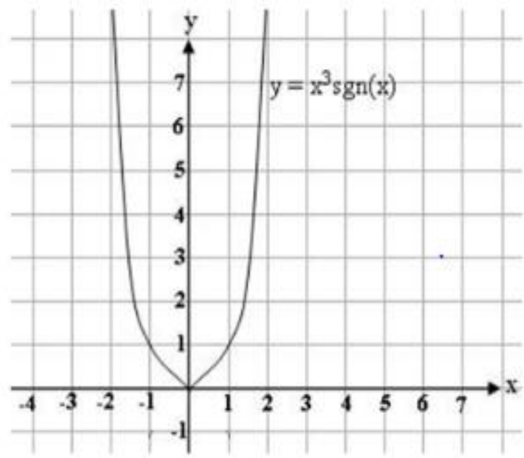
a.



c.



b.



d.

Example

Define the function $f(x) = 2 + x\operatorname{sgn} x$ piece wisely.

Solution

$$f(x) = 2 + x\operatorname{sgn} x = 2 + x\begin{cases} 1, \ if \ \ x > 0 \\ 0, \ \ if \ \ x = 0 \\ -1, \ \ if \ x < 0 \end{cases}$$

$$= \begin{cases} x + 2 \ \ if \ \ x > 0 \\ 2, \qquad if \ \ x = 0 \\ -x + 2 \ \ if \ x < 0 \end{cases}$$

4. Greatest Integer Function

Definition

The greatest integer function, denoted by $f(x) = \lfloor x \rfloor$, is defined as the greatest integer $\leq x$.

Note

Domain of $f(x) = \lfloor x \rfloor$ is the set of real number and its range is the set of integers.

Example

Find $\lfloor x \rfloor$ when

a. $x = 2.3$ b. $x = -3.1$ c. $x = 5$

Solution

a. 2 is the largest integer ≤ 2.3. i.e. $\lfloor 2.3 \rfloor = 2$.
b. -3 is the largest integer ≤ -3.1. i.e. $\lfloor -3.1 \rfloor = -3$.
c. 5 is the largest integer ≤ 5. i.e. $\lfloor 5 \rfloor = 5$.

Example

Let $f(x) = \lfloor x \rfloor$. Then

a. Complete the table

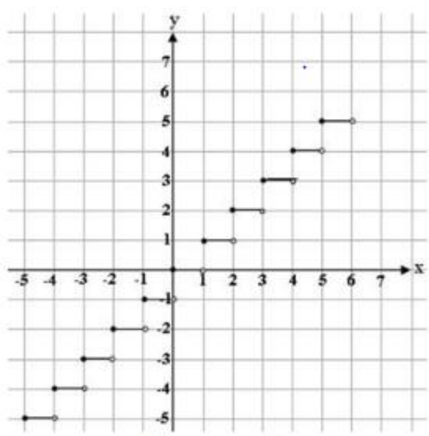
x	$-3 \leq x < -2$	$-2 \leq x < -1$	$-1 \leq x < 0$	$0 \leq x < 1$	$1 \leq x < 2$	$2 \leq x < 3$
$f(x) = \lfloor x \rfloor$						

b. Sketch the graph of $f(x) = \lfloor x \rfloor$

Solution

a.

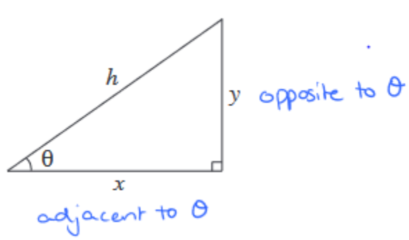
x	$-3 \leq x < -2$	$-2 \leq x < -1$	$-1 \leq x < 0$	$0 \leq x < 1$	$1 \leq x < 2$	$2 \leq x < 3$
$f(x) = \lfloor x \rfloor$	-3	-2	-1	0	1	2



b. Graph of f (x)

2.6. Trigonometric Function

Consider a right-angled triangle with angle θ and side lengths x, y and h as shown:



The trigonometric functions sine, cosine and tangent of θ are defined as:

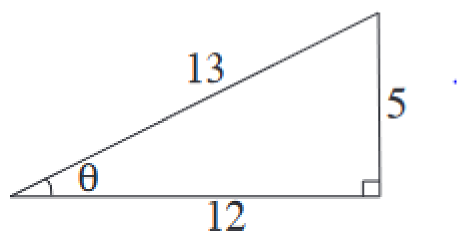
- $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{y}{h}$
- $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{x}{h}$
- $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{y}{x} = \dfrac{\sin\theta}{\cos\theta}$

Reciprocal Trigonometric Functions

The reciprocal trigonometric functions secant, cosecant and cotangent are defined as:

- $\csc\theta = \dfrac{\text{hypotenuse}}{\text{opposite}} = \dfrac{h}{y} = \dfrac{1}{\sin\theta}$
- $\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent}} = \dfrac{h}{x} = \dfrac{1}{\cos\theta}$
- $\cot\theta = \dfrac{\text{adjacent}}{\text{opposite}} = \dfrac{x}{y} = \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$

Example



For the above right-angled triangle, find:

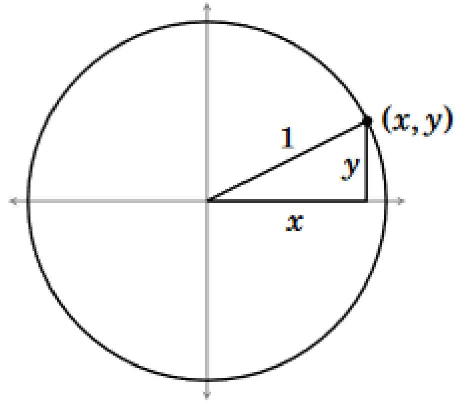
a. $\sin\theta$
b. $\cos\theta$
c. $\tan\theta$
d. $\csc\theta$
e. $\sec\theta$
f. $\cot\theta$

Solution

a. $\sin\theta = \frac{5}{13}$
b. $\cos\theta = \frac{12}{13}$
c. $\tan\theta = \frac{5}{12}$
d. $\csc\theta = \frac{13}{5}$
e. $\sec\theta = \frac{13}{12}$
f. $\cot\theta = \frac{12}{5}$

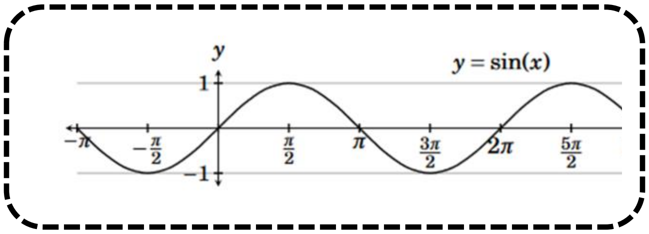
The Unit Circle

We can form right-angled triangles in a unit circle (circle of radius 1). If θ is the anticlockwise angle between the positive x-axis and the ray $\overrightarrow{OP}$ then for all $\theta \in R$:



$$\begin{cases} \sin\theta = \dfrac{y}{1} = y \\ \cos\theta = \dfrac{x}{1} = x \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

Graphs of trigonometric functions



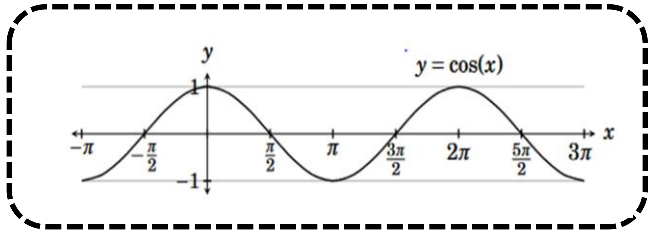
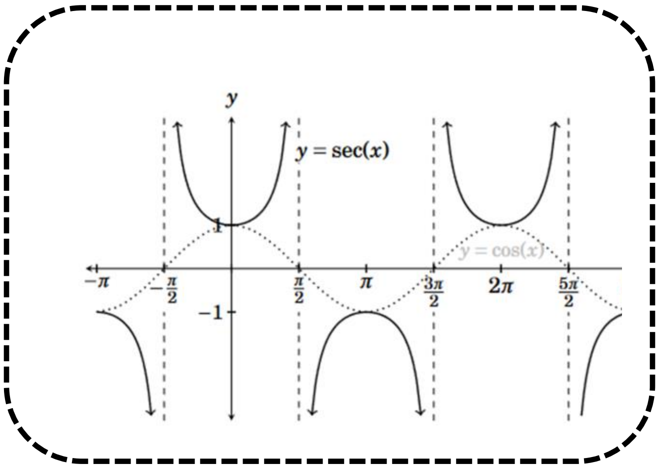


Fig. Graph of $\boldsymbol{y = sinx}$

Fig. Graph of $\boldsymbol{y = cosx}$



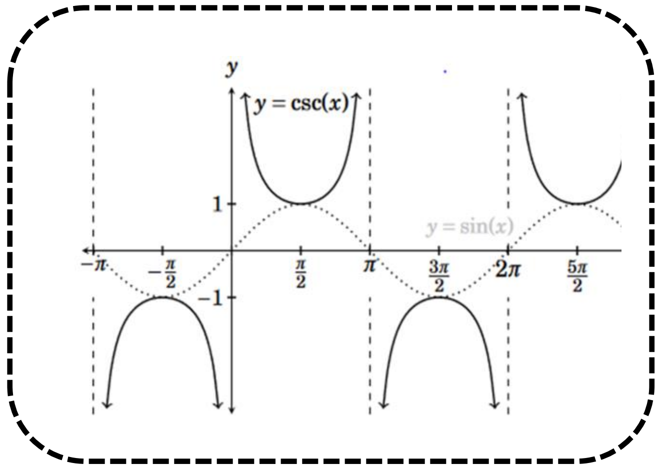
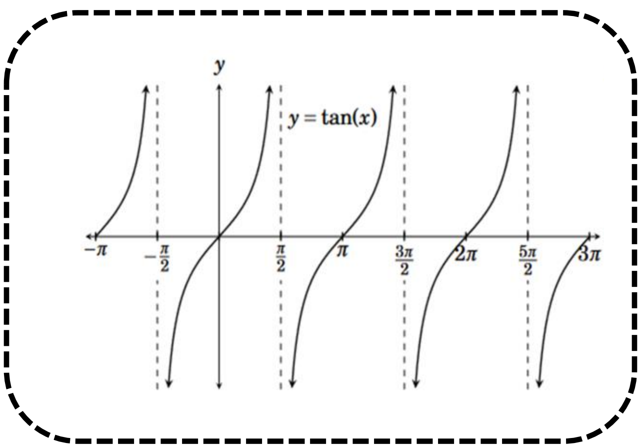


Fig. Graph of $\boldsymbol{y = secx}$

Fig. Graph of $\boldsymbol{y = cscx}$



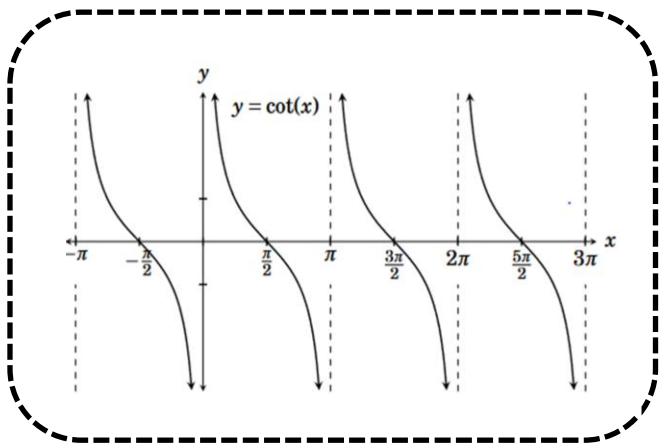
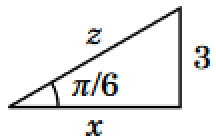


Fig. Graph of $\boldsymbol{y = tanx}$

Fig. Graph of $\boldsymbol{y = cotx}$

Review questions

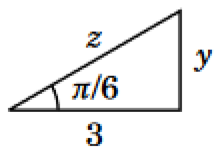
1. Find the missing sides.



To find z: $\sin\frac{\pi}{6} = \frac{\text{OPP}}{\text{HYP}}$, so $\frac{1}{2} = \frac{3}{z}$. Therefore $\boxed{z = 6.}$

To find x: $\tan\frac{\pi}{6} = \frac{\text{OPP}}{\text{ADJ}}$, so $\frac{1}{\sqrt{3}} = \frac{3}{x}$. Therefore $\boxed{x = 3\sqrt{3}.}$

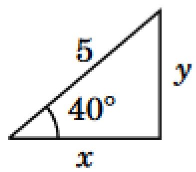
2. Find the missing sides.



To find z: $\cos\frac{\pi}{6} = \frac{\text{ADJ}}{\text{HYP}}$, so $\frac{\sqrt{3}}{2} = \frac{3}{z}$. Therefore $\boxed{z = \frac{6}{\sqrt{3}}.}$

To find y: $\tan\frac{\pi}{6} = \frac{\text{OPP}}{\text{ADJ}}$, so $\frac{1}{\sqrt{3}} = \frac{y}{3}$. Therefore $\boxed{y = \frac{3}{\sqrt{3}}.}$

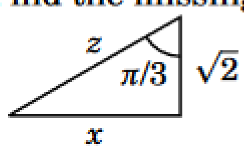
3. Find the missing sides.



To find x: $\cos 40^\circ = \frac{\text{ADH}}{\text{HYP}}$, so $\cos 40^\circ = \frac{x}{5}$, so $\boxed{x = 5\cos 40^\circ.}$

To find y: $\sin 40^\circ = \frac{\text{OPP}}{\text{HYP}}$, so $\sin 40^\circ = \frac{y}{5}$, so $\boxed{y = 5\sin 40^\circ.}$

4. Find the missing sides.



To find x: $\tan\frac{\pi}{3} = \frac{\text{OPP}}{\text{ADJ}}$, so $\sqrt{3} = \frac{x}{\sqrt{2}}$. Thus $\boxed{x = \sqrt{6}.}$

To find z: $\cos\frac{\pi}{6} = \frac{\text{ADJ}}{\text{HYP}}$, so $\frac{1}{2} = \frac{\sqrt{2}}{z}$. Thus $\boxed{z = 2\sqrt{2}.}$

Multiple Choice questions

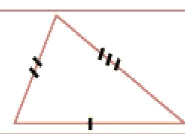
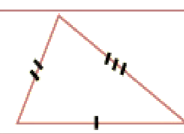
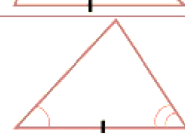
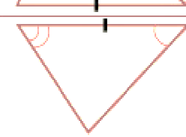
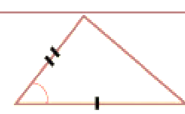
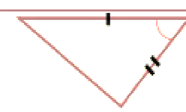
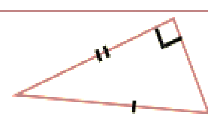
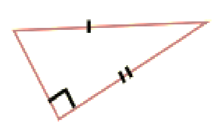
1. Which one of the following is not a polynomial function?
   A. $f(x) = 5x^3 + 3x + 7$ C. $f(x) = |2x^2 + x - 7|$
   B. $f(x) = -\frac{1}{\pi}$ D. $f(x) = |x^2 + x + 5|$
2. Which of the following is a factor of the polynomial $f(x) = x^3 - 3x + 2$?
   A. $x - 2$ B. $x + 3$ C. $x - 1$ D. $x + 2$
3. What is the remainder when $4x^5 + 2x^3 - 5x^2 + 7$ is divided by $x + 1$?
   A. 6 B. – 4 C. 4 D. 0
4. If f and g are polynomial functions of degree 4, then which of the following is necessarily true?
   A. fg is of degree 8 C.$3f$ is of degree 12 B. f + g is of degree 4
   D. f – g is of degree 4
5. If $x+1$ is the factor of $f(x) = ax^3 - 2x + b$ and $f(x)$ is divided by $x-1$ the remainder is 2, then the formula for $f(x)$ will be:
   A. $f(x) = 3x^3 - 2x - 1$ C. $f(x) = -3x^3 - 2x + 1$
   B. $f(x) = 3x^3 - 2x + 1$ D. $f(x) = -3x^3 - 2x - 1$
6. If α *and* β are the zeros of the quadratic polynomial $f(x) = 3x^2 - 7x - 7 - c$, find the value of $(\alpha + 1)(\beta + 1)$
   A. $\frac{c}{3} - 7$ B. $\frac{c}{3} + 7$ C. $\frac{c}{3} - 1$ D. $1 - \frac{c}{3}$

Chapter 3

Geometry & Measurement

1. A polygon is said to be:
    a. Convex if each interior angle is less than 180.
    b. Concave if at least one of its interior angles is greater than 180.
2. The sum S of all the interior angles of an n-sided polygon is given by the formula $\mathbf{S = (n-2)180}$
3. The sum of all the exterior angles of an n-sided polygon is given by $\mathbf{S = 360^0}$.
4. A regular polygon is a convex polygon with all sides equal and all angles equal.
    a) Each interior angle of a regular n-sided polygon is $\frac{\mathbf{(n-2)180}}{\mathbf{n}}$
    b) Each exterior angle of a regular n-sided polygon is $\frac{\mathbf{360^0}}{\boldsymbol{n}}$
    c) Each central angle of a regular n-sided polygon is $\frac{\mathbf{360^0}}{\boldsymbol{n}}$
5. A figure has a line of symmetry, if it can be folded so that one half of the figure coincides with the other half. A figure that has at least one line of symmetry is called a symmetrical figure.
6. An n-sided regular polygon has n lines of symmetry.
7. A circle can be always inscribed in or circumscribed about any given regular polygon.
8. The apothem is the distance from the centre of regular polygon to a side of the polygon.
9. Formulae for the length of a side **s**, apothem **a**, perimeter **P** and area **A** of a regular polygon with n-sides and radius r are given by
    a) $\mathrm{S} = 2\mathrm{r}\sin\left(\frac{180^0}{n}\right)$
    b) $\mathrm{P} = 2\mathrm{n}\ \mathrm{r}\sin\left(\frac{180^0}{n}\right)$
    c) $\mathrm{a} = \mathrm{r}\cos\left(\frac{180^0}{n}\right)$
    d) $\mathrm{A} = \frac{1}{2}\mathrm{aP} = \frac{1}{2}\mathrm{n}r^2\sin\left(\frac{360^0}{n}\right)$
10. **Congruency**

    Two triangles are congruent, if the following corresponding parts of the triangles are congruent.

i	Three sides (SSS)		
ii	Two angles and the included side (ASA)		
iii	Two sides and the included angle (SAS)		
iv	A right angle, hypotenuse and a side (RHS)		

11. **Similarity**

i. Two polygons of the same number of sides are similar, if their corresponding angles are congruent and their corresponding sides have the same ratio.

ii. Similarity of triangles

a. SSS-similarity theorem :If three sides of one triangle are proportional to the three sides of another triangle, then the two triangles are similar.

b. SAS-similarity theorem: Two triangles are similar, if two pairs of corresponding sides of the triangles are proportional and the included angles between the sides are congruent.

c. AA–similarity theorem: If two angles of one triangle are correspondingly congruent to two angles of another triangle, then the two triangles are similar.

12. If the ratio of the lengths of any two corresponding sides of two similar polygons is k then

a. the ratio of their perimeters is k.

b. the ratio of their areas is k^2

13. **Heron's formula**

The area A of a triangle with sides a, b and c units long and semi-perimeter $\mathbf{S} = \frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ is given by $\mathbf{A} = \sqrt{\mathbf{S}(\mathbf{S} - \mathbf{a})(\mathbf{S} - \mathbf{b})(\mathbf{S} - \mathbf{c})}$

14. If h is the height of the triangle perpendicular to base b, then the area A of the triangle is $\mathbf{A} = \frac{1}{2}\mathbf{bh}$

15. If the angle between the sides a and b is θ, then the area A of the triangle is $\mathbf{A} = \frac{1}{2}\mathbf{absin\theta}$

16. A circle is symmetrical about every diameter.

17. A diameter perpendicular to a chord bisects the chord.

18. The perpendicular bisector of a chord passes through the centre of the circle.

19. In the same circle, equal chords are equidistant from the centre.

20. A tangent is perpendicular to the radius drawn at the point of contact.

21. Line segments that are tangents to a circle from an outside point are equal.

22. Angle properties of a circle

a. The measure of an angle at the centre of a circle is twice the measure of an angle at the circumference subtended by the same arc.

b. Every angle at the circumference subtended by the diameter of a circle is a right angle.

c. Inscribed angles in the same segment of a circle are equal.

23. The length l of an arc that subtends an angle θ at the centre of a circle with radius r is $\boldsymbol{l} = \frac{\boldsymbol{\pi r\theta}}{\mathbf{180^0}}$.

24. The area A of a sector with central angle θ and radius r is given by $\mathbf{A} = \frac{\mathbf{\pi r^2\theta}}{\mathbf{360^0}}$.

25. The area A of a segment associated with a central angle θ and radius r is given by $\mathbf{A} = \frac{\mathbf{\pi r^2\theta}}{\mathbf{360^0}} - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{r^2sin\theta}$.

26. The perimeter P of a segment is given by $P = \frac{\boldsymbol{\pi r\theta}}{\mathbf{180^0}} + 2r\sin\frac{\theta}{2}$

27. If A_L is the lateral surface area of a prism, A_T is the total surface area of the prism, A_B is base area of the prism and Vis the volume of the prism, then

a. $\mathbf{A_L = Ph}$, where P is the perimeter of the base and h the altitude or height of the prism.

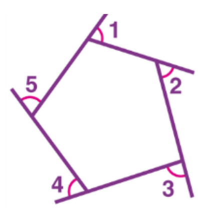
b. $\mathbf{A_T = 2A_B + A_L}$

c. $\mathbf{V = A_Bh}$

As we know, there are different types of polygons. Therefore, the number of interior angles and the respective sum of angles is given below in the table.

Polygon Name	Number of Interior Angles	Sum of Interior Angles = (n-2) x 180°
Triangle	3	180°
Quadrilateral	4	360°
Pentagon	5	540°
Hexagon	6	720°
Septagon	7	900°
Octagon	8	1080°
Nonagon	9	1260°
Decagon	10	1440°

Exterior angles



Solved examples

Q.1: If each interior angle is equal to 144°, then how many sides does a regular polygon have?

Solution:

Given: Each interior angle = 144°

We know that,

Interior angle + Exterior angle = 180°

Exterior angle = 180°-144°

Therefore, the exterior angle is 36°

The formula to find the number of sides of a regular polygon is as follows:

Number of Sides of a Regular Polygon = 360° / Magnitude of each exterior angle

Therefore, the number of sides = 360° / 36° = 10 sides

Hence, the polygon has 10 sides.

Q.2: What is the value of the interior angle of a regular octagon?

Solution: A regular octagon has eight sides and eight angles.

n = 8

Since, we know that, the sum of interior angles of octagon, is;

Sum = (8-2) x 180° = 6 x 180° = 1080°

A regular octagon has all its interior angles equal in measure.

Therefore, measure of each interior angle = 1080°/8 = 135°.

Q.3: What is the sum of interior angles of a 10-sided polygon?

Answer: Given,

Number of sides, n = 10

Sum of interior angles = (10 – 2) x 180° = 8 x 180° = 1440°.

Q.4: What is the measure of each angle of a regular decagon?

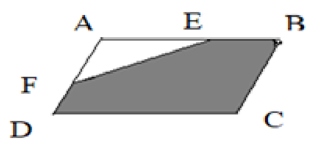
A decagon has 10 sides and 10 angles.
Sum of interior angles = (10 – 2) x 180°
= 8 × 180°
= 1440°
A regular decagon has all its interior angles equal in measure. Therefore,
Each interior angle of decagon = 1440°/10 = 144°

Review Questions

1. Find the perimeter and area of a segment of a circle of radius 8 cm, cut off by a chord that subtends a central angle of:
   a. 120^0 b. $\frac{3}{4}\pi$ radians
2. Calculate the volume and total surface area of a right circular cylinder of height 1m and radius 70 cm.
3. A 40 m deep well with radius $3\frac{1}{2}$m is dug and the earth taken out is evenly spread to form a platform of dimensions 28 m by 22 m. Find the height of the platform.

4. A glass cylinder with a radius of 7 cm has water up to a height of 9 cm. A metal cube of $5\frac{1}{2}$ cm edge is immersed in it completely. Calculate the height by which the water rises in the cylinder.
5. An agriculture field is rectangular, with dimensions 100 m by 42 m. A 20 m deep well of diameter 14 m is dug in a corner of the field and the earth taken out is spread evenly over the remaining part of the field. Find the increase in the level of the field.
6. The areas of two similar triangles are 144 cm^2 and 81 cm^2. If one side of the first triangle is 6 cm, what is the length of the corresponding side of the second?
7. A chord of a circle of radius 6 cm is 8 cm long. Find the distance of the chord from the centre.
8. Two chords, AB and CD, of a circle intersect at right angles at a point inside the circle. If m(∠BAC) = 35^0, find m(∠ABD).
9. If ABCD is a parallelogram with area 64 sq. units, AE = $\frac{3}{4}$AB, AF = $\frac{2}{3}$AD, then the area the shaded region is equal to:

   A. 24   *B. 48*   C. 16   D. 56



1. The area of the upper and the lower bases of frustum of a pyramid are 3 square units and 12 square units respectively, If the volume of the frustum is 42 cubic units, then what is the height of the pyramid? **(1993 E.C)**

   (A) *12 units*   (B) 9 units   (C) 6 units   (D) 15 units

2. If the edge of the cube is 1 cm long, then what is its total surface area? **(1993 E.C)**

   (A) $\frac{1}{4}$ cm^2 (B) $\frac{3}{2}$ cm^2 (C) $\frac{1}{6}$ cm^2 (D) 4 cm^2

3. If the diagonals of a cube is 6 units, then which of the following is true? **(1994 E.C)**

   (A) Total surface area of the cube is 48 square units.

   (B) The volume of the cube is $24\sqrt{3}$ cubic units.

   (C) Lateral surface area of the cube is 24 square units.

   (D) None of the above.

4. A frustum of height 6 cm is formed from a right circular cone of height 12 cm and base radius 4 cm, then the volume of the frustum is (1994 E.C)

   (A) $56\pi\ cm^3$ (B) $72\pi\ cm^3$ (C) $64\pi\ cm^3$ (D) $8\pi\ cm^3$

5. A regular square pyramid is cut by a plane parallel to the base so that the plane bisects the altitude of the pyramid. If the slant height of the pyramid is 12 units and its lateral edge 13 units long, then the volume of the frustum in cubic units is: (1995 E.C)

   (A) $\frac{175\sqrt{119}}{6}$ (B) $\frac{175\sqrt{119}}{24}$ (C) 2275   (D) 325

6. If the radius of the sphere is doubled, what happens to the surface area?

(A) It becomes four times as long as the original surface area.

(B) It becomes eight times as long as the original surface area.

(C) It is doubled

(D) It is increased by four square units.

7. What is the lateral surface area a regular square pyramid of height 6 m if its base has edge of length 16 m?

(A) 192 m^2 (B) 512 m^2 (C) *320* m^2 (D) 576 m^2

8. A solid is made up of a hemispherical top superimposed on a frustum of right circular cone of radii 3 units and 15 units with slant height of 20 units as shown in the figure. What is the volume of the solid in cubic units?

(A) 1938π (B) 2310π (C) 3738π (D) 4110π

9. A right pyramid with a rectangular base of length 8 m and width 6 m has a 13 m long lateral edge. What is the volume of the pyramid in cubic meter?

(A) 208 (B) *192* (C) $64\sqrt{10}$ (D) 160

10. A water tank fork is in the shape of frustum of a right circular cone with radii of bases 8m and 4m. If its slant height is 5m, what is the capacity (volume) of the tank cubic meter?

(A) 112π (B) 128π (C) $\frac{320}{3}\pi$ (D) 64π

11. Which of the following is true about the volume V of a cone determined by doubling both the base radius and the height of a right circular cone while volume was V_0?

(A) $V = 8V_0$ (B) $V = 6V_0$ (C) $V = 4V_0$ (D) $V = 2V_0$

12. A 9 cm high box is in the form of the frustum of a regular square pyramid. What is the volume of the box if the areas of the bases are $4cm^2$ and $36cm^2$?

(A) $108cm^3$ (B) 96cm3 (C) $156cm^3$ (D) $120cm^3$

13. The slant height of a square pyramid is the distance measured from its vertex to

(A) One of the vertices of its base.

(B) Any point inside its base.

(C) one of the sides of its base.

(D) the center of its base.

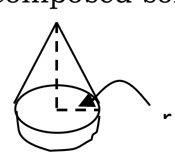
14. A right circular cone of height r cm is made to fit with hemisphere whose radius is also r cm as shown in the figure below. What is the volume of the composed solid in cm^3?

(A) $\frac{1}{3}\pi r^3$

(B) $\frac{5}{3}\pi r^3$

(C) $\frac{2}{3}\pi r^3$

(D) πr^3



15. The volume of a pyramid that has a height of 8 in and a rectangular base of dimension 6 in by 4 in is

(A) 96 in^2 (B) *64* in^2 (C) 576 in^2 (D) 192 in^2

16. A glass in a shape of a frustum of a right circular cone has slanted height 10 cm, and bases of radii 3 cm and 9 cm. what is the volume of the glass?

(A) 240 cm^3 (B) $312\,\pi\,cm^3$ (C) $192\,\pi\,cm^3$ (D) $204\,\pi\,cm^3$

17. The diameter of the base and the height of a circular cone are found to be a and 2b units long respectively. What is the formula for the volume V of the cone? 2000

(A) $V = \frac{4}{3}\,\pi\,a^2b$ (B) $V = \frac{1}{6}\,\pi\,a^2b$ (C) $V = \frac{1}{3}\,\pi\,a^2b$ (D) $V = \frac{2}{3}\,\pi\,a^2b$

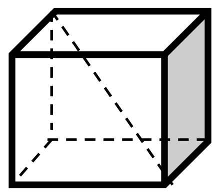
18. What is the volume of the cube whose diagonal has length 4 cm and makes 300 with its adjacent edge as shown in the figure below?

(A) $12\sqrt{3}\ cm^3$

(B) $\frac{64}{3\sqrt{3}}\ cm^3$

(C) $8\ cm^3$

(D) $24\sqrt{3}\ cm^3$



19. A hemisphere is extracted from the top of a right circular cylindrical wood of height 5 units and base radius 3 units. If the hemisphere has the same radius as the base of the cylinder, then what is the volume of the remaining objects in cubic units?

(A) $39\,\pi$ (B) $9\,\pi$ (C) $15\,\pi$ (D) $27\,\pi$

20. A right prism has a rectangular base of length L and width W units. If its height is h units long, then the lateral surface area of the prism is

(A) $2(LW + Lh + hW)$ (B) $2h(W + L)$ (C) $2L(W + h)$ (D) $LW + 2h(L + W)$

21. A right circular cylinder can which is open at the top has base radius of 5 cm and height of 10 cm. what is its total surface area? 2002

(A) $125\pi\ cm^2$ (B) $150\pi\ cm^2$ (C) $250\pi\ cm^2$ (D) $100\pi\ cm^2$

22. The height of the right circular cone is 8 cm from this cone a frustum is made with upper and lower bases of radii 3 cm and 6 cm respectively. What is the volume of this frustum?

(A) $48\,\pi\ cm^3$ (B) $96\,\pi\ cm^3$ (C) $84\,\pi\ cm^3$ (D) $42\,\pi\ cm^3$