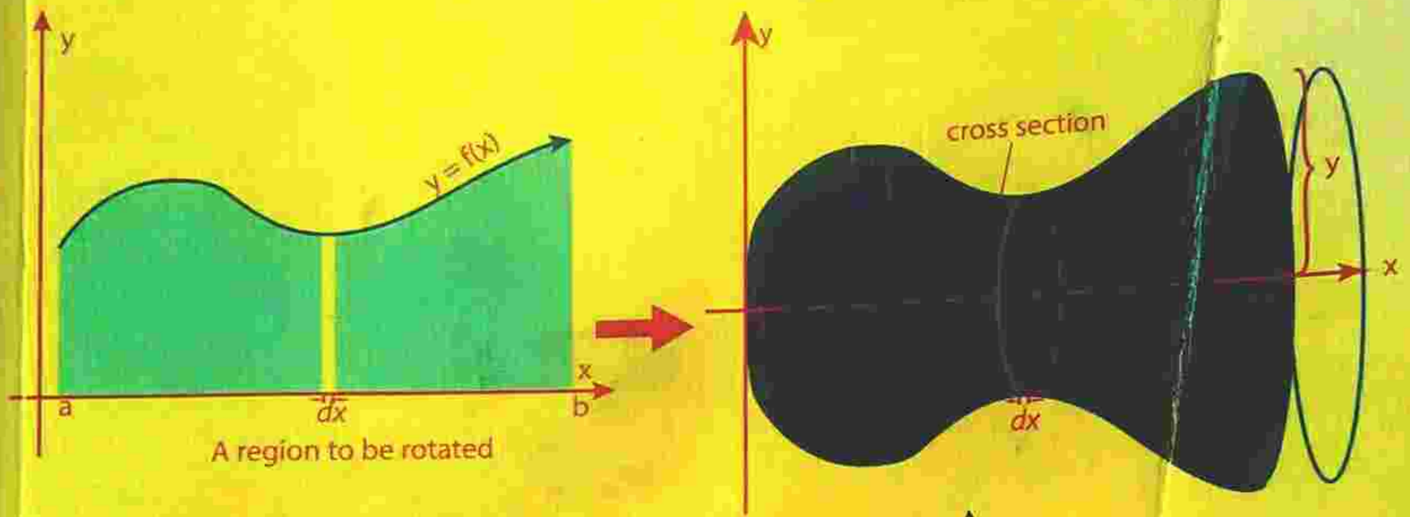
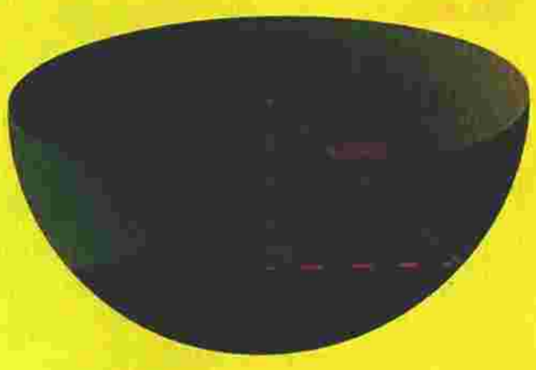
Extreme Series

MATHEMATICS

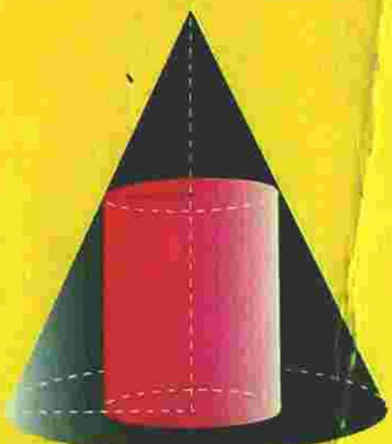
GRADE 11



A region to be rotated



Hemispherical bowl



Based on Common Currently Used

Table of Contents

UNIT ONE

Revision on Relations

Definition: A relation is a set whose elements are ordered pairs.

- Relation the way things are related.

Usually we use relating phrase like:

"is smaller than"
"is greater than"
"is multiple of"
"is factor of"
"is father of"........etc

Definition: If A and B are sets then a relation R from A to B is any subset of $A \times B$ if and only if $R \subseteq A \times B$

Definition: If R is relation from A to A, then R is relation on A. i.e. $R \subseteq A \times A$

Let R be relation from A to B. Then

- Domain of $R = \{x \in R: (x, y) \in R, \text{ for some } y \in B\}$
- Range of $R = \{y \in B: (x, y) \in R, \text{ for some } x \in A\}$

Inverse of a relation

Let R be a relation from A to B. The inverse of R, denoted by R^{-1} is relation from B to A, given by: $R^{-1} = \{(y, x)(x, y) \in R\}$.

- **Note:** $(x, y) \in R \Rightarrow (y, x) \in R^{-1}$

Illustrative Example

1. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ $B = \{2, 4, 6, 8\}$
If R is relation from A to B such that: $R = \{(x, y): y = 2x - 4\}$
Then:
i) a) List all elements of relation R
b) Find the domain and range of relation R.
c) List all elements of R^{-1}
d) Domain and range of R^{-1}

Solution: We know that $R \subseteq AxB \Rightarrow x \in A$ and $y \in B$ satisfies the relation $R= \{(x, y):y = 2x - 4\}$

Let $x = 1 \in A \Rightarrow y = 2(1) - 4 = -2 \notin B$ $\therefore (1, 2) \notin R$
Let $x = 2 \in A \Rightarrow y = 2(2) - 4 = 0 \notin B$ $\therefore (1, 2) \notin R$
Let $x = 3 \in A \Rightarrow y = 2(3) - 4 = 2 \in B$ $\therefore (3, 2) \in R$

Let $x = 4 \in A \Rightarrow y = 2(4) - 4 = 4 \in B$ $\therefore (4, 4) \in R$
Let $x = 5 \in A \Rightarrow y = 2(5) - 4 = 6 \in B$ $\therefore (5, 6) \in R$
Let $x = 6 \in A \Rightarrow y = 2(6) - 4 = 8 \in B$ $\therefore (6, 8) \in R$
Let $x = 7 \in A \Rightarrow y = 2(7) - 4 = 10 \notin B$ $\therefore (7, 10) \notin R$
Let $x = 8 \in A \Rightarrow y = 2(8) - 4 = 12 \notin B$ $\therefore (8, 12) \notin R$

a) Thus, all elements of R from A to B is
$R = \{(3, 2), (4, 4), (5, 6), (6, 8)\}$

b) Domain of $R = \{3, 4, 5, 6\}$
Range of $R = \{2, 4, 6, 8\}$

c) All element of R from B to A i.e.
$R^{-1} = \{(2, 3), (4, 6), (6, 5), (8, 6)\}$

d) Domain of $R^{-1} = \{2, 4, 6, 8\} =$
Range of $R = \{2, 4, 6, 8\}$
Range of $R^{-1} = \{3, 4, 5, 6\} =$
Domain of $R = \{3, 4, 5, 6\}$

> ***Explanation***
> To find the inverse of R
> Interchange x and y
> $x \to y$
> $y \to x$ then solve for y
> $\therefore R^{-1} = \{(y,x): y = 2x-4\}$
> $= \{(x,y): x = 2y-4\}$
> $= \{(x,y): y=\frac{1}{2}x-2\}$
> Solve for y

2. Let A = the set of natural number.
If R is relation from A to A such that $R = \{(x, y): 2x + y = 10\}$ then
   a) find all elements of R
   b) find the domain and range of R
   c) find R^{-1}
   d) find the domain and range of R^{-1}

Solution: We know $R \subseteq A\text{x}A \Rightarrow x \in A$ and $y \in A$, satisfies the relation
$R = \{(x,y): 2x + y = 10\}$

a) Let $x = 1 \in A \Rightarrow y = 8 \in A \Rightarrow (1, 8) \in R$
Let $x = 2 \in A \Rightarrow y = 6 \in A \Rightarrow (2, 6) \in R$
Let $x = 3 \in A \Rightarrow y = 4 \in A \Rightarrow (3, 4) \in R$
Let $x = 4 \in A \Rightarrow y = 2 \in A \Rightarrow (4, 2) \in R$
Let $x = 5 \in A \Rightarrow y = 0 \notin A$

Because 0 is not natural number $\therefore (5, 0) \notin R$
Hence $R = \{(1, 8), (2, 6), (3, 4), (4, 2)\}$

b) domain $= \{1, 2, 3, 4\}$ range of $R = \{2, 4, 6, 8\}$
c) all elements of $R^{-1} = \{(2, 4),(4,3),(6,2),(8,1)\}$
d) domain of $R^{-1} = \{2, 4, 6, 8\}$ = Range of R
range of $R^{-1} = \{(1, 2, 3, 4\}$ = Domain of R

> **Note:** $R^{-1} = \{(y, x): 2x + y = 10\} = \{(x, y): x + 2y = 10\}$
> Solving for $y = \{(x,y): y = 5 - \frac{1}{2}x\}$

3. If A = {0, 1, 2, 3, 4, 5}
B = {0, 1, 2, 3, 4} and R is a relation from A to B such that
$R = \{(x, y) : y = \sqrt{x-1}$ then what is the range of R^{-1} $y^2 = x - 1$

A. {1, 2, 5} C. $\{0, 1, \sqrt{2}, \sqrt{3}, 2\}$ $x^2 + 1 = y$

B. {0, 1, 2} D. {0, 1, 2, 3}

Solution: $R \subseteq AxB \Rightarrow x \in A$ and $y \in B$ satisfies the relation

$R = \{(x, y) : y = \sqrt{x-1}$ $x \geq 1$

- Let $x = 0 \in A \Rightarrow y = \sqrt{0-1} = \sqrt{-1} \notin B$
- Let $x = 1 \in A \Rightarrow y = \sqrt{1-1} = 0 \in B$
  $\therefore (1, 0) \in R \Rightarrow (0, 1) \in R^{-1}$
- Let $x = 2 \in A \Rightarrow y = \sqrt{2-1} = 1 \in B$ $\therefore (2, 1) \in R$
  $\Rightarrow (1, 2) \in R^{-1}$
- Let $x = 3 \in A \Rightarrow y = \sqrt{3-1} = \sqrt{2} \notin B$
  $\therefore (3, \sqrt{2}) \notin R \Rightarrow (\sqrt{2}, 3) \notin R^{-1}$
- Let $x = 4 \in A \Rightarrow y = \sqrt{4-1} = \sqrt{3} \notin B$
  $\therefore (4, \sqrt{3}) \notin R \Rightarrow (\sqrt{3}, 4) \notin R^{-1}$
- Let $x = 5 \in A \Rightarrow y = \sqrt{5-1} = 2 \in B$
  $\therefore (5, 2) \in R \Rightarrow (2, 5) \in R^{-1}$

$\therefore$ Element of R = {(1, 0),(2, 1),(5, 2)}
$\therefore$ Element of R^{-1} = {(0, 1), (1, 2), (2, 5)}

- Domain of R^{-1} = {0, 1, 2} = Range of R
- Range of R^{-1} = {1, 2, 5} = Domain of R

Answer: A

Note: $R^{-1} = \{(y, x): y = \sqrt{x-1}\} = \{(x, y): x = \sqrt{y-1}\}$
$= \{(x, y): y = x^2 + 1\}$ solving for y.

4. Let A = {1, 2, 3, 4, 5, 6, 7}
If R is relation from A to A such that R = {(x,y):y = 2x + 1}
Then a) List element of R
b) List domain and range of R
c) List element of R^{-1}
d) List domain and range of R^{-1}

Solution: $R \subseteq A \times B \Rightarrow x \in A$ and $y \in A$ satisfies $y = 2x + 1$
Let $x = 1 \in A \Rightarrow y = 3 \in A$

$\therefore (1, 3) \in R \Rightarrow (3, 1) \in R^{-1}$

Let $x = 2 \in A \Rightarrow y = 5 \in A$

$\therefore (2, 5) \in R \Rightarrow (5, 2) \in R^{-1}$

Let $x = 3 \in A \Rightarrow y = 7 \in A$

$\therefore (3, 7) \in R \Rightarrow (7, 3) \in R^{-1}$

Let $x = 4 \in A \Rightarrow y = 9 \notin A$

$\therefore (4, 9) \notin R \Rightarrow (9, 4) \notin R^{-1}$ b/c $9 \notin A$

a) $\therefore R = \{(1, 3), (2, 5),$

b) domain = $\{1, 2, 3\}$

Range = $\{3, 5, 7\}$

c) $R^{-1} = \{(3, 1), (5, 2),$

d) domain of $R^{-1} = \{3,$

Range of $R^{-1} = \{1, 2,$

Explanation

$R^{-1} = \{(y,x): y = 2x+1\}$

$= \{(x,y): x = 2y+1\}$

$= \{(x,y): y = \frac{1}{2}x - \frac{1}{2}\}$

5. Let $R = \{(x, y): 2x - 3y > 3$ then which of the following points satisfied the relation R^{-1}. $y > 3x + 3$

A. $(-2, 3)$ B. $(1, 3)$ C. $(2, -1)$ D. $(-3, -2)$

Solution: A) $(-2, 3) \in R^{-1} \Rightarrow (3, 2) \in R$

$\Rightarrow 2(3) - 3(-2) > 3 \Rightarrow 12 > 3$ = True

$\therefore (-2, 3) \in R^{-1}$

Explanation

If $(a, b) \in R$ then $(b,a) \in R^{-1}$

B) $(1, 3) \in R^{-1} \Rightarrow (3, 1) \in R$

$\Rightarrow 2(3) - 3(1) > 3 \Rightarrow 3 > 3$ = False

$\therefore (1, 3) \notin R^{-1}$

C) $(2, -1) \in R^{-1} \Rightarrow (-1, 2) \in R$

$\Rightarrow (2)(-1) - 3(2) > 3 \Rightarrow -8 > 3$, False

$\therefore (2, -1) \notin R^{-1}$

D) $(-3, -2) \in R^{-1} \Rightarrow (-2, -3) \in R$

$\Rightarrow -2(-1) - 3(-3) > 3 \Rightarrow 5 > 3$, True

$\therefore (-3, -2) \in R^{-1}$

$\therefore (-2, 3)$ and $(-3, -2)$ satisfies the relation R^{-1} and $(3, -2)$ and $(-2, -3)$ satisfies the relation R.

6. Let $R = \{(x, y): y \geq x^2 - 4$ and $y \leq 5\}$

a) Sketch the graph of R

b) Find the domain and range of R

c) Find R^{-1}

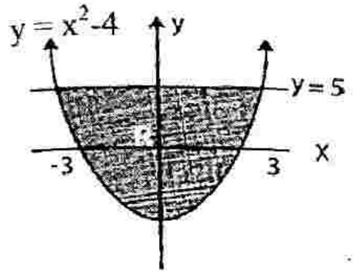
d) Find the domain and range of R^{-1}

Solution:

a) To sketch the graph, first we find the x-intercept set $y = 0$

$\Rightarrow x^2 - 4 = 0 \Rightarrow x = 2$, and $x = -2$

b) Domain of R = $\{x: x^2 - 4 \leq 5\}$



$= \{x: x^2 \leq 9\} = \{x: |x| \leq 3\}$

$= \{x: -3 \leq x \leq 3\}$

Range = $\{y: -4 \leq y \leq 5\}$

$x^2 - 4 \leq 5.$

$x^2 \leq 9$

c) $R^{-1} = \{(y, x): y \geq x^2 - 4 \text{ and } y \leq 5\}$

$= \{(x, y): x \geq y^2 - 4 \text{ and } x \leq 5\}$

$= \{(x, y): y^2 \leq x + 4 \text{ and } x \leq 5\}$

d) Domain of $R^{-1} = \{x: -4 \leq x \leq 5\}$

Range of $R^{-1} = \{y: -3 \leq y \leq 3\}$

7. Let $R = \{(x, y): x^2 + y^2 \leq 2 \text{ and } y \geq x^2\}$

Find a) Domain of R c) Domain of R^{-1}

b) Range of R d) Range of R^{-1}

Solution: The boundary of $x^2 + y^2 \leq 2$ and $y \geq x^2$ are $x^2 + y^2 = 2$ and $y = x^2$

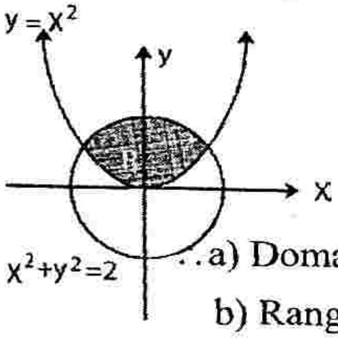
$y \geq x - 3$

$y > \frac{1}{2}x + 3$

Solving together $\begin{cases} x^2 + y^2 = 2 \\ y = x^2 \end{cases}$

$y \geq -3$

$y > 3$



$\Rightarrow x^4 + x^2 - 2 = 0$

$\Rightarrow (x^2 - 1)(x^2 + 2) = 0$

$\Rightarrow x^2 - 1 = 0 \text{ or } x^2 + 2 = 0$

$\Rightarrow x = 1, -1$

$-3 \leq y > 3$

$(-3, \infty)$

$\therefore$ a) Domain of R = $\{x: -1 \leq x < 1\}$

b) Range of R = $\{y: 0 \leq y \leq \sqrt{2}\}$

c) Domain of $R^{-1} = \{x: 0 \leq x \leq \sqrt{2}\}$

d) Range of $R^{-1} = \{y: -1 \leq y \leq 1\}$

8. Find the domain and range of the inverse of each of the following

a) $R = \{(x, y): y < 0\}$

b) $R = \{(x, y): y \leq 2, y > x - 3 \text{ and } y > -x - 3\}$

c) $R = \{(x, y): y = \sqrt{1 - |x|}$

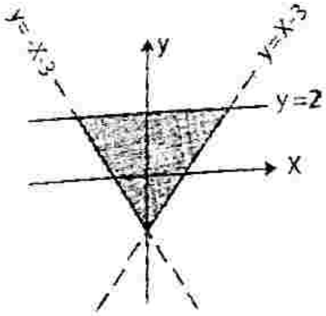
d) $R = \{(x, y): y = 1 - 3^x\}$

Solution: a) $R^{-1} = \{(x, y): x < 0\} = \{(y, x): y < 0\}$

Domain of $R^{-1} = \{x: x < 0\}$

Range of $R^{-1} = \{y: y \in \mathbb{R}\}$

b) To find the intersection point solving together $\begin{cases} y = 2 \\ y = x - 3 \\ y = -x - 3 \end{cases}$



$\Rightarrow$ when $y = 2$, $x = 5$ and $x = -5$

$\Rightarrow$ when $x = 0$, $y = -3$

$\therefore$ Domain of $R = \{x: -5 < x < 5\}$

Range of $R = \{y: -3 < y \leq 2\}$

$\therefore$ Domain of $R^{-1} = \{x: -3 < x \leq 2\}$

Range of $R^{-1} = \{y: -5 < y < 5\}$

c) $R = \{(x, y): y = \sqrt{1 - |x|}$

$R^{-1} = \{(x,y): x = \sqrt{1 - |y|}$ or $R^{-1} = |y| = 1 - x^2\}$

$\therefore$ Domain of $R^{-1} = \{x: 0 \leq x \leq 1\}$

Range of $R^{-1} = \{y: -1 \leq y \leq 1\}$

d) $R = \{(x, y): y = 1 - 3^x\}$

$R^{-1} = \{(x, y): x = 1 - 3^y\} = \{(x,y): y = \log_3^{1-x}\}$

Domain of $R^{-1} = \{x: x < 1\}$ and Range of $R^{-1} = \{y: y \in \mathbb{R}\}$

Revision on Function

Definition: A function is a relation in which each element of the domain is paired with exactly one element in the range.

Note: Let A and B two non – empty sets.

- ✓ Then a **function** f from A to B denoted by $f: A \longrightarrow B$ is a rule for assigning to each element of **A** *exactly one element of* **B**.
- ✓ Domain of f = A and
- ✓ f is single valued (well – defined) i.e., if $(a, b) \in f$ and $(a, c) \in f$ imply that $b = c$, we write $f(a) = b$

Illustrative Example

9. Determine and justify which of the following sets are **function from A to B**: ($f: A \longrightarrow B$) given by:

a) $f = \{(1, 2), (1, 3), (2, 3), (3, 4)\}$; where $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$

b) $f = \{(a, b), (b, c), (c, d), (d, a)\}$
   Where, $A = \{a, b, c, d\}$, $B = \{a, b, c, d\}$

c) $f = \{(a, b), (b, c), (c, d)\}$;
   Where, $A = \{a, b, c, d\}$; $B = \{a, b, c, d\}$

d) $f = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} | \ y = x + 4\}$; $A = B = \mathbb{Z}$

e) $f = \{(x, y) \in \mathbb{R} \times \mathbb{R}\} | \ y = \sqrt{x}\ \}$; $A = B = \mathbb{R}$

Solution:

a) Here f is not function because $f(1) = 2$ and $f(1) = 3$ thus $f(1)$ is not unique (not single valued).

b) f is function from A to B, because each element of A has exactly one image.

c) In this case, f is not a function from A to B, because, the domain of this relation f is $\{a, b, c\} \subset A$. But $A = \{a, b, c, d\}$

d) f is function from A to B.

e) f is not function, since it is not defined for negative real number.

10. What the domain and range of $f(x) = \sqrt{x - |x|}$?

Solution: Domain $= \{x: x - |x| \geq 0\} = \{x: |x| \leq x\} = \{x: x \geq 0\}$
Range $= \{y: y = 0\}$

Even and odd function

> **Definition**: consider a function f: $A \longrightarrow B$. Then f is called
> i. **Even**, if and only if, for any $x \in A$; $f(-x) = f(x)$.
> ii. Odd, if and only if for all $x \in A$, $f(-x) = -f(x)$.

Illustrative Example

11. Determine whether the following functions are even or odd or neither.

a) $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^3 - x$ $\quad -x^3 + x$

b) $f: \mathbb{R} \longrightarrow [-3, \infty)$ defined by $f(x) = x^2 - 3$ $\quad -(x^3 - x)$

c) $f: \mathbb{R} \longrightarrow [-3, \infty)$ defined by $f(x) = x^2 + 2x - 2$

d) $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = |x|x$

e) $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = |x| + x$

Solution:

a) $f(-x) = (-x)^3 - x = -(x^2 + x) = -f(x)$, therefore f **is odd**.

b) $f(-x) = (-x)^2 - 3 = x^2 - 3 = f(x)$, therefore f **is even.**

c) $f(-x) = (-x)^2 + 2(-x) - 2 = x^2 - 2x - 2$, which is neither equal to $f(x)$ nor equal to $-f(x)$. Hence f is **neither** even nor an odd function.

d) $f(-x) = |-x|(-x) = -x|x| = -f(x)$, therefore f is odd

e) $f(-x) = |-x| + (-x) = |x| - x$, neither.

12. Which of the following function is an odd function?

A. $f(x) = |x|$ C. $f(x) = [x] - x$

B. $f(x) = |x - 2|$ D. $f(x) = \ell n\left(\dfrac{1-x}{1+x}\right)$

Solution: We need to show $f(-x) = -f(x)$

A. $f(-x) = |-x| = |x| = f(x)$, therefore f is even

B. $f(-x) = |-x - 2| = |x + 2|$ neither

C. $f(-x) = [-x] + x$, neither

D. $f(-x) = \ell n\left(\dfrac{1+x}{1-x}\right) = \ell n(1+x) - \ell n(1-x)$

$= [\ell n\,(1 - x) - \ell n(1 + x)]$

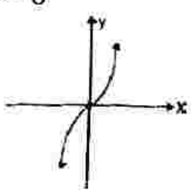
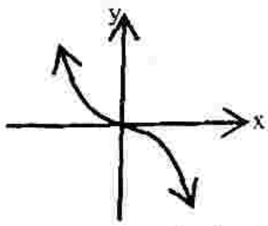
$= -\ell n\left(\dfrac{1-x}{1+x}\right) = -f(x)$ therefore, f is odd. **Answer: D**

Some Types of Function

These are • Power function
- Absolute value (modulus) function
- Signum function
- Greatest integer function

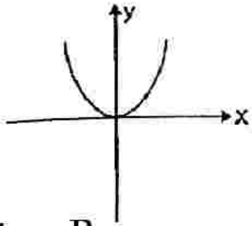
Power function

A function of the form $f(x) = ax^n$ where n is constant and $a \in R$

i) $f(x) = ax^n$	n is odd and $n > 0$
If $a > 0$  • domain R • Range R • odd function • Increasing Ex. $f(x) = x^3\ x^5, \ldots$	If $a < 0$  • Domain = $\mathbb{R}$ • Range = $\mathbb{R}$ • odd • decreasing Example, $f(x) = -x^3, -x^5, -x^7, \ldots$

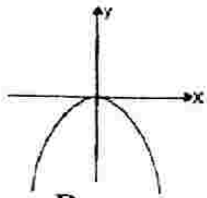
ii) $f(x) = ax^n$, n is an even and $n > 0$

If $a > 0$



- Domain = R
- Range = $[0, \infty)$
- Even function

If $a < 0$



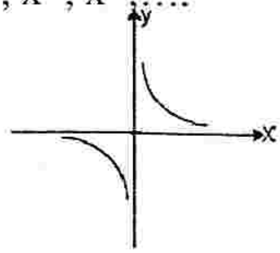
- Domain = R
- Range = $(-\infty, 0]$
- Even function

Example: $f(x) = -x^2, -x^4, \ldots$

iii) $f(x) = ax^n$, n is odd and n <

If $a > 0$

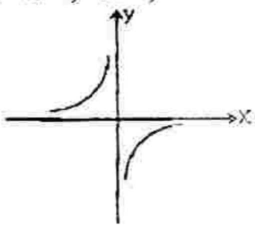
Example: $f(x)=x^{-3}, x^{-5}, x^{-7}, \ldots$



- Domain = $R\backslash\{0\}$
- Range = $R\backslash\{0\}$
- odd function
- decreasing function

If $a < 0$

Example: $f(x)=-x^{-3}, -x^{-5}, -x^{-7}, \ldots$

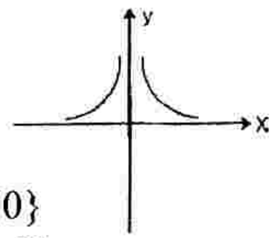


- Domain = $R\backslash\{0\}$
- Range = $R\backslash\{0\}$
- odd function
- Increasing function

iv) $f(x) = ax^{\frac{n}{m}}$ n is even and $n < 0$ m is odd

If $a > 0$

Example: $f(x)= x^{\frac{-2}{3}}, x^{\frac{-4}{3}}, x^{\frac{-6}{5}}, \ldots$

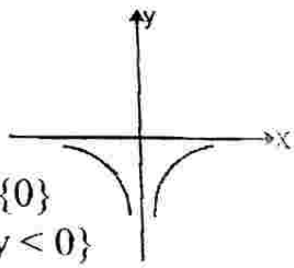


- Domain = $R\backslash\{0\}$
- Range = $\{y: y>0\}$
- Even function

If $a < 0$

Example: $f(x)= -x^{\frac{-2}{3}}, -x^{\frac{-4}{3}}, -x^{\frac{-6}{5}}, \ldots$

$\frac{1}{x^{-2/3}}$



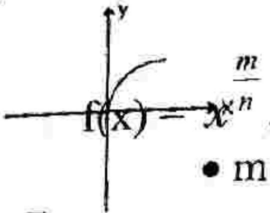
- Domain = $R\backslash\{0\}$
- Range = $\{y: y < 0\}$
- even function

$\frac{1}{x}$

v) If f(x) = $x^{\frac{m}{n}}$, for m is odd, n is even the graph of f

- not symmetric
- neither even nor odd
- has range $[0, \infty)$
- has domain $[0, \infty)$
- increasing

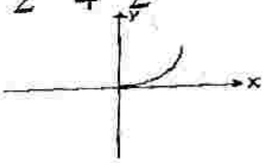
Example: $\frac{m}{n} = \frac{1}{2}, \frac{3}{4}, \ldots$



f(x) = $x^{\frac{m}{n}}$, for m < n

- m odd
- n even

Example: $\frac{m}{n} = \frac{3}{2}, \frac{5}{4}, \frac{7}{2}, \ldots$



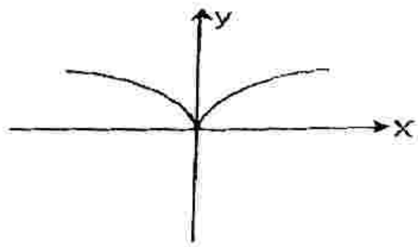
f(x) = $x^{\frac{m}{n}}$, for m > n

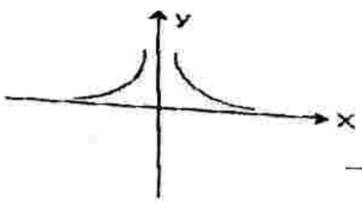
- m odd
- n even

vi) f(x) = $x^{\frac{m}{n}}$ for m is even, n is odd then the graph of f

- symmetric w.r.t y axis
- even function
- has domain real number
- has range: $0 \le y < \infty$

Example: $f(x) = x^{\frac{2}{3}}, x^{\frac{4}{5}}, \ldots$





$\longrightarrow \frac{m}{n} < 0$

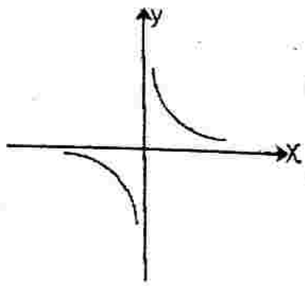
f(x) = $x^{\frac{m}{n}}$

Example, f(x) = $x^{\frac{-2}{3}}$ .....

vii) $f(x)=x^{\frac{-1}{n}}$, n is odd, then the graph of f

- symmetric
- odd function
- increasing
- Domain R\{0}

Example, f(x) = x^{-1}, $x^{-\frac{1}{3}}, x^{\frac{-1}{5}}, x^{\frac{-1}{7}}$ .....



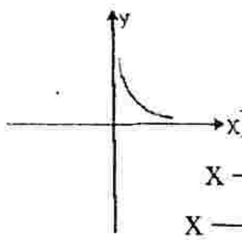
Explanation

- $x \longrightarrow 0^+$
  $y \longrightarrow \infty$
- $x \longrightarrow 0^-$
  $y \longrightarrow -\infty$
- $x \longrightarrow \infty$
  $y \longrightarrow 0$
- $x \longrightarrow -\infty$
  $y \longrightarrow 0^-$

viii) $f(x)=x^{\frac{-1}{n}}$, n is even, then the graph of f

- not symmetric
- neither even nor odd
- has domain {x: x > 0}
- Range = {y: y > 0}

Example, f(x) = $x^{\frac{-1}{2}} = \dfrac{1}{\sqrt{x}}$



Explanation

$x \longrightarrow 0^+$, $y \longrightarrow \infty$

$x \longrightarrow \infty$, $y \longrightarrow 0^+$

Illustrative Example

13. Which of the following are power functions and which are not (Justify)

a) $f(x)=\sqrt{\frac{1}{x}}$

b) $f(x)=\sqrt[3]{\frac{1}{2x}}$

c) $f(x)=\frac{x^2}{2}$

d) $f(x)=2x^{\frac{-4}{5}}$

e) $f(x)=2^x$

f) $f(x)=3x^2+4$

Solution: Power function is written in the form $f(x)=ax^r$

Thus, a) $f(x)=\sqrt{\frac{1}{x}}=\left(\frac{1}{x}\right)^{\frac{1}{2}}=x^{\frac{-1}{2}}$ ← power function

b) $f(x)=\sqrt[3]{\frac{1}{2x}}=\left(\frac{1}{2x}\right)^{\frac{1}{3}}=\left(\sqrt[3]{\frac{1}{2}}\right)x^{\frac{1}{3}}$ ← power function

with $a=\frac{1}{\sqrt[3]{2}}$

c) $f(x)=\frac{x^2}{2}=\frac{1}{2}x^2$ ← power function with $a=\frac{1}{2}$

d) $f(x)=2^x$ ← Not power function, it is exponential function

f) $f(x)=3x^2+4$ ← Not power function

Note: All power function of the form $f(x)=\mathbf{ax^r}$ with $a=1$ satisfies multiplicative property of $f(xy)=f(x).f(y)$

Illustrative Example

14. Which of the following power function does not satisfies the condition $f(xy)=f(x)\cdot f(y)$?

a) $f(x)=\sqrt{\frac{1}{x}}$

c) $f(x)=x^{\cdot}$

b) $f(\mathrm{x}) = \mathrm{x}^3$ d) $f(\mathrm{x}) = 2\mathrm{x}^4$

Solution: a) $f(xy) = \sqrt{\dfrac{1}{xy}} = \sqrt{\dfrac{1}{x}} \cdot \sqrt{\dfrac{1}{y}} = f(x).f(y)$

b) $f(\mathrm{xy}) = (\mathrm{xy})^3 = \mathrm{x}^3 \cdot \mathrm{y}^3 = f(\mathrm{x}) \cdot f(\mathrm{y})$

c) $f(xy) = (xy)^{\frac{2}{3}} = x^{\frac{2}{3}} \cdot y^{\frac{2}{3}} = f(x) \cdot f(y)$

d) $f(xy) = 2(xy)^4 = 2x^4 \cdot y^4$

but $f(x) \cdot f(y) = 2x^2 \cdot 2y^4 = 4x^4 y^4$

hence $\mathrm{f(xy)} \neq \mathrm{f(x) \cdot f(y)}$ **Answer:** D

Absolute value (Modulus) Function

Definition: The absolute value (Modulus) of x is defined by

$$f(\mathrm{x}) = |\mathrm{x}| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

- Domain of $f = \mathbb{R}$
- Range of $f = [0, \infty)$

Illustrative Example

15. Evaluate each of the following

a) $|-3|$ c) $|\sqrt{2} - 2|$ e) $|\sqrt{6} - 3|$

b) $|\pi - 4|$ d) $|\mathrm{x} - \mathrm{y}|$ f) $|3 - \sqrt{7}|$

Solution:

a) $|-3| = -(-3) = 3$, since $-3 < 0$, thus $|-3| = 3$

b) $|\pi - 4| = -(\pi - 4) = 4 - \pi$ since $\pi - 4 < 0$

c) $|\sqrt{2} - 2| = 2 - \sqrt{2}$

d) $|\mathrm{x} - \mathrm{y}| = \begin{cases} x - y \text{ if } x \geq y \\ y - x \text{ if } x < y \end{cases}$

e) $|\sqrt{6} - 3| = 3 - \sqrt{6}$

f) $|3 - \sqrt{7}| = 3 - \sqrt{7}$

Explanation

$\sqrt{9} = 3$

$\therefore |\sqrt{6} - \sqrt{9}| = \sqrt{9} - \sqrt{6}$

$= 3 - \sqrt{6}$

Note: $|x| = a$ iff $x = \pm a$, if $a < 0$ then $|x| = a$ has no *Solution*.

16. Find the *Solution* set of each of the following equations.

a) $|x - 1| = \pi$ c) $|x| = -1$ e) $|4x + 3| = 5$

b) $|x| = \sqrt{2} - 1$ d) $|4x + 3| = 0$ f) $|3 - 5x| = 6$

g) $|x^2 - 2| = 1$ h) $|x^3 - 2| = 6$

Solution: Here

a) $|x - 1| = \pi \Leftrightarrow x - 1 = \pi$ or $x - 1 = -\pi \Rightarrow x = 1 + \pi$ or $x = 1 - \pi$

$\therefore$ S.S $= \{1+\pi, 1-\pi\}$

b) $|x| = \sqrt{2} - 1 \Leftrightarrow x = \sqrt{2} - 1$ *or* $x = 1 - \sqrt{2}$

c) $|x| = -1$ is $\varnothing$ because $|x| \geq 0$

d) $|4x + 3| = 0 \Leftrightarrow 4x + 3 = 0 \Rightarrow 4x = -3$

$$\therefore \text{S.S} = \left\{\frac{-3}{4}\right\}$$

e) $|4x + 3| = 5 \Leftrightarrow 4x + 3 = 5$ or $4x + 3 = -5$

$\Rightarrow 4x = 2$ or $4x = -8$

$$\therefore x = \frac{1}{2} \text{ or } x = -2$$

f) $|3 - 5x| = 6 \Leftrightarrow 3 - 5x = 6$ or $3 - 5x = -6$

$$\Rightarrow -5x = 3 \text{ or } -5x = -9 \Rightarrow x = \frac{-3}{5} \text{ or } x = \frac{9}{5}$$

$$\therefore \text{S.S} = \left\{\frac{-3}{5}, \frac{9}{5}\right\}$$

g) $|x^2 - 2| = 1 \Leftrightarrow x^2 - 2 = 1$ or $x^2 - 2 = -1$

$\Rightarrow x^2 = 3$ or $x^2 = -1 + 2 = 1$

$\Rightarrow x = \pm\sqrt{3}$ or $x = \pm 1$

$\therefore$ S.S $= \{\sqrt{3}, -\sqrt{3}, 1, -1\}$

h) $|x^3 - 2| = 6 \Leftrightarrow x^3 - 2 = 6$ or $x^3 - 2 = -6$

$\Rightarrow x^3 = 8$ or $x^3 = -4$

$\Rightarrow x = 2$ *or* $x = \sqrt[3]{-4}$

$\therefore$ S.S $= \{2, \sqrt[3]{-4}\}$

17. For what real number x true and solve each of the following.

a) $|2x + 3| = 4x + 8$

b) $|3x-4| = 4-x^2$

Solution: Universe $|2x+3| \geq 0 \Rightarrow 4x+8 \geq 0$

$$\Rightarrow 4x \geq -8 \Rightarrow x \geq -2$$

$\therefore$ Universe, $\{x: x \geq -2\} \longleftarrow$ domain

And $|2x+3| = 4x+8 \Leftrightarrow 2x+3 = 4x+8$ or $2x+3 = -4x-8$

$$\Rightarrow -2x = 5 \text{ or } 6x = -11$$

$$\Rightarrow x = \underset{\substack{\uparrow \\ \text{Reject}}}{\frac{-5}{2}} \text{ or } x = \frac{-11}{6} \in [-2, \infty)$$

The only solution set is $\left\{x : x = \frac{-11}{6}\right\}$ because $\frac{-5}{2} \notin [-2, \infty)$

b) $|3x-4| = 4-x^2 \Leftrightarrow 3x-4 = 4-x^2$ or $3x-4 = x^2-4$

$$\Rightarrow x^2+3x-8 = 0 \text{ or } x^2-3x = 0$$

$$\Rightarrow x(x-3) = 0$$

$$\therefore 0, x = 3$$

Since the universe $4-x^2 \geq 0 \Leftrightarrow 4 \geq x^2$

$$\Leftrightarrow x^2 \leq 4$$

$$\Leftrightarrow -2 \leq x \leq 2$$

$\therefore$ The only S.S is $\{0\}$, but -4 and 3 are not ***Solution*** because $4-x^2$ is negative at $x = 3$.

Graph of Absolute value

Definition: The graph of absolute value is the pictorial representation of set of points $(x, f(x))$ in a plane of absolute value function.

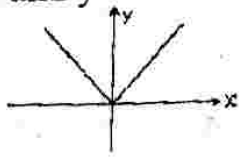
Illustrative Example

18. Sketch the graph find the domain and range of each of the following.

a)	$f(x) = \lvert x\rvert$	e)	$f(x) = \lvert x^2-4\rvert$
b)	$f(x) = \lvert x\rvert + 2$	f)	$f(x) = \lvert x\rvert - x + 1$
c)	$f(x) = \lvert x+2\rvert$	g)	$f(x) = \lvert x-3\rvert$
d)	$f(x) = \left\lvert \frac{1}{x+2} \right\rvert$	h)	$f(x) = \lvert x-3\rvert + 1$

Solution: a) We have $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$ and we know $y = f(x)$

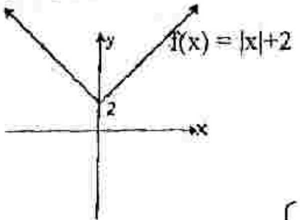
$\Rightarrow y = x$ if $x \geq 0$ and $y = -x$ if $x < 0$



- Domain = R
- Range = $[0, \infty)$

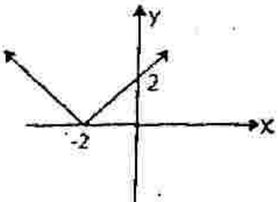
$f(x) = |x|$

b) Adding 2 from each value of $f(x) = |x|$



- domain = R
- range = $[2, \infty)$

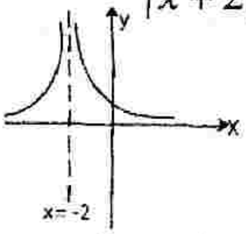
c) We have $f(x) = |x + 2| = \begin{cases} x+2 \text{ if } x+2 \geq 0 \\ -x-2 \text{ if } x+2 < 0 \end{cases}$



$f(x) = |x+2|$

- Domain = R
- Range = $[0, \infty)$
- Symmetry line $x = -2$

d) $f(x) = \left|\dfrac{1}{x+2}\right| = \begin{cases} \dfrac{1}{x+2} \text{ if } x \geq -2 \\ \dfrac{-1}{x+2} \text{ if } x < -2 \end{cases}$



- Domain = R\{-2}
- Range = {y: y > 0}
- Symmetry, the line x = -2

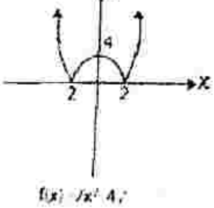
$f(x) = \left|\dfrac{1}{x+2}\right|$

$x^2 = 4$

e) $f(x) = |x^2 - 4| = \begin{cases} x^2 - 4 \text{ if } x^2 - 4 \geq 0 \quad i.e \\ \quad x \leq -2 \text{ or } x \geq 2 \\ 4 - x^2 \text{ if } -2 < x < 2 \end{cases}$

$x^2 \geq 4$

$-2 \geq x$



- Domain = R

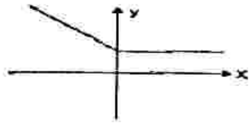
$f(x) = |x^2 - 4|$

- Range = $[0, \infty)$

$f(x) = |x^2 - 4|$

f) we have $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$

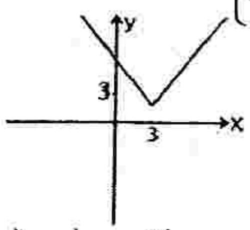
Hence $f(x) = |x| - x + 1 = x - x + 1 = 1$ if $x \geq 0$ and
$f(x) = -x - x + 1 = -2x + 1$ if $x < 0$



- Domain = R
- Range = $[1, \infty)$

$f(x) = |x| - x + 1$

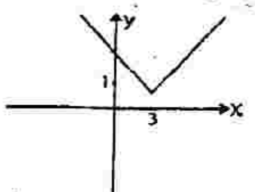
g) $f(x) = |x - 3| = \begin{cases} x - 3 \text{ if } x \geq 3 \\ -x + 3 \text{ if } x < 3 \end{cases}$



- Domain = R
- Range = $[0, \infty)$
- axis of symmetry line x=3

$f(x) = |x - 3|$

h) we shift 1 unit up the graph of $|x - 3|$



- Domain = R
- Range = $[1, \infty)$
- symmetry, x = 3

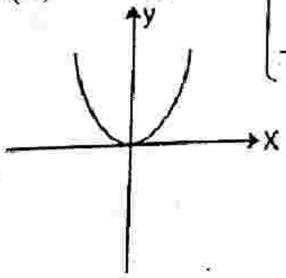
$f(x) = |x - 3| + 1$

Signum function

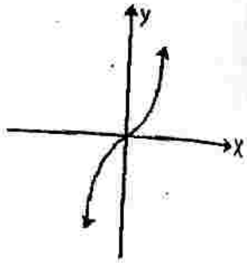
The signum function, read as **signum (x)** is written as sign x and is defined by $f(x) = \text{sgn}x = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x = 0 \\ -1 \text{ for } x < 0 \end{cases}$

Or

c) $f(x) = x^n \text{sgn}x = \begin{cases} x^n & if\ x > 0 \\ 0 & if\ x = 0 \\ -x^n & if\ x < 0 \end{cases}$

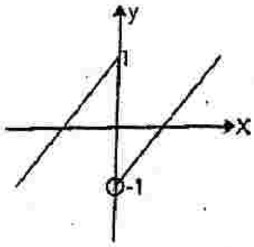


$f(x) = x^n\text{sgn}x$, if n is odd
- Domain = R
- Range = $[0, \infty)$
- even function



$f(x) = x^n\text{sgn}x$ if n is even

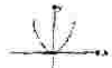
d) $f(x) = x - \text{sgn}x = \begin{cases} x-1 & if\ x > 0 \\ 0 & if\ x = 0 \\ x+1 & if\ x < 0 \end{cases}$



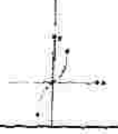
- Domain = R
- Range = R\{1,-1}
- f in neither even nor odd

$f(x) = x - \text{sgn}x$

Note: i) sgnx is an odd function



ii) xsgnx, x^3sgnx, x^5sgnx …. are even function of graph of the form



iii) x^2sgnx, x^4sgnx, …..are **odd function** of graph of the form

22. If $f(x) = 3(\text{sgn}x + 1)$ then find $f(-x) + f(x)$

Solution: Here

$\Rightarrow f(-x) = 3(\text{sgn}(-x) + 1) = 3[-\text{sgn}x + 1]$

$= -3\text{sgn}x + 3$

Thus $f(-x) + f(x) = -3\text{sgn}x + 3 + 3\text{sgn}x + 3$

$= 6$

$\therefore f(-x) + f(x) = 6$

Explanation
Sgn(-x) = -sgnx

Explanation
f(-2) + f(2) = 6
f(-1) + f(1) = 6
$\therefore$ f(-x) + f(x) = 6

23. If $h(x) = \frac{1}{2}(\text{sgn}\, x + 2)$, then

Find a) h(x) + h(-x)

b) h (2) + h(-2)

c) $h\left(\frac{1}{3}\right) + h\left(-\frac{1}{3}\right)$

Solution: Here

a) $h(-x) = \frac{1}{2}(\text{sgn}(-x) + 2)$

$$= \frac{-1}{2}\text{sgn}\, x + \left(\frac{1}{2}\right)(2) = \frac{-1}{2}\text{sgn}\, x + 1$$

Explanation
sgn(-x) = -sgnx
because
signum function is
odd function

And $h(x) = \frac{1}{2}(\text{sgn}\, x + 2)$

$$= \frac{1}{2}\text{sgn}\, x + (2)\left(\frac{1}{2}\right) = \frac{1}{2}\text{sgn}\, x + 1$$

$$\therefore h(x) + h(-x) = \frac{1}{2}\text{sgn}\, x + 1 \ - \frac{1}{2}\text{sgn}\, x + 1 = 2$$

b) $h(2) + h(-2) = \left(\frac{1}{2}\text{sgn}(2) + 1)\right) + \left(-\frac{1}{2}\text{sgn}(2) + 1\right)$

$$= \frac{1}{2} + 1 - \frac{1}{2} + 1 = 2$$

Explanation
sgn(2) = 1
sgn(-2) = -1

c) $h\left(\frac{1}{3}\right) + h\left(-\frac{1}{3}\right) = \left(\frac{1}{2}\text{sgn}\left(\frac{1}{3}\right) + 1)\right) + \left(-\frac{1}{2}\text{sgn}\left(\frac{1}{3}\right) + 1\right)$

$$= \frac{1}{2} + 1 - \frac{1}{2} + 1 = 2$$

Greatest integer function

The greatest integer function is denoted by [x] and is defined by $f(x) = [x]$ = the greatest integer less than or equal to x. Note $[x] \leq x$

Illustrative Example

Evaluate each of the following

a)	$[2\cdot64]$	d)	$[-1\cdot24]$	g)	$[-1\cdot5]$		
b)	$[1\cdot24]$	e)	$[\pi]$	h)	$\left\lfloor -\sqrt{3} \right\rfloor$		
c)	$[-2\cdot64]$	f)	$[-2]$	o)	$[0\cdot25]$		
p)	$[-0\cdot5]$	q)	$[3]$	r)	$[2\cdot47]$		
s)	$[-31\cdot004]$						

Solution: Here

a)	$[2\cdot64] = 2$	e)	$[\pi] = 3$	o)	$[0\cdot25] = 0$		
b)	$[1\cdot24] = 1$	f)	$[-2] = -2$	p)	$[-0\cdot5] = -1$		
c)	$[-2\cdot64] = -3$	g)	$[-1\cdot5] = -2$	q)	$[3] = 3$		
d)	$[-1\cdot24] = -2$	h)	$\left\lfloor -\sqrt{3} \right\rfloor = -2$	r)	$[2\cdot47] = 2$		

Equation involving greatest integer

24. Find the value of x, for each part

a) $[x] = a$ d) $[x - 2] = 6$

b) $[x] = -3$ e) $[2x + 1] = 10$

c) $[x] = 3$

Solution: Here

a) $[x] = a$ if $a \leq x < a + 1$, where a is integer

b) $[x] = -3$ if $-3 \leq x < -3 + 1 \Rightarrow -3 \leq x < -2$

c) $[x] = 3$ if $3 \leq x < 3 + 1 = 3 \leq x < 4$

d) $[x - 2] = 6$ if $6 \leq x - 2 < 7$

adding both sides 2

e) $[2x + 1] = 10$ if $10 \leq 2x + 1 < 11$

subtracting (1) and dividing by 2 $\Rightarrow \frac{9}{2} \leq x < 5$

Special properties of greatest integer function

If $f(x) = [x]$, then

i) $[x] \leq x$

ii) $f(x + k) = f(x) + k$, where k is integer and x is real number

iii) $f(x) + f(y) \leq f(x + y) \leq x + y$

iv) $f(x) \leq x < f(x) + 1$

Example: Let $x = 2.54$, $y = -3.46$, $k = 2$

i) $(2.54) \leq 2.54 \Rightarrow 2 \leq 2.54$

ii) $f(2.45 + 2) = f(2.54) + 2 = [2.54] + 2 = 2 + 2 = 4$

iii) $f(2.54) + f(-3.46) \leq f(2.54 + -3.46) \leq 2.54 + -3.46$

$\Rightarrow [2.54] + [-3.46] \leq [-0.92] \leq -0.92$
$\Rightarrow 2 + -4 \leq -1 \leq -0.92$
$\Rightarrow -2 \leq -1 \leq -0.92$

$\therefore f(x) + f(y) \leq f(x + y) \leq x + y$ since $f(x) = [x]$
$\therefore [x] + [y] \leq [x + y] \leq x + y$ and $f(y) = [y]$

iv) $[2.54] \leq 2.54 < [2.54] + 1$
$\Rightarrow 2 \leq 2.54 < 2 + 1$
$\therefore f(x) \leq x < f(x) + 1$, if $f(x) = [x]$
i.e. $[x] \leq x < [x] + 1$

25. Let $f(x) = x - [x]$
Find a) the domain
b) the possible value of f(x) will be $0 \leq f(x) < 1$
i.e. all the out put will be lie between 0 and 1

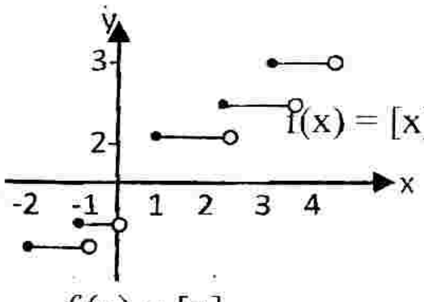
Example: $f(2.9) = 2.9 - [2.9] = 0.9$
$f(-2.9) = -2.9 - [-2.9] = -2.9 + 3 = 0.1$
$f(-2.1) = -2.1 + [-2.1] = -2.1 + 3 = 0.9$
$f(-2) = -2 - [-2] = -2 + 2 = 0$

Thus, conclude that
$0 \leq x - [x] < 1$ for every value of x

Graph of greatest integer function f(x) = [x]

- Domain = R
- Range = the set of integer (z)

Hence f: R $\longrightarrow$ z given by $f(x) = [x]$



$$f(x) = [x] = \begin{cases} -2 \text{ if } -2 \leq x < -1 \\ -1 \text{ if } -1 \leq x < 0 \\ 0 \text{ if } 0 \leq x < 1 \\ 1 \text{ if } 0 \leq x < 2 \\ 2 \text{ if } 2 \leq x < 3 \end{cases}$$

$f(x) = [x]$

> **Note:** The graph of the greatest integer function y = [x] lies on below the line y = x, so it provides an integer **floor for x.**

26. Let $y = 2[x] + 3$ and $y = 3[x - 2] + 5$
Then find the value of $[x + y]$

Solution: we have

$\Rightarrow 2[x] + 3 = 3[x - 2] + 5$

$\Rightarrow 2[x] + 3 = 3[x] - 6 + 5$

$\Rightarrow 3[x] - 2[x] = 3 + 1 = 4 \Rightarrow [x] = 4$ if $4 \le x < 5$

Therefore, $y = 2[x] + 3 = 2(4) + 3$, since $[x] = 4$

$\therefore y = 11$

And $x = 4 +$ fractional part $= 4 + f$

Hence $[x + y] = [4 + f + 11] = [15 + f] = 15$

27. Indicate whether the statement is true for all real number x, y, and z. If false give a counter example

a) $[x + y] = [x] + [y]$

b) $[x] + [y] \le [x + y]$

c) $[x] - [y] = [x - y]$

d) $[x]([y] + [z]) = [xy] + [xz]$

e) $[x]^2 = [x^2]$

f) $[x]^2 \le [x^2]$

g) $\frac{[x]}{2} = \left[\frac{x}{2}\right]$

h) $[-x] = -[x]$

i) $[x] + \frac{1}{2} =$

j) $[xy] = [x][y]$

k) $[xy] \ge [x][y]$

Solution: Here

a) False: counter example
Let $x = 3.6$, and $y = 2.4$
$\Rightarrow [x] + [y] \ne [x + y] \Rightarrow [3.6] + [2.4] \ne [3.6 + 2.4]$
$\Rightarrow 3 + 2 \ne 6$
$\Rightarrow 5 \ne 6 \quad \therefore [x] + [y] \ne [x + y]$

b) True, $[x] + [y] \le [x + y]$

c) False, counter example, Let $x = -3.8$, $y = -5.2$
then $[x] = [-3.8] = -4$ and $[y] = [-5.2] = -6$
thus, $[x] - [y] = -4 - 6 = -10$
and $[x - y] = [-3.8 - (-5.2)] = [-3.8 + 5.2] = [1.4] = 1$
$\therefore [x] - [y] \ne [x - y]$

d) False

e) False, counter example, Let $x = 2.4$
$[2.4]^2 \ne [(2.4)^2] \Rightarrow 2^2 \ne [5.76]$
$4 \ne 5$
$\therefore [x]^2 \ne [x^2]$

And $[x]^2 \leq [x^2]$

f) True

g) False, counter example, Let x = 7.6 then $\frac{[7.6]}{2} = \frac{7}{2} = 3.5$

and $\left[\frac{7.6}{2}\right] = [3.8] = 3$

$\therefore \frac{[x]}{2} \neq \left[\frac{x}{2}\right]$

h) False, counter example, Let x = 3.7 then [-3.7] = -4 and $-[3.7] = -3$

$\therefore [-x] \neq -[x]$ note $[-x] \leq -[x]$

i) False, counter example, Let x = 3.6

then $[3.6] + \frac{1}{2} = 3 + \frac{1}{2} = 3.5$ and

$\left[3.6 + \frac{1}{2}\right] = [3.6 + 0.5] = [4.1] = 4$

$\therefore [x] + \frac{1}{2} \neq \left[x + \frac{1}{2}\right]$

j) False, counter example, Let x = 2.4, and y = 3.4
then, $[(2.4)\cdot(3.4)] = [8.16) = 8$
and $[2.4][3.4] = (2)(3) = 6 \therefore [xy] \neq [x][y]$

k) $[xy] \geq [x][y]$ is true

Classification of Function

Based on some relating situation between the domain and its value 'range' function can classify as:

i. One – to – one (injective)
ii. On to (surjective)
iii. One – to – one correspondence (bijective)

One – to – one function

Definition: A function f: A $\longrightarrow$ B is said to be **one – to – one** (injective) if and only if, each element of the range is paired with exactly one element of the domain. That means;

a) if $f(a_1) = f(a_2)$, then $a_1 = a_2$

b) if $f(a_1) \neq f(a_2)$, then $a_1 \neq a_2$ for all $a_1, a_2 \in A$.

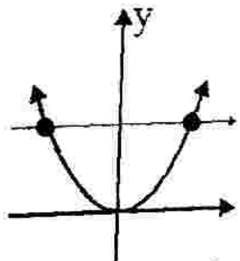
Horizontal – Line Test

If the graph of a function f is known, there is a simple test, called the **horizontal – line test**, to determine whether f is one – to – one.

Theorem: A function f: A $\longrightarrow$ B is one – to – one if and only if every horizontal line crosses the graph of a function f in at most one point.

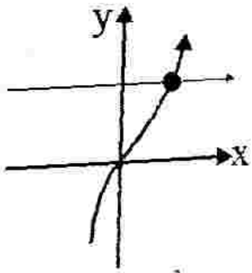
Consider the following figure

a) A horizontal line crosses the graph of f twice; f is not – one – to – one.



$\rightarrow f(x) = ax^2$

b) Every horizontal line crosses the graph of g exactly once. Therefore, g is one – to – one.



$g(x) = ax^3$

Theorem:

A function that is increasing over its domain is one – to – one function.

A function that is decreasing over its domain is one – to – one.

Study the following Example

✓ Quadratic function with domain on set of real number is not **one – to – one.**

- f: $\mathbb{R} \longrightarrow [0, \infty)$ given by $f(x) = x^2$ is not one – to – one but:

i) $f: [0, \infty) \longrightarrow [0, \infty)$ given by $f(x) = x^2$ is one – to – one

ii) $f: (-\infty, 0] \longrightarrow [0, \infty)$ given by $f(x) = x^2$ is one – to – one.

✓ Absolute value function with domain set of real number is **not one – to – one.**

- $f(x) = |x|$ is not one – to – one when $x \in (-\infty, \infty)$, but
- $f(x) = |x|$ is one – to – one when $x \in (-\infty, 0]$ or $x \in [0, \infty)$

✓ Linear function $f(x) = mx + c$, for $m \neq 0$ are one – to – one.

✓ Logarithm function and exponential function are one – to – one.

- $f(x) = \log_a^x$ is one – to – one when $x \in (0, \infty)$
- $f(x) = a^x$ is one – to – one $x \in (-\infty, \infty)$

Illustrative Example

28. Determine which of the following functions are one – to – one

A. $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = 2x - 3$

B. $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = |x| + x$

C. $f: \mathbb{R} \longrightarrow \mathbb{Z}$ defined by $f(x) = [x]$

D. $f: (2, \infty) \longrightarrow \mathbb{R}$ defined by $f(x) = \log_5^{x-2}$ $5^y = x - 2$

E. $f: \mathbb{R} \longrightarrow (0, \infty)$ defined by $f(x) = 3^x$

Solution:

A. Linear function are one – to – one function.

B. Not – one to one function, because $f(-2) = |-2| + (-2) = 2 + (-2) = 0$ and $f(-3) = |-3| + (-3) = 3 - 3 = 0$

$$\Rightarrow f(-2) = f(-3) = 0, \text{ but } -2 \neq -3$$

C. Not one – to – one, let $a_1 = 2.1$ and $a_2 = 2.7$

$$\Rightarrow f(2.1) = [2.1] = 2 \text{ and } f(2.7) = [2.7] = 2$$
$$\Rightarrow f(2.1) = f(2.7) = 2, \text{ but } 2.1 \neq 2.7$$

D. Logarizm function of the form $f(x) = \log_a^x$ are one – to – one function, because a horizontal line crosses its graph one time only.

E. Exponential of the form $f(x) = a^x$ are one to one function, because a horizontal line crosses its graph at most once.

29. Determine which of the following functions are one – to – one

A. $f: \mathbb{R} \longrightarrow [3, \infty)$, defined by $f(x) = (x + 2)^2 + 3$

B. $f: [-2, \infty] \longrightarrow [3, \infty)$ defined by $f(x) = (x + 2)^2 + 3$

C. $f: \mathbb{R} \longrightarrow [0, \infty)$ defined by $f(x) = |x - 4|$

D. $f: [4, \infty) \longrightarrow (0, \infty)$ defined by $f(x) = |x - 4|$

Solution:

A. Let $a_1 = 0$ and $a_2 = -4$

$f(0) = (0 + 2)^2 + 3 = 7$ and $f(-4) = (-4 + 2)^2 + 3 = 7$

$\Rightarrow f(0) = f(-4)$, but $0 \neq -4$

For different domain, with the same y – value.

Therefore f is not – one – to one.

B. For all, $a_1, a_2 \in [-2, \infty)$, $f(a_1) \neq f(a_2)$ and $a_1 \neq a_2$ for all a_1, a_2. Therefore, f is one – to – one.

C. Let $a_1 = 5$ and $a_2 = 3$, thus $f(5) = |5 - 4| = 1$, and $f(3) = |3 - 4| = 1$

$\Rightarrow f(5) = f(3) = 1$, but $5 \neq 3$

$\Rightarrow$ Different domain with the same y – value.

$\therefore f$ is not – one – to one.

D. f is one – to – one for all $x \in [4, \infty)$

30. Find the set A, for which f is one – to – one function.

a) $f: A \longrightarrow \mathbb{R}$, given by $f(x) = \log_5^{2x-3}$

b) $f: A \longrightarrow [-6, \infty)$ given by $f(x) = x^2 + 8x + 10$

c) $f: A \longrightarrow [2, \infty)$ given by $f(x) = |4x + 3| + 2$

Solution:

a) f is one – to – one on the domain:

$$A = \{x: 2x - 3 > 0\} = \left\{x : x > \frac{3}{2}\right\}$$

b) f is one – to – one on the interval.

$$A = \left\{x : x \geq \frac{-b}{2a} \text{ or } x \leq \frac{-b}{2a}\right\}$$

Then, $A = \left\{x : x \geq \frac{-8}{2} \text{ or } x \leq \frac{-8}{2}\right\}$

$\therefore A = (-\infty, -4]$ or $[-4, \infty)$

c) f is one – to – one on the interval

$A = \{x: 4x + 3 \geq 0 \text{ or } 4x + 3 \leq 0\}$

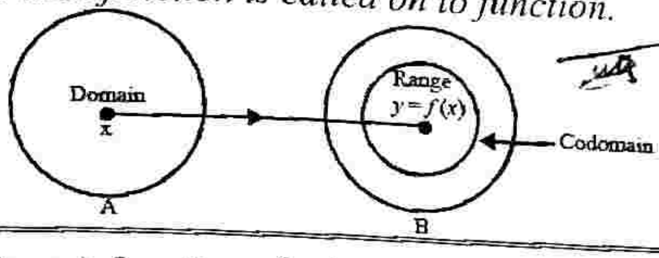
$$\therefore A = \left(-\infty, \frac{-3}{4}\right) \text{or} \left[\frac{-3}{4}, \infty\right)$$

On to (Surjective function)

Function is mapping say A to B denoted by $f: A \longrightarrow B$,
for each $x \in A$ paired with exactly one element $y \in B$, such that $f(x) = y$.

- The element y is called **the image of x** under f.
- The set A is called the **domain of the function f**.
- The set B is called the **codomain of f**.
- All the image of the element in A under f is a subset of B, which is called the range of the function f.

In some cases, range of a function may be equal to the whole codomain, in which case such function is called on to function.

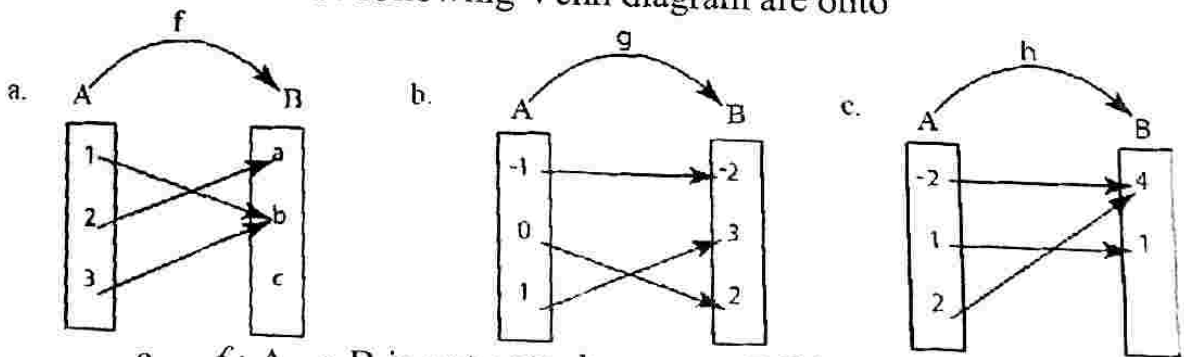


Definition: A function $f: A \longrightarrow B$ is on to if and only if:

- Range if $f = B$, i.e., $f(A) = B$
- For every $b \in B$, the exist $a \in A$ such that $f(a) = b$
- Every element of B has pre-image in A.

Illustrative Example

31. Which of the following Venn diagram are onto



a. $f: A \to B$ is not onto, because $c \in B$ has no pre- image in A, i. e., image of $f \neq B$, and image of $f = \{a, b\}$ but $B = \{a, b, c\}$

b. $g: A \to B$ is onto, because image of $g = B = \{-2, 3, 2\}$

c. $h: A \to B$ is onto, since image of $h = B = \{4, 1\}$

32. Determine which of the following sets are onto function from A to B,

$\{(-1,2), (0,1) (1,2), (2, 0)\}$,
where $A = \{-1, 0, 1, 2\}$ and $B = \{0, 1, 2, 5\}$

b. $g = \{(-2, 4) (-1,1) (1, 3), (2, 4)\}$
where $A = \{-2, -1, 1, 2\}$ and $B = \{1, 3, 4\}$

Solution: f is onto, if im $(f) = B$.

a. f is not onto, because $f(A) \neq B$,
$\Rightarrow$ image of $f = \{0, 1, 2\}$, but $B = \{0, 1, 2, 5\}$
i.e $5 \in B$, has no pre- image in A.

b. g is onto, because $g(A) = B = \{1, 3, 4\}$

Note: If $f: A \to B$ is onto iff the range of $f = B$

Illustrative Example

33. Determine which of the following functions are onto:

a. $f: \mathbb{R} \to \{-1, 0, 1\}$, given by $f(x) = (-1)^x$
b. $f: \mathbb{R}^+ \to (0, \infty)$, given by $f(x) = x^2$, for all $x \in \mathbb{R}$
c. $f: \mathbb{R} \to \mathbb{Q}$, given by $f(x) = 2x$, for all $x \in \mathbb{R}$
d. $f: \mathbb{R} \to \mathbb{R}$, given by $f(x) = |x| + x$, for all $x \in \mathbb{R}$
e. $f: \mathbb{R} \to [3, \infty]$ given by $f(x) = x^2 - 4x + 7$
f. $f: [1, \infty) \to \mathbb{R}$, given by $f(x) = 4 - \sqrt{x-1}$
g. $f: \mathbb{R} \to [1, \infty)$, given by $f(x) = |x - 3| + 1$

Solution: Let $f: A \to B$, then

a. $B = \{-1,0,1\}$ and range of $f = \{-1, 1\}$
$\Rightarrow 0 \in B$, has no pre – image in A, such that $f(x) = 0$
$\Rightarrow$ Range of $f \neq B \Leftrightarrow \{-1, 1\} \neq \{-1, 0, 1\}$
$\therefore f$ is not onto.

b. $B = (0, \infty)$ and range of $f = (0, \infty)$
$\Rightarrow$ Range of $f = B$, $\Rightarrow (0, \infty) = (0, \infty)$ $\therefore f$ is onto

c. $B = \mathbb{Q}$ and range of $f = (0, \infty)$ since $-1 \in Q$, but there does not exist any $x \in \mathbb{R}$ such that $f(x) = 2^x = -1$
$\therefore f$ is not onto
Method II, Range of $f \neq B \Rightarrow \mathbb{Q} \neq (0, \infty)$
$\therefore f$ is not onto.

d. $B = \mathbb{R}$, and range of $f = (0, \infty)$ since $-2 \in \mathbb{R}$, but there does not exist any $x \in \mathbb{R}$, such that $f(x) = |x| + x = -2$
$\therefore f$ is not onto
Method II, Range of $f \neq B$
$\therefore f$ is not on to

e. $B = [3, \infty)$ and range of $f(x) = (x - 2)^2 + 3$ is $[3, \infty)$

$\Rightarrow$ Range of f = B $\Rightarrow [3, \infty) = [3, \infty)$

$\therefore f$ is onto

f. B = $\mathbb{R}$ and range of $f(x) = 4 - \sqrt{x-1}$ is $(-\infty, 4]$ since $5 \in \mathbb{R}$, but there does not exist any $x \in [1, \infty)$. Such that

$$f(x) = 4 - \sqrt{x-1} = 5 \Rightarrow \sqrt{x-1} = -1$$

$\therefore f$ is not onto.

Method II. Range of $f \neq$ B $\Rightarrow \mathbb{R} \neq (-\infty, 4]$

$\therefore f$ is not onto

g. B = $[1, \infty)$ and range of $f(x) = |x-3| + 1$ is $[1, \infty)$.

$\Rightarrow$ Range of f = B = $[1, \infty)$

$\therefore f$ is onto.

34. For each of the following function, find the set B for which $f: \mathbb{R} \to B$ is onto

a.	$f(x) = x^2 + 4$	d.	$f(x) = 2 - 5\,|x|$
b.	$f(x) = \text{s gn}x$	e.	$f(x) = |x| + 3$
c.	$f(x) = [x]$	f.	$f(x) = x^2 + 6x + 10$

$y = 2x -$

Solution:

Remember, $f: \mathbb{R} \to B$ is onto if range of f = B.

a. Range of $f(x) = x^2 + 4$ is $[4, \infty]$ for all $x \in \mathbb{R}$

$\therefore B = [4, \infty)$

b. Range of $f(x) = \text{sgnx}$ is $\{-1, 0, 1\}$ for all $x \in \mathbb{R}$

$\therefore B = \{-1, 0, 1\}$

c. Range of $f(x) = [x]$ is z for all $x \in \mathbb{R}$

$\therefore B = z$ = the set of integer

d. Range of $f(x) = 2 - 5\,|x|$ is $(-\infty, 2)$ for all $x \in \mathbb{R}$

$\therefore B = (-\infty, 2]$

e. Range of $f(x) = |x| + 3$ is $[3, \infty)$, for all $x \in \mathbb{R}$

$\therefore B = [3, \infty)$

f. Range of $f(x) = x^2 + 6x + 5 = (x+3)^2 - 4$ is $[-4, \infty)$

$\therefore B = [-4, \infty)$

One – to – one Correspondence

Two sets are said to be equivalent set if there exists one- to- one correspondence (bijective) between them.

Definition: A function f: A $\to$ B is a one – to – one correspondence (bijective) if and only if f is both onto and one – to one

Illustrative Example

35. Determine and justify which of the following sets are one – to – one, onto or one – to – one correspondence from A to B.

a. $f = \{(x, 1), (y, 1)\ (z, 2), (w, 3)\}$, $A = \{x, y, z, w\}$ and $B = \{1,2,3\}$

b. $g = \{(x, 1), (y, 2), (z, 3)\}$, $A = \{x, y, z\}$ and $B = \{1, 2, 3,4\}$

c. $h = \{(x, 1), (y, 2)\ (z, 3)\}$, $A = \{x, y, z\}$ and $B = \{1, 2, 3\}$

d. $k = \{(x, 1), (y, 2), (z, 3), (w, 3)$, $A = \{x, y, z\}$ and $B = \{1, 2, 3, 4\}$,

Solution:

a. f is onto function but not one- to – one

b. g is one- to- one, but not onto, because range of $g \neq B$

c. h is one – to – one correspondence, because h is both onto and one – to –one

d. k is neither onto nor one – to – one

36. Determine which of the following functions are onto, one – to – one or one – to – one correspondence .

a. $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|\, x$ for all $x \in \mathbb{R}$

b. $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{4x+3}{7}$ for all $x \in \mathbb{R}$

c. $f: [2, \infty) \to [0, \infty)$ defined by $f(x) = \sqrt{x-2}$, for all $x \geq 2$

d. $f: \mathbb{R} \to \mathbb{Q}$, defined by $f(x) = 3^x$, for all $x > \mathbb{R}$

e. $f: (1, \infty) \to (0, \infty)$ defined by $f(x) = \log_5^{x-1}$, for all $x > 1$

f. $f: [3, \infty) \to [1, \infty)$ defined by $f(x) = (x-3)^2 + 1$ for all $x \geq$ 3

g. $f: \mathbb{R} \to \{-1, 0, 1\}$ defined by $f(x) = \operatorname{sgn} x$.

Solution: Let $f: A \longrightarrow B$

a) i) Let $a_1, a_2 \in (-\infty, \infty) \Rightarrow f(a_1) \neq f(a_2)$ then $a_1 \neq a_2$

$\therefore f$ is one – to – one

ii) $B = \mathbb{R}$ and range of $f = \mathbb{R}$

$\Rightarrow$Range of $f = B = \mathbb{R}$, then f is on to

iii) Since f is both one – to – one and on to, then f is one to one correspondence

b) Linear functions are one – to – one and on to.

$\Rightarrow f$ is one to one correspondence

c) i) For all $a_1, a_2 \in [2, \infty)$, then $f(a_1) \neq f(a_2)$

$$\Rightarrow \sqrt{a_1 - 2} \neq \sqrt{a_2 - 2} \text{ then } a_1 \neq a_2$$

$\Rightarrow f$ is one to one

ii) $B = [0, \infty)$ and range of $f = [0, \infty)$ for all $x \geq 2$

$\Rightarrow$Range of $f = B = [0, \infty) \Rightarrow f$ is on to

iii)Since f is both one to one and onto

$\Rightarrow f$ is one to one correspondence

d) i) For all $a_1, a_2 \in Q \Rightarrow f(a_1) \neq f(a_2) \Rightarrow 3^{a_1} \neq 3^{a_2}$ then $a_1 \neq a_2, \Rightarrow f$ is one to one

ii) $B = \mathbb{Q}$ and range of $f = (0, \infty)$ for all $x \in \mathbb{R}$

$\Rightarrow$Range of $f \neq B$, therefore f is not on to

e) i) For all $x > 1$, $f(x) = \log_5^{x-1}$ is one to one

ii) $B = (0, \infty)$ and range of $f = (0, \infty)$

$\Rightarrow$Range of $f = B = (0, \infty)$ thus f is on to

iii) Since f is both one – to – one and on to

Therefore f is one – to – one correspondence.

f) i) For all $a_1, a_2 \in [3, \infty) \Rightarrow f(a_1) \neq f(a_2)$

$\Rightarrow (a_1 - 3)^2 + 1 \neq (a_2 - 3)^2 \Rightarrow a_1 \neq a_2$

Therefore f is one – to – one correspondence

ii) $B = [1, \infty)$ and range of $f = [1, \infty)$, for all $x \geq 3$

$\Rightarrow$Range of $f = B = [1, \infty)$ thus f is on to

iii) Since f is both one to one and on to therefore f is one to one correspondence

g) i) Let $a_1 = 3$ and $a_2 = 4 \Rightarrow f(3) = 1$ and $f(4) = 1$, so that $f(3) = f(4)$ but $3 \neq 4$, therefore f is no one to one

ii) $B = \{-1, 0, 1\}$ and range of $f = \{-1, 0, 1\} \Rightarrow f$ is on to

7. Which one of the following is a one – to – one correspondence function from $A = [0, 1]$ to $B = [2, 3]$?

A. $f(x) = x + 1$ C. $f(x) = 2x + 3$

B. $f(x) = x^3 + 1$ D. $f(x) = x^2 + 2$

Solution: $f : A \longrightarrow B$

A. $B = [2, 3]$ and range of $f = [1, 2]$ for all $x \in [0, 1]$

$\Rightarrow$ Range of $f \neq B$, thus f not on to

B. $B = [2, 3]$ and range of $f = [1, 2]$ for all $x \in [0, 1]$

$\Rightarrow$Range of $f \neq B$ Therefore, f is not on to.

C. $B = [2, 3]$ and range of $f = [3, 5]$ for all $x \in [0, 1]$

$\Rightarrow$Range of $f \neq B$ Therefore, f is not on to.

D. i) $B = [2, 3]$ and range of $= [2, 3]$ for all $x \in [0, 1]$

$\Rightarrow$Range of $f = B = [2, 3]$ Therefore, f is on to

ii) For all $a_1, a_2 \in [0, 1]$ $f(a_1) \neq f(a_2)$

$$\Rightarrow a_1^2 + 2 \neq a_2^2 + 2 \Rightarrow a_1 \neq a_2$$

$\therefore f$ is one – to – one

iii) since f is both one – to – one and on to

$\Rightarrow f$ is one – to – one correspondence

Combination function

(Sum, difference, and quotient) of function

Term	Functional value	Domain
i) $f + g$	$(f+g)(x) = f(x) + g(x)$	Domain of $(f \cap g)$
ii) $f - g$	$(f-g)(x) = f(x) - g(x)$	
iii) $f.g$	$(f.g)(x) = f(x).g(x)$	
iv) $\frac{f}{g}$	$\frac{f}{g}(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$	

Illustrative Example

32. Let $f(x) = \sqrt{9 - x^2}$ and $g(x) = 4x + 5$

Find a) $(f+g)(x)$ b) $(f-g)(x)$ c) $(fg)(x)$

d) $\left(\frac{f}{g}\right)(x)$ e) $\left(\frac{g}{f}\right)(x)$

Solution: i) The domain of $f = \{x: 9 - x^2\} = \{x: -3 \leq x \leq 3\}$

ii) The domain of $g = \{x: x \in R\} = (-\infty, \infty)$

The domain of $f + g$, $f - g$ and $f.g$ are

{the domain of f} $\cap$ {the domain of g} = $\{x: -3 \leq x \leq 3\}$

Therefore,

	Value	Domain
a)	$(f+g)(x) = \sqrt{9-x^2} + 4x + 5$	$\{x: -3 \leq x \leq 3\}$
b)	$(f-g)(x) = \sqrt{9-x^2} - (4x+5)$	$\{x: -3 \leq x \leq 3\}$
c)	$(f\,g)(x) = \sqrt{(9-x^2)}(4x+5)$	$\{x: -3 \leq x \leq 3\}$

d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{\sqrt{9-x^2}}{4x+5}$ $\qquad \left\{x : -3 \le x \le 3 \text{ and } x \ne \dfrac{-5}{4}\right\}$

e) $\left(\dfrac{g}{f}\right)(x) = \dfrac{4x+5}{\sqrt{9-x^2}}$ $\qquad \{x: -3 < x < 3\}$

Composition of Function

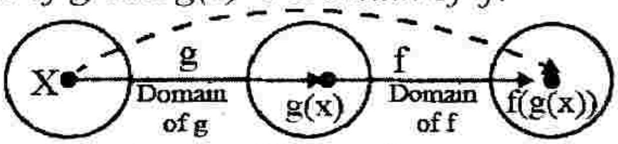
The composition of f and g is denoted by $(fog)(x) = f(g(x))$ provided that:

- $g(x)$ is in the domain of f.
- Domain of fog $\sqsubseteq$ Domain of g.
- Range of fog $\sqsubseteq$ Range of f.

The domain of fog is the set of all real number

x in the domain of g such that g(x) in the domain of f. i.e.

$x \in$ Domain of g and $g(x) \in$ Domain of f.



Illustrative Example

38. Let $f(x) = 3x - 4$ and $g(x) = x^2 + 3x$. **Find**

a)	(fog)(1)	c)	(fog)(x)	e)	(fof)(2)
b)	(gof)(1)	d)	(gof)(x)	f)	(fof)(x)

Solution:

a) $(fog)(1) = f(g(1)) = f(4) = 12 - 4 = 8$

b) $(gof)(1) = g(f(1)) = g(-1) = 1 - 3 = -2$

c) $(fog)(x) = f(g(x)) = f(x^2 + 3x) = 3(x^2 + 3x) - 4$

d) $(gof)(x) = g(f(x)) = g(3x - 4) = (3x - 4)^2 + 3(3x - 4)$

$= 9x^2 - 24x + 16 + 9x - 12 = 9x^2 - 5x + 4$

39. Find f and g such that $(fog)(x) = h(x)$ if:

a) $h(x) = \sqrt{x^2 + 3x}$ $\qquad$ b) 3^{2x^2-1}

Solution: $(fog)(x) = f(g(x)) = h(x)$

a) Let $g(x) = x^2 + 3x$ and $f(x) = \sqrt{x}$

$\Rightarrow f(g(x)) = f(x^2 + 3x) = \sqrt{x^2 + 3x} = h(x)$

b) Let $g(x)=2x^2-1$ and $f(x)=3^x$

$\Rightarrow f(g(x)) = f(2x^2-1) = 3^{2x^2-1} = h(x)$

40. If $f(x) = 3x^2 - 7$ and $g(x) = 4x + k$, for what value of k, so that the graph of $(fog)(x)$ crosses the y-axis at 8?

A. $k = 0$ B. $k = \pm\sqrt{3}$ C. $k = \pm\sqrt{5}$ D. $k = \pm 5$

Solution:

$(fog)(x)$ crosses, y-axis when $x = 0 \Rightarrow f(g(0)) = f(k) = 3k^2 - 7 = 8$

$$k^2 = \frac{15}{3} = 5 \Leftrightarrow k = \pm\sqrt{5}$$ **Answer: C**

41. Let $f(x) = 3x - 1$. Then find $g(x)$:

a) if $f(g(x)) = 9x^2 + 12x + 8$

b) if $g(f(x)) = x^2 - 1$

Solution: $f(x) = 3x - 1$

a) $f(g(x)) = 3g(x) - 1 = 9x^2 + 12x + 8 \Rightarrow 3g(x) = 9x^2 + 12x + 9$

$\therefore g(x) = 3x^2 + 4x + 1$

b) Let $g(x) = ax^2 + bx + c$, then $g(g(x)) = g(3x - 1)$

$\Rightarrow g(3x - 1) = a(3x - 1)^2 + b(3x - 1) + c = x^2 - 1$

$\Rightarrow 9ax^2 - 6ax + a + 3bx - b + c = x^2 - 0x - 1$

Comparing like coefficient

$$\Rightarrow 9ax^2 = x^2 \Rightarrow a = \frac{1}{9} \text{ and } -6ax + 3bx = 0$$

$$\Rightarrow 3b = 6a = 6\left(\frac{1}{9}\right) \Rightarrow b = \frac{2}{9}$$

$$\text{Also, } a - b + c = -1 \Rightarrow \frac{1}{9} - \frac{2}{9} + c = -1 \Rightarrow c = \frac{-8}{9}$$

$$\therefore g(x) = \frac{1}{9}x^2 + \frac{2}{9}x - \frac{8}{9}$$

42. The function $f(x)$ is defined in $[0, 1]$.
What are the domains of the function $f(2x^2)$.

A. $\mathbb{R}$ B. $-\frac{1}{2} \le x \le \frac{1}{2}$ C. $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$ D. $[0, \infty)$

Explanation: Let $u = 2x^2$

$\Rightarrow f(u)$ is defined for $0 \le u \le 1$

$\Rightarrow 0 \le 2x^2 \le 1$, since $2x^2 \ge 0$ for all x

$$\Rightarrow x^2 \le \frac{1}{2} \Rightarrow -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$$ **Answer: C**

43. If $f(x) = \log(4 - x^2)$ and $g(x) = 3x + 7$ then what is the domain of $(fog)(x)$?

Solution: $(fog)(x) = f(g(x))$

$x \in D_g$ and $g(x) \in D_f$

The function $f(x)$ is defined in $4 - x^2 > 0 = \{x: -2 < x < 2\}$

$\Rightarrow f(g(x))$ is defined for: $-2 < g(x) < 2$

$\Rightarrow -2 < 3x + 7 < 2$

$$\Rightarrow -9 < 3x < -5 \Rightarrow -3x < x < -\frac{5}{3}$$

$$\therefore \text{Domain of } (fog)(x) = \{x: -3 < x < -\frac{5}{3}\}$$

44. If $f(x) = \dfrac{1}{x-2}$ and $g(x) = \dfrac{1}{x+4}$

a) What is the domain of $(fog)(x)$?

b) What is the domain of $(gof)(x)$?

Solution:

a) Domain of $g(x) = \{x: x \neq -4\}$ and domain of $f(x) = \{x: x \neq 2\}$

$$\Rightarrow f(g(x)) = \frac{1}{g(x) - 2}, g(x) - 2 \neq 0$$

Domain of $f(g(x)) = \{x \in D_g \text{ and } g(x) \in D_f\}$

$= \{x: x \neq -4 \text{ and } g(x) \neq 2\}$

$$= \{x: x \neq -4 \text{ and } \frac{1}{x+4} \neq 2\}$$

$= \{x: x \neq -4 \text{ and } 2x + 8 \neq 1\}$

$$= \{x: x \neq -4 \text{ and } x \neq \frac{-7}{2}\}$$

$$\therefore \text{Domain of } (fog)(x) \text{ is } \mathbb{R} \setminus \left\{-4, \frac{-7}{2}\right\}$$

b) Exercise left for you. (Ans: $\mathbb{R} \setminus \left\{2, \frac{7}{4}\right\}$.

45. If $f(x) = \ln(x + 3)$ and $g(x) = x^3 - 4$, then what is the domain of $(fog)(x)$?

A. $(-1, \infty)$ B. $(1, \infty)$ C. $[1, \infty]$ D. $[-1, \infty)$

Solution: $(fog)(x) = f(g(x))$

Domain of $f(x) = \{x: x + 3 > 0\} = \{x > -3\}$

$\Rightarrow f(g(x))$ is defined for $-3 < g(x) < \infty \Rightarrow g(x) > -3$

$\therefore$ Domain of $(fog)(x) = \{x: g(x) > -3\} = \{x: x^3 - 4 > -3\}$

$\therefore$ Domain of $(fog)(x) = \{x: x^3 > 1\} = (1, \infty)$ Answer: B

INVERSE FUNCTIONS

The inverse of a one to one function f, denoted f^{-1}, is a function whose ordered pairs are obtained from f by interchanging the x and y – coordinates.

That means:

If $(a, b) \in f$, then $(b, a) \in f^{-1}$ or if $(a, b) \in f^{-1}$, then $(b, a) \in f$.

Note:
1. A function is invertible if and only if one – to – one.
2. If f is strictly monotonic on its entire domain, then it is invetible.

Illustrative Example

46. Determine whether each function is invertible. If it is invertible, then find the inverse function.

a) $f = \{(-2,1), (-1,0), (1,2), (3, 2)\}$ b) $g = \{(3, 1), (0, 2), (2, 3)\}$

Solution:

a) Since (1, 2) and (3, 2) have the same y coordinate, this function is not one – to – one, and f is not invertible.

b) This function is one – to – one, and so it is invertible $g^{-1} = \{(1, 3), (2, 0), (3, 2)\}$.

To find the inverse of $y = f(x)$ (when it exists):
1. Replace $f(x)$ by y
2. Interchange x and y
3. Solve for y then replace y by $f^{-1}(x)$

Note: Domain of f^{-1} = Range of f.
Range of f^{-1} = Domain of f.

HORIZONTAL – LINE TEST FOR INVERS FUNCTION

A function f has an inverse if and only if no horizontal line crosses its graph more than once.

Illustrative Example

47. Determine whether each function is invertible. If it is invertible, then find the inverse function.

a) $f = \{(x, y): y = 3x - 2\}$ b) $g = \{(x, y): y \geq 3x - 2\}$

c) $h = \{(x, y): y = \log_3^{4x}\}$ d) $k = \{(x, y): y = e^{2x}\}$

Solution:

a) $f = \{(x, y): y = 3x - 2\}$ is linear function and linear functions are one-to-one and so f is invertible.

To find the inverse:

Let $y = f(x) = 3x - 2$, interchange x and y

$\Rightarrow x = 3y - 2 \Leftrightarrow 3y = x + 2$

$\Rightarrow y = \dfrac{x+2}{3}$, therefore, $f^{-1}(x) = \dfrac{x+2}{3}$

b) $g = \{(x, y): y \geq 3x - 2\}$. This function is not invertible because a horizontal line crosses the graph at two point $(-2, 1)$ and $(0, 1)$.

c) $h = \{(x, y): y = \log_3^{4x}\}$. This function is invertible because every horizontal line that crosses the graph only once.

To find the inverse: h^{-1}, let $y = h(x) = \log_3^{4x}$ interchange x and y.

$$\Rightarrow x = \log_3^{4y} \Leftrightarrow 4y = 3^x \Rightarrow y = \frac{3^x}{4}$$

$$\therefore h^{-1}(x) = \frac{3^x}{4}$$

d) $k = \{(x, y): y = e^{2x}\}$. This function is one-to-one and so that k has an inverse.

To find k^{-1}, let $y = k(x) = e^{2x}$, $x \longleftrightarrow y$

$$\Rightarrow x = e^{2y} \Leftrightarrow 2y = \ln x \Rightarrow y = \frac{\ln x}{2}$$

$$\therefore k^{-1}(x) = \frac{\ln x}{2}$$

Note:
If the point (a, b) lies on the graph of f, then the point (b, a) must lie on the graph of f^{-1}, and vice versa. This means that the graph of f^{-1} is a reflection of the graph of f in the line $y = x$.

48. If the point (4, -5) is on the graph of $y = f(x)$, which point is on the graph of $y = f^{-1}(x)$.

A. $\left(\frac{1}{4}, -5\right)$ B. $(-4, 5)$ C. $(-5, 4)$ D. $\left(4, -\frac{1}{5}\right)$

Answer: C

Identify Inverse Functions

Functions f and **g** are inverses of each other if and only if
i) $(f \circ g)(x) = x$ for every value of x in the domain of g.
ii) $(g \circ f)(x) = x$ for every value of x in the domain of f.

Illustrative Example

49. Determine whether each pair of functions f and **g** are inverses of each other.

a) $f(x) = 3x + 7$ and $g(x) = \dfrac{x-7}{3}$

b) $f(x) = x^2$ and $g(x) = \sqrt{x}$

c) $f(x) = \sqrt[3]{2x+1}$ and $g(x) = \dfrac{x^3 - 1}{2}$

d) $f(x) = \sqrt{x+2}$ and $g(x) = x^2 - 2$

e) $f(x) = \sqrt{x+2}$ and $g(x) = x^2 - 2$, where $x \geq 0$

Solution: We must verify that $f(g(x)) = x$ and $g(f(x)) = x$.

a) $f(g(x)) = 3g(x) + 7 = 3\left(\dfrac{x-7}{3}\right) + 7 = x$

$$g(f(x)) = \frac{f(x) - 7}{3} = \frac{3x + 7 - 7}{3} = x, \text{ for all } x.$$

$\therefore f$ and g are inverses of each other.

b) If $x \in (-\infty, \infty)$, we can write:

$$f(g(x)) = (g(x))^2 = \left(\sqrt{x}\right)^2 = \sqrt{x^2} = |x|$$

The domain of f is $(-\infty, \infty)$, and $|x| \neq x$ if x is negative.

$\therefore f$ and g are not inverse of each other.

c) $f(g(x) = \sqrt[3]{2g(x)+1} = \sqrt[3]{\frac{2(x^3-1)}{2}+1} = x$

$\therefore f$ and g are inverse of each other.

d) If x ∈ (−∞, ∞), then:

$f(g(x)) = \sqrt{g(x)+2} = \sqrt{(x^2-2)+2} = \sqrt{x^2} = |x|$

Since x ∈ (−∞, ∞), therefore, $|x| \neq x$

$\therefore f$ and g are not inverses of each other.

e) If x ∈ [0, ∞) then:

$f(g(x)) = \sqrt{g(x)+2} = \sqrt{x^2-2+2} = \sqrt{x^2} = |x| = x$

For all $x \geq 0, f(g(x)) = g(f(x)) = x$

$\therefore f$ and g are inverse of each other.

50. Find the inverse of the following function and determine the domain and range of f^{-1}.

a) $f(x) = \dfrac{1}{3x-2}$ b) $f(x) = \dfrac{4x}{x+3}$

c) $f(x) = \dfrac{3x+2}{4x-1}$ d) $f(x) = 2x^3 + 5$

Solution: Replace $y = f(x)$

a) $y = \dfrac{1}{3x-2} \Leftrightarrow 3xy - 2y = 1 \Rightarrow 3xy = 2y + 1 \Rightarrow x = \dfrac{2y+1}{3y} = f^{-1}(y)$

$$\therefore f^{-1}(x) = \frac{2x+1}{3x}$$

Domain of $f^{-1} = \mathbb{R} \setminus \{0\}$ and

Range of $f^{-1} = \mathbb{R} \setminus \left\{\frac{2}{3}\right\}$

b) $y = \dfrac{3x+2}{4x-1}$, interchange x to y

$$x = \frac{3y+2}{4y-1} \Leftrightarrow 4xy - x = 3y + 2$$

$$\Rightarrow 4xy - 3y = x + 2 \Leftrightarrow y\,(4x-3) = x + 2$$

$$\Rightarrow y = \frac{x+2}{4x-3} \Rightarrow f^{-1}(x) = \frac{x+2}{4x-3}$$

$\therefore$Domain of $f^{-1} = \mathbb{R} \setminus \left\{\frac{3}{4}\right\}$ and

Range of $f^{-1} = \mathbb{R} \setminus \left\{\frac{1}{4}\right\}$

c) $y = \dfrac{4x}{x+3} \longleftarrow$ interchange x and y

$$x = \frac{4y}{y+3} \Leftrightarrow xy + 3x = 4y$$

$$\Rightarrow xy - 4y = -3x \Leftrightarrow y(x-4) = -3x$$

$$y = \frac{-3x}{x-4} \Rightarrow f^{-1}(x) = \frac{3x}{4-x}$$

Domain of $f^{-1} = \mathbb{R}\setminus\{4\}$ and
Range of $f^{-1} = \mathbb{R}\setminus\{-3\}$

d) $y = 2x^3 + 5 \longleftarrow$ interchange x to y

$\Rightarrow x = 2y^3 + 5 \Rightarrow 2y^3 = x - 5$

$\Rightarrow y = \sqrt[3]{\dfrac{x-5}{2}}$, therefore $f^{-1}(x) = \sqrt[3]{\dfrac{x-5}{2}}$

Domain of $f^{-1} = \mathbb{R}$ and Range of $f^{-1} = \mathbb{R}$

51. Find the inverse of each of the following function.

a) $f(x) = \ell n(2x + 1)$ c) $f(x) = \log_2^{\sqrt[3]{4x-1}}$

b) $f(x) = 1 + \dfrac{1}{2}\ell n(x-3)$ d) $f(x) = \ell n\left(\dfrac{x}{x-1} + 2\right)$

Solution:

a) $y = \ell n(2x + 1) \longleftarrow$ interchange x to y

$$\Rightarrow x = \ell n(2y+1) = \log_e^{(2y+1)}$$

$$\Rightarrow 2y + 1 = e^x \Rightarrow y = \frac{e^x - 1}{2}$$

$$\therefore f^{-1}(x) = \frac{1}{2}(e^x - 1)$$

b) $y = 1 + \dfrac{1}{2}\ell n(x-3) \longleftarrow$ interchange x to y

$$\Rightarrow x = 1 + \frac{1}{2}\ell n(y-3) \Leftrightarrow x - 1 = \frac{1}{2}\ell n(y-3)$$

$$\Rightarrow 2x - 2 = \ln(y-3) = \log_e^{(y-3)} \Rightarrow e^{2x-2} = y - 3$$

$$\therefore y = e^{2x-2} + 3 \Rightarrow f^{-1}(x) = e^{2x-2} + 3$$

c) $y = \log_2^{\sqrt[3]{4y-1}}$, interchange x to y

$$\Rightarrow x = \log_2^{\sqrt[3]{4y-1}} \Rightarrow \sqrt[3]{4y-1} = 2^x$$

$$\Rightarrow 4y - 1 = (2^x)^3 \Leftrightarrow 4y = 2^{3x} + 1$$

$$\therefore y = \frac{2^{3x}+1}{4} \Rightarrow f^{-1}(x) = \frac{1}{4}(2^{3x}+1)$$

d) $y = \ell n\left(\frac{x}{x-1} + 2\right) \longleftarrow$ interchange x to y

$$x = \ell n\left(\frac{y}{y-1} + 2\right) = \ln\left(\frac{3y-2}{y-1}\right) \Rightarrow e^x = \frac{3y-2}{y-1} \Leftrightarrow e^x y - e^x = 3y - 2$$

$$\Rightarrow e^x y - 3y = e^x - 2 \Leftrightarrow y(e^x - 3) = e^x - 2$$

$$\therefore y = \frac{e^x - 2}{e^x - 3} \Rightarrow f^{-1}(x) = \frac{e^x - 2}{e^x - 3}$$

52. Find the inverse of the following function:

a) $f(x) = \frac{1}{e^x - 1}$ b) $f(x) = \frac{1}{e^x + 1}$

c) $f(x) = \sqrt[3]{1 + e^{-x}}$ d) $f(x) = \frac{1}{e^{-x} + 1}$

Solution:

a) $y = \frac{1}{e^x - 1}$ then $x = \frac{1}{e^y - 1}$

$$\Rightarrow xe^y - x = 1 \Rightarrow xe^y = x + 1 \Rightarrow e^y = \frac{x+1}{x} \Rightarrow y = \ell n\left(\frac{x+1}{x}\right)$$

$$\therefore f^{-1}(x) = \ln\left(\frac{x+1}{x}\right) = \ell n(x+1) - \ln x$$

b) Exercise left for you. **Ans:** $\ell n\left(\frac{1-x}{x}\right)$.(UEE)

c) Exercise left for you. **Ans:** $\ell n\left(\frac{1}{x^3 - 1}\right)$.(UEE)

Let $y = f(x) = |x + 2| + 3 \Rightarrow y = x + 2 + 3 = x + 5$, then
$x = y + 5 \Rightarrow y = x - 5$
Therefore $f^{-1}(x) = x - 5$

57. Which of the following functions are invertible?
A. $f: \mathbb{R} \longrightarrow$ be defined by $f(x) = [x]$
B. $f: \mathbb{R} \longrightarrow \{-1, 0, 1\}$ be defined by $f(x) = \text{sgn}(x)$
C. $f: \mathbb{R} \longrightarrow [-9, \infty)$ be defined by $f(x) = x^2 + 2x - 8$
D. $f: [-2, 0) \longrightarrow [-4, 0]$ be defined by $f(x) = x^2 - 4$

Solution:
When $x \in [-2, 0], f(x) = x^2 - 4$ is one to one so that invertible.

Answer: D

58. What is the set of value of B, for which $f: \mathbb{R} \longrightarrow B$ defined by $f(x) = x^2 + 6x + 7$ is on to?
A. $[-4, \infty)$ B. $[-2, \infty)$ C. $[-3, \infty)$ D. $(-\infty, -1]$

Solution:
$f: \mathbb{R} \longrightarrow B$ is said to be on to if and only if range of $f = B$
Hence $f(x) = x^2 + 6x + 7 = x^2 + 6x + 9 + 7 - 9$
$\Rightarrow f(x) = (x + 3)^2 - 2$
$\therefore$ The range of $f = [-2, \infty)$

Answer: B

59. For what values of A, $f: A \longrightarrow [5, \infty)$ be defined by $f(x) = x^2 + 2x + 6$ is $1 - 1$.
A. $\mathbb{R}$ B. $[-1, \infty)$ C. $(-\infty, -1]$ D. B and C

Solution:
$f: A \longrightarrow [5, \infty)$ is $1 - 1$ correspondence if and only if one - to - one and on to.
A. $f: \mathbb{R} \longrightarrow [5, \infty)$ given by $f(x) = x^2 + 2x + 6$ is on to because image of $f(x)$ $x^2 + 2x + 6$ is $[5, \infty)$. But not one to one since $(-2, 6)$ and $(0, 6)$.
B. $f: [-1, \infty) \longrightarrow [5, \infty)$, given by $f(x) = x^2 + 2x + 6$ is onto and one - to - one
C. $f: (-\infty, -1] \longrightarrow [5, \infty)$, given by $f(x) = x^2 + 2x + 6$ is onto and one - to - one.

60. Let $F(x) = f(2g(x))$ where $f(x) = x^4 + x^3 + 1$ for $0 \le x \le 2$ and $g(x) = f^{-1}(x)$. Find $F(3)$

Solution: We have $f(x) = x^4 + x^3 + 1$, for $0 \le x \le 2$.
$f(0) = 0^4 + 0^3 + 1 = 1 \Rightarrow f^{-1}(1) = 0$ g(1)

$f(1) = 1^4 + 1^3 + 1 = 3 \Rightarrow f^{-1}(3) = 1 = g(3)$
$f(2) = 2^4 + 2^3 + 1 = 25$
$\therefore F(3) = f(2g(3)) = f(2) = 25$

61. Let $f(x) = 2x^3 + 5x + 8$ and $f^{-1}(2x) = -1$. Find the value of x.
Solution: If $f^{-1}(a) = b \Rightarrow f(b) = a$

Then, $f^{-1}(2x) = -1 \Rightarrow f(-1) = 2x \Rightarrow -2 + -5 + 8 = 2x \Leftrightarrow x = \frac{1}{2}$

62. Let $f^{-1}(3x - 1) = -2$ and $f(-2) = 11$. Find x
Solution: $f^{-1}(3x - 1) = -2 \Rightarrow f(-2) = 3x - 1$

$\Rightarrow 3x - 1 = 11 \Rightarrow 3x = 12$ $\qquad \therefore x = \frac{12}{3} = 4$

63. If $y = f(x)$ and $f(x) = \frac{1}{1-x}$ and $z = f(y)$, then $f(z)$ is equal to:

A. $-x$ B. x C. $x-1$ D. 2

Solution:

Now since $f(x) = \frac{1}{1-x}$, we have,

$$f(z) = \frac{1}{1-z} = \frac{1}{1-f(y)} = \frac{1}{1-\frac{1}{1-y}}$$

$$= \frac{1-y}{1-y-1} = \frac{y-1}{y} = \frac{\frac{1}{1-x}-1}{\frac{1}{1-x}} = \frac{1-(1-x)}{1-x}.\left(\frac{1-x}{1}\right)$$

$$= \frac{1-1+x}{1} = x$$

Answer: B

64. Let $f(x) = \frac{x}{1+x^2}$, what is the range of the function;

A. $\left[-\frac{1}{2}, \frac{1}{2}\right]$ B. $[-2, 2]$ C. $(-1, 1]$ D. $\mathbb{R}$

Solution: Let $f(x)$=y then $y = \frac{x}{1+x^2} \Rightarrow x = y + yx^2 \Leftrightarrow yx^2 - x + y = 0$

We observe that the expression is a quadratic in x. So that

$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, Where b = 1, a = y and c = y

$$\therefore x = \frac{-1 \pm \sqrt{1-4y^2}}{2y}$$

Since x is real numbers, so $y \neq 0$ and $1 - 4y^2 \geq 0$.

$\therefore$ Range of $f = \{y: 1 - 4y^2 \geq 0\} = \{y: (1 + 2y)(1 - 2y) \geq 0\}$

$$= \{y: \left(y+\frac{1}{2}\right)\left(y-\frac{1}{2}\right) \leq 0\}$$

Hence, range $\left\{y : -\frac{1}{2} \leq y \leq \frac{1}{2}\right\}$ **Answer: A**

65. Let $f(x) = (x-3)^2 + 2$ for $x \leq 3$. Find $f^{-1}(x)$

Solution: Let $y = f(x)$, then $y = (x-3)^2 + 2$, $x \leq 3$

$\Rightarrow x = (y-3)^2 + 2$ for $y \leq 3 \longleftarrow$ interchange x and y

$\Rightarrow (y-3)^2 = x - 2$, for $y \leq 3 \longleftarrow$ solve for y

$\Rightarrow y = 3 \pm \sqrt{x-2}$, for $y \leq 3 \longleftarrow y = f^{-1}(x)$

$\therefore f^{-1}(x) = 3 - \sqrt{x-2}$

$\therefore$Domain of $f^{-1} = \{x: x \geq 2\}$ and range of $f^{-1} = \{y: y \leq 3\}$

66. Which one of the following is a one – to – one correspondence function from $A = \{-1, 0, 1\}$ to $B = \{1, 2, 3\}$?

A. $f(x) = 2x + 1$
B. $f(x) = x^3 + 2$
C. $f(x) = |x| + 2$
D. $f(x) = x^2 + 2$

Solution:

A function f: $A \longrightarrow B$ is one – to – one correspondence if f is both one-to-one and on to.

A. For $2 \in B$, there does not exist any $x \in \{-1, 0, 1\}$ such that

$$f(x) = 2 \Rightarrow 2x + 1 = 2 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2} \notin \{-1, 0, 1\}$$

B. $f(x) = x^3 + 2$ is on to because for every $y \in B$, there is $x \in A$ such that $f(x) = y$. That means image of f is B. Also f is one-to – one. Therefore, f is a one – to – one correspondence.

C. $f(x) = |x| + 2$ is not on to because for $1 \in B$, there does not exist any $x \in \{-1, 0, 1\}$ such that $f(x) = 1$ for which $|x|+2 = 1$ $\Rightarrow |x| = 1 - 2 = -1$

D. $f(x) = x^2 + 2$ is not on to, because for $1 \in B$, there does not exist any $x \in \{-1, 0, 1\}$ such that $f(x) = 1$ for which $x^2+2 = 1$

Answer: B

67. If $f(x) = x + \sqrt{1+x^2}$, then what will be $f^{-1}(x)$?

Solution: Let $y = f(x) \Rightarrow y = x + \sqrt{1+x^2}$

$\Rightarrow y - x = \sqrt{1+x^2} \Leftrightarrow (y - x^2) = 1 + x^2$

$\Rightarrow y^2 - 2xy + x^2 = 1 + x^2 \Leftrightarrow y^2 - 2xy = 1 \Rightarrow 2xy = y^2 - 1$

$\Rightarrow x = \dfrac{y^2-1}{2y} = f^{-1}(y)$ $\qquad \therefore f^{-1}(x) = \dfrac{1}{2}\dfrac{x^2-1}{x} = \dfrac{1}{2}\left(x - \dfrac{1}{x}\right)$

Review Exercise on Unit – 1

1. Determine which of the following sets are functions from A to B;
   a. $f = \{(a, b), (a,c), (b,d), (c,a)$; $A = \{a, b, c\}$, $B = \{a, b, c, d\}\}$
   b. $f = \{(1, 3), (2, 2), (3, 4)\}$; $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$
   c. $f = \{(a, b) \in \mathbb{R} \times \mathbb{R} / b = \sqrt{a}$, $A = \mathbb{R}$, $B = \mathbb{R}\}$
   d. $f = \{(1,2), (2,3), (3,4)$, where , $A=\{1,2,3,4\}$, $B = \{1,2,3,4\}$
2. Determine whether the given function is even, odd or neither.
   a) $f(x) = x^3 + 5x$ $\qquad$ c. $f(x) = \dfrac{1+x^2}{x^3}$
   b) $f(x) = \dfrac{x}{1+x^2}$ $\qquad$ d. $f(x) = x^2 - 4|x| + 3$
3. Determine which of the following are **onto, one – to – one or one – to – one correspondence** function.
   a. $f: \mathbb{R} \to \mathbb{R}$, defined by $f(x) = 4x + 1$
   b. $f: \mathbb{R} \to [-3, \infty)$ defined by $f(x) = 2(x - 1)^2 - 3$
   c. $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = f(x) = |x| - x$
   d. $f: [1, \infty) \to [5, \infty)$ defined by $f(x) = 3(x - 1)^2 + 5$
   e. $f: [0, \infty) \to [-1, \infty)$ defined by $f(x) = \sqrt{x} - 1$
   f. $f: \mathbb{R} \to \mathbb{R}^+$, defined by $f(x) = 4^x$, $x \in \mathbb{R}$
   g. $f: \mathbb{Z} \to \{-1, 1\}$ defined by $f(x) = (-1)^{x+}$, $x \in \mathbb{Z}$
   h. $f: \mathbb{R} \to [-4, \infty)$ defined by $f(x) = |x| - 4$
4. Determine which of the following are **onto, one – to – one or one – to – one correspondence** function from $A = [0, 1]$ to $B = [1, 2]$
   a. $f(x) = x$ $\qquad$ b) $f(x) = 2x^3 + 1$ $\qquad$ c. $f(x) = x^3 + 1$
5. Find the domain of the composite function fog.
   a. $f(x) = \sqrt{x-4}$ and $g(x) = 2x + 3$

d) Exercise left for you. **Ans:** $\ell nx - \ell n(1-x)$

53. Find the inverse of $f(x) = \log_5^{3x-4}$.

Solution: Let $y = f(x) \Rightarrow y = \log_5^{3x-4} \longleftarrow$ Interchange x to y

$$\Rightarrow x = \log_5^{3y-4} \Leftrightarrow 5^x = 3y - 4 \Rightarrow 3y = 5^x + 4 \Leftrightarrow y = \frac{1}{3}(5^x + 4)$$

$$\therefore f^{-1}(x) = \frac{1}{3}(5^x + 4)$$

54. Find the inverse of $f(x) = 3 + \sqrt{x}$.

Solution: Let $y = f(x)$

$$y = 3 + \sqrt{x} \text{ then } x = 3 + \sqrt{y} \Rightarrow \sqrt{y} = x - 3 \Rightarrow y = (x-3)^2$$

$$\therefore f^{-1}(x) = (x-3)^2 \text{ for } x \geq 3$$

Domain of $f^{-1} = \{x: x \geq 3\}$ and range of $f^{-1} = \{y: y \geq 0\}$

55. Find the inverse of $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

Solution: Let $y = f(x)$

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ then } x = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$\Rightarrow xe^y + xe^{-y} = e^y - e^{-y} \Leftrightarrow xe^{-y} + e^{-y} = e^y - xe^y$$

$$\Rightarrow e^{-y}(x+1) = e^y(1-x) \Leftrightarrow \frac{e^y}{e^{-y}} = \frac{x+1}{1-x}$$

$$\Rightarrow e^{2y} = \frac{x+1}{1-x} \Rightarrow y = \frac{1}{2}\ell n\left(\frac{x+1}{1-x}\right)$$

$$\therefore f^{-1}(x) = \frac{1}{2}\ell n\left(\frac{x+1}{1-x}\right) = \frac{1}{2}[\ell n(x+1) - \ell n(1-x)]$$

Inverse of Even power and Quadratic function

As such quadratic function and even power has no inverse because horizontal line test shows this function is not one to one, and hence cannot have inverse.

However, by restricting their domains to some "suitable" intervals, one is attained and the function becomes invertible.

b. $f(x) = x^2 + 4$ and $g(x) = \sqrt{x-2}$

c. $f(x) = \ln(x-3)$ and $g(x) = x^3 - 5$

6. If $f: A \rightarrow B$ and $g: B \rightarrow C$ are functions, then which of the following statement are **True or false** about the composition function.

a. Domain of $(g\circ f) \sqsubseteq$ Domain of f.

b. Domain of $(g\circ f) \sqsubseteq$ Domain of g.

c. Range of $(g\circ f) \sqsubseteq$ Range of f.

d. Range of $(g\circ f) \sqsubseteq$ Range of g.

7. Find the inverse of the following function.

a. $f(x) = 3 + \dfrac{1}{2}\ell n(x-1)$ c. $f(x) = 2(x-3)^2 + 1$, for $x \leq 3$

8. If $f(x) = 2x^2 + 5$ and $g(x) = 3x + k$, find k so that the graph of **fog** crosses the y-axis at 45.

9. If $f(x+2) = 2x^2 - 3x - 1$, then find $f(x+1)$.

10. If the domain of $f = \{x \in \mathbb{R}: 0 \leq x \leq 8\}$ and the domain of $g = \{x \in \mathbb{R}: -3 \leq x \leq 4\}$ then what is the domain of a) $f + g$, b) $f - g$.

11. Find the inverse of each function:

a) $f(x) = (x+5)^2$ for $x \geq -5$

b) $f(x) = x^2 + 3$ for $x \geq 0$

c) $f(x) = \dfrac{x+1}{3x-4}$ e) $f(x) = \sqrt{x+2}$

d) $f(x) = \sqrt[3]{x-4}$ f) $f(x) = x^3 - 2$

Unit Two

Rational function

Definition: A rational function of x is a function which can be defined by $f(x)=\dfrac{N(x)}{D(x)}$, where N(x) and D(x) are polynomials and $D(x) \neq 0$

Domain of Rational function

Domain of rational function is the set of all real numbers x for which $D(x) \neq 0$.

Illustrative example

1. In each part, find the domain and the range

a) $f(x)=\dfrac{x+2}{x-3}$ b) $f(x)=\dfrac{3}{x^2+1}$

c) $f(x)=\dfrac{x^2+2x}{x^2-1}$ d) $f(x)=\dfrac{1}{x^2+3x+2}$

Solution: a) domain = $\{x: x-3 \neq 0\} = \{x: x \neq 3\}$ or $R\backslash\{3\}$

range = $\{y \in R: y \neq 1\}$ or $R\backslash\{1\}$

b) domain = R, range = $\{y: 0 < y \leq 3\}$

c) domain = $\{x: x^2-1 \neq 0\} = \{x: x \neq 1, -1\}$

range = R

d) domain = $\{x: x^2+3x+2 \neq 0\} = \{x: x \neq -1, -2$

range = $R\backslash\{0\}$

2. Let $f(x)=\dfrac{x+1}{x-2}$ then, Find domain and range of

a) f(x) b) f(x – 1) c) $f(x^2-2)$

Solution:

a) domain = $\{x: x-2 \neq 0\} = \{x: x \neq 2\}$ Range = $\{y: y \neq 1\}$

b) we have $f(x)=\dfrac{x+1}{x-2}$ then

$f(x-1)=\dfrac{x-1+1}{x-1-2}=\dfrac{x}{x-3}$. Therefore, domain of

$\dfrac{x}{x-3}$ is $R\setminus\{x=3\}$ range $=\{y: y\neq 1\}$

c) $f(x)=\dfrac{x+1}{x-2}$ then $f(x^2-2)=\dfrac{x^2-2+1}{x^2-2-2}=\dfrac{x^2-1}{x^2-4}$

then domain of $f(x^2-2)=\{x: x^2-4\neq 0\}=\{x: x\neq 2,-2\}$

Range $=\{R\setminus y\neq 1\}$

Simplification of rational expression

Definition:

A rational expression is said to be reduced to lowest terms (or simplified). If the numerator and denominator have no common factor except -1 and 1.

Illustrative example

3. Simplify the following

a) $\dfrac{x}{x^2-1}+\dfrac{2}{x^2+2x+1}$

c) $\dfrac{y-1}{1-\dfrac{1}{1-\dfrac{1}{y}}}$

b) $\dfrac{\dfrac{x}{x^2-1}-\dfrac{2}{x^2-1}}{x-\dfrac{2}{x-1}}$

d) $\dfrac{\dfrac{1}{x-3}+\dfrac{3}{x-1}}{\dfrac{2}{x^2-4x+3}+\dfrac{4}{x-3}}$

Solution:

a) $\dfrac{x}{x^2-1}+\dfrac{2}{x^2+2x+1}=\dfrac{(x+1)x}{(x-1)(x+1)}+\dfrac{2(x-1)}{(x+1)(x-1)}$

$$= \frac{x^2 + 3x - 2}{(x+1)^2 (x-1)}, x \neq -1, 1$$

b) $$\frac{\frac{x}{x^2-1} - \frac{2}{x^2-1}}{x - \frac{2}{x-1}} = \frac{\frac{x}{x^2-1} - \frac{2}{x^2-1}}{\frac{x(x-1)}{x-1} - \frac{2}{x-1}} = \frac{\frac{x-2}{x^2-1}}{\frac{x^2-x-2}{x-1}} = \frac{x-2}{x^2-1} \cdot \frac{x-1}{x^2-x}$$

$$= \frac{x-2}{(x+1)(x-1)} \cdot \frac{x-1}{(x-2)(x+1)} = \frac{1}{(x+1)^2}, x \neq -$$

c) $$\frac{y-1}{1 - \frac{1}{1-\frac{1}{y}}} = \frac{y-1}{1 - \frac{1}{\frac{y-1}{y}}} = \frac{y-1}{1 - \frac{y}{y-1}} = \frac{y-1}{\frac{y-1-y}{y-1}} = \frac{y-1}{\frac{1}{y-1}} = -(y-1)^2$$

d) $$\frac{\frac{1}{x-3} + \frac{3}{x-1}}{\frac{2}{x^2-4x+3} + \frac{4}{x-3}} = \frac{\frac{1(x-1)+3(x-3)}{(x-3)(x-1)}}{\frac{2}{(x-1)(x-3)} + \frac{4}{(x-3)} \cdot \frac{(x-1)}{(x-1)}}$$

$$= \frac{\frac{x-1+3x-9}{(x-3)(x-1)}}{\frac{2+4(x-1)}{(x-3)(x-1)}} = \frac{4x-10}{(x-3)(x-1)} \cdot \frac{(x-3)(x-1)}{(4x-2)} = \frac{2x-5}{2x-1}, x \neq \frac{1}{2}$$

Rational Equations

The easiest way to solve most equations involving rational expressions is to **multiply all the terms in the equations** by the least common denominator.

Illustrative Example

1. Solve the equation

a) $\dfrac{2}{3x+1} = \dfrac{1}{x} - \dfrac{6x}{3x+1}$ c) $\dfrac{2}{m-3} - \dfrac{3}{3+m} = \dfrac{12}{m^2-9}$

b) $\dfrac{3}{x-2} + \dfrac{21}{x^2-4} = \dfrac{14}{x+2}$ d) $\dfrac{2}{x-4} - \dfrac{3}{x+1} = \dfrac{6}{x-1}$

Solution: To solve $\frac{2}{3x+1}=\frac{1}{x}-\frac{6x}{3x+1}$

Multiply both sides by x(3x + 1) the resulting equation ..

$\Rightarrow 2x = 3x + 1 - 6x^2$

$\Rightarrow 6x^2 + 2x - 3x - 1 = 0$ .... Standard form

$\Rightarrow (3x + 1)(2x - 1) = 0$

$\Rightarrow 3x + 1 = 0$ or $2x - 1 = 0 \Rightarrow x = -\frac{1}{3}$ or $x = \frac{1}{2}$

Using $-\frac{1}{3}$ in the original equation causes the denominator ...

1 to equal zero. So it is not a ***Solution.***

$\therefore$ The ***Solution*** set is {½}

b) $\frac{3}{x-3}+\frac{21}{x^2-4}=\frac{14}{x+2}$

Multiply both sides by (x – 2)(x + 2)

$\Rightarrow 3(x + 2) + 21 = 14(x - 2) \Rightarrow 3x + 6 + 21 = 14x - 28$

$\Rightarrow 27 + 28 = 14x - 3x \Rightarrow 55 = 11x \Rightarrow x = 5$

c) $\frac{2}{m-3}-\frac{3}{3+m}=\frac{12}{m^2-9}$

Multiply both sides by (m – 3)(m + 3)

$\Rightarrow 2(m + 3) - 3(m - 3) = 12$

$\Rightarrow 2m + 6 - 3m + 9 = 12 \Rightarrow -m = 12 - 15 = -3$

$\Rightarrow m = 3 \longleftarrow$ not in the domain

$\therefore$ m = 3 is not ***Solution*** set $\quad\therefore$ Solution set is $\varnothing$

d) $\frac{2}{x-4}-\frac{3}{x+1}=\frac{6}{x-1}$

Multiply both sides by (x + 1)(x – 1)(x – 4)

$\Rightarrow 2(x + 1)(x - 1) - 3(x - 1)(x - 4) = 6(x + 1)(x - 4)$

$\Rightarrow 2(x^2 - 1) - 3(x^2 - 5x + 4) = 6(x^2 - 3x - 4)$

$\Rightarrow 2x^2 - 2 - 3x^2 + 15x - 12 = 6x^2 - 18x - 24$

$\Rightarrow -x^2 + 15x - 14 = 6x^2 - 18x - 24$

$\Rightarrow 7x^2 - 33x - 10 = 0 \Rightarrow (7x + 2)(x - 5) = 0$

$\Rightarrow 7x + 2 = 0$ or $x - 5 = 0$

$\therefore x = \frac{-2}{7}$ or $x = 5$ $\therefore S \cdot S = \left\{\frac{-2}{7}, 5\right\}$

Decomposition of rational expressions into partial fractions

Decomposition of rational expressions in to sums of simpler expressions is called **Partial fractions decomposition**

We may verify that $\frac{2}{x^2-1} = \frac{1}{x-1} + \frac{-1}{x+1}$

Rule of Finding partial fraction

Decomposition of $\frac{P(x)}{Q(x)}$

1. **Linear factor Rule (Degree of P(x) < degree of Q(x).** For each factor of the form (ax +b), the partial fraction decomposition contains the following sum of k, partial fraction.

$$\frac{1}{(ax+b)^k} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3} \ldots\ldots + \frac{D}{(ax+b)^k}$$

Illustrative Example

1. Find the partial fraction decomposition of the following:

a) $\frac{1}{x^2-9}$ c) $\frac{x^2+10x-36}{x(x-3)^2}$

b) $\frac{4x^2+13x-9}{x^3+2x^2-3x}$ d) $\frac{2x-3}{(x-1)^2}$

Solution: Here

$$\frac{1}{x^2-9} = \frac{1}{(x-3(x+3)} = \frac{A}{x-3} + \frac{B}{x+3} = \frac{A(x+3)+B(x-3)}{(x-3)(x+3)}$$

Now, 1 = A(x+3) + B (x-3)

To find for A, Let x =3 ⇒ 1= A (3+3) + B (3-3)

$\Rightarrow 1 = 6A \therefore A = \frac{1}{6}$

To find for B, Let x = -3⇒=A(-3+3) + B(-3-3)⇒1=-6B

$$\therefore B = -\frac{1}{6}$$

$$\therefore \frac{1}{x^2-9} = \frac{1}{6(x-3)} + \frac{-1}{6(x+3)}$$

b) $\frac{4x^2+13x-9}{x^3+2x^2-3x}$, Since the degree of numerator less than degree of denominator so, long division is not required.

Thus, the partial fraction decomposition has the form:

$$\frac{4x^2+13x-9}{x^3+2x^2-3x} = \frac{4x^2+13x-9}{x(x+3)(x-1)} = \frac{A}{x} + \frac{B}{x+3} + \frac{c}{x-1}$$

$\Rightarrow 4x^2 + 13x\text{-}9 = A\ (x+3)(x-1) + Bx(x-1) + Cx(x+3)$

To find for A, Let X=0

$\Rightarrow 4(2)^2 + 13(0) - 9 = A\ (0+3)\ (0-1) + 0 + 0 \Rightarrow -9 = -3A \qquad \therefore A = 3$

To find for B, Let x =-3

$\Rightarrow 4(-3)^2 + 13(-3)\text{-}9 = A\ (-3 + 3) + B(-3)(-3-1) + C\ (-3+3)$

$\Rightarrow 36\text{-}39\text{-}9 = 0 + 12B + 0 \Rightarrow -12 = 12B \qquad \therefore B = -1$

To find for C, Let x =1

$\Rightarrow 4(1)^2 + 13(1)\ \text{-}9 = A\ (1+3)(1-1) + B(1)(1-1) + C(1)(1+3)$

$\Rightarrow 8 = 4C\ \therefore C = 2$

$$\Rightarrow \frac{4x^2+13x-9}{x(x+3)(x-1)} = \frac{3}{x} + \frac{-1}{x+3} + \frac{2}{x-1}$$

C) $\frac{x^2+10x-36}{x(x-3)^2} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$

$\Rightarrow x^2 + 10x\text{-}36 = A\ (x-3)^2 + Bx\ (x-3) + Cx$

To find A, Let x = 0

$\Rightarrow 0^2 + (10)(0)\ \text{-}36 = A\ (0-3)^2 + 0 + 0 \Rightarrow -36 = 9A \qquad \therefore A = -4$

To find C, Let x=3

$\Rightarrow 3^2 + 10(3)\ \text{-}36 = A\ (3-3)^2 + B\ (3)(3-3) + 3C \Rightarrow 3 = 3C \therefore C = 1$

To find B, Let x = 1

$\Rightarrow 1^2 + 10(1)\ \text{-}36 = A\ (1-3)^2 + B(1)(1-3) + C\ (1)$

$\Rightarrow -25 = 4A\ \text{-}2B + C$

But A + -4 and C = 1

Hence, $\Rightarrow -25 + 4(-4) + -2B + 1$

$\Rightarrow -25 + 15 = -2B \Rightarrow -10 = -2B \quad \therefore B = 5$

Thus, $\dfrac{x^2+10x-36}{x(x-3)^2} = \dfrac{-4}{x} + \dfrac{5}{x-3} + \dfrac{1}{(x-3)^2}$

d) $\dfrac{2x-3}{(x-1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} \Rightarrow 2x-3 = A(x-1)^2 + B$

$\Rightarrow 2x-3 = Ax^2 - 2Ax + A + B \ldots \Rightarrow -2A = 2 \therefore A = -1$

And $A + B = -3; -1 + B = -3 \Rightarrow B = -3 + 1 = -2$

Explanation: Squaring $(x-1)^2 = x^2 - 2x + 1$

$\therefore A(x-1)^2 = Ax^2 - 2Ax + A$

Thus, $\dfrac{2x-3}{(x-1)^2} = \dfrac{-1}{x-1} + \dfrac{-2}{(x-1)^2}$

Rule2, when the degree of numerator is greater than or equal to the degree of denominator, we first perform along division, then decompose into partial fraction.

Illustrative example

2. Find partial decomposition.

a) $\dfrac{x^2-x-2}{x^2-9}$ b) $\dfrac{x^3}{x^2-3x+2}$

Solution: Here

a) The degree of numerator is equal to the degree of denominator, hence by long division

$$\frac{x^2-x-2}{x^2-9} = 1 + \frac{7-x}{x^2-9} \text{ and } \frac{7-x}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3}$$

$\Rightarrow 7-x = A(x+3) + B(x-3)$

To find A, Let $x = 3 \Rightarrow 7-3 = A(3+3) + B(3-3)$

$\Rightarrow 4 = 6A \quad \therefore A = \dfrac{4}{6} = \dfrac{2}{3}$

To find B, Let $x = -3$

$\Rightarrow 7-(-3) = A(-3+3) + B(-3-3) \Rightarrow 10 = -6B$

$\therefore B = \dfrac{-10}{6} = \dfrac{-5}{3}$

$$\Rightarrow \frac{x^2-x-2}{x^2-9} = 1 + \frac{2}{3(x-3)} + \frac{-5}{3(x+3)}$$

b) $\frac{x^3}{x^2-3x+2}$, The degree of numerator is greater than the degree of denominator. We perform long division.

$$\frac{x^3}{x^2-3x+2} = x+3+\frac{7x-6}{x^2-3x+2} = x+3+\frac{7x-6}{(x-2)(x-1)}$$

Again: $\frac{7x-6}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$

To find B, Let x = 1 ⇒7-6 = B (1-2) ⇒ 1 = -B ∴ B = -1

To find A, Let x = 2 ⇒ 7(2)-6 = A (2-1) + B (2-2) ⇒ 8 = A

$$\therefore \frac{x^3}{x^2-3x+2} = x+3+\frac{8}{x-2}+\frac{-1}{x-1}$$

Rule 3 for prime quadratic factor: $ax^2 + bx + C$, (with b^2-4ac)

A prime quadratic factor of the form ax^2+bx+c (with $b^2 - 4ac < 0$) in the denominator require partial fraction $\frac{Bx+c}{ax^2+bx+c}$

Illustrative example

3. Find partial fraction decomposition.

a) $\frac{8x^3+13x}{(x^2+2)^2}$ b) $\frac{2x^3-4x-8}{(x^2-x)(x^2+4)}$

Solution:

a) $\frac{8x^3+13x}{(x^2+2)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2}$

$8x^3 + 13x = (Ax + B)(x^2 + 2) + Cx + D$

Expanding the basic equation and collecting like terms on opposite sides of the equation.

$\Rightarrow 8x^3+13x = Ax^3 + 2Ax + Bx^2 + 2B + Cx + D$

$\Rightarrow 8x^3 + 13x = Ax^3 + Bx^2 + (2A + C)x + 2B + D$

∴ A = 8, B = 0, 2A+C = 13 and 2B + D = 0

In 2A + C = 13

∴ 2(8) + C = 13, ∴C = 13-16 = -3 ∴ D = 0

Thus, $\dfrac{8x^3+13x}{(x^2+2)^2} = \dfrac{8x}{x^2+2} + \dfrac{-3x}{(x^2+2)^2}$

Graph of rational function

A graph of a rational function is pictorical representation of rational function.

When sketching the graph of rational function f, it is important to investigating

- The intercepts, x and y
- The asymptote
- The behavior of function near the vertical asymptote (functional value of f(x) when x is close to a zero of the denominator)
- The behavior of function as x goes to very large number
- The party i.e. f is odd or even or neither.

Asymptote

There are three:

- Vertical asymptote
- Horizontal asymptote
- Oblique asymptote

Definition: The line x = a is a vertical asymptote for the graph of a function f if f(x) $\to\infty$ or f(x) $\to -\infty$ as x approaches **a** form either the left or the right.

Notation	Terminology
$\to a^+$	x approaches a from the right
$\to a^-$	x approaches a from the left
(x) $\to\infty$	f(x) increase with out bound
(x) $\to -\infty$	f(x) decrease with out bound

Illustrative example

8. Find all vertical asymptotes of each rational function

a) $f(x)=\dfrac{2x-1}{x^2+3x}$ d) $f(x)=\dfrac{4}{x-3}$

b) $f(x)=\dfrac{3x^2+5}{x^2-4}$ e) $f(x)=\dfrac{x^2-x-6}{x^2-9}$

c) $f(x)=\dfrac{x-2}{x^2+2x-8}$

Solution: To find vertical asymptote simplify the common factor then set denominator zero

a) $x^2+3x=0 \Rightarrow x(x+3)=0 \Rightarrow x=0$ and $x=-3$

$\therefore$ V.A, $x=0$ and $x=-3$

b) $x^2-4=0 \Rightarrow x=2$ and $x=-2$

$\therefore$ V.A, $x=-2$ and $x=2$

c) $\dfrac{x-2}{x^2+2x-8}=\dfrac{(x-2)}{(x-2)(x+4)}=\dfrac{1}{x+4}$ simplifying common factor

$\therefore$ V.A is only $x=-4$, but not $x=2$

$\therefore$ the function has hole at $x=2$

d) $x-3=0 \Rightarrow x=3 \longleftarrow$ V.A

e) $\dfrac{(x-3)(x+2)}{(x-3)(x+3)}=\dfrac{x+2}{x+3}$, simplifying common factor

$\therefore$ V.A is only $x=-3$ but not $x=3$

$\therefore$ The graph of function has hole at $x=3$.

Definition of Horizontal asymptote

The line $y=b$ is a horizontal asymptote of the graph of function f if $f(x) \to b$ as $x \to \infty$ or as $x \to -\infty$

Rule of Horizontal asymptote

Let $f(x)=\dfrac{N(x)}{D(x)}=\dfrac{ax^n+......+a_0}{bx^m+......+b_0}$ be rational function

1. If $n<m$, then x-axis ($y=0$) is the horizontal asymptote

2. If n = m, then the line $y = \frac{a}{b}$ is the horizontal asymptote
3. If n > m, then the graph has no horizontal asymptote. It has oblique asymptote if n = m + 1
4.

Illustrative Example

9. Find the horizontal asymptote for the graph of f if it exist

a) $f(x) = \frac{2x+3}{x^2+1}$ c) $f(x) = \frac{2x-4}{5x+6}$

b) $f(x) = \frac{4x^2+1}{3x^2+4}$ d) $f(x) = \frac{x^3+1}{x-2}$

Solution:

a) $f(x) = \frac{2x+3}{x^2+1}$, here, the degree of the numerator 2x + 3 is less than the degree of the denominator $x^2 + 1$

$\therefore$ y = 0 ← H.A

b) $f(x) = \frac{4x^2+1}{3x^2+4}$, since the degree of numerator equal to the degree of denominator

$\therefore$ The line $y = \frac{4}{3}$ ← H.A

c) since n = m $\therefore$ The line $y = \frac{2}{5}$ ← H.A

d) since n > m $\therefore$ It has no horizontal asymptote

The zeros of rational function

Definition: Let $f(x) = \frac{N(x)}{D(x)}$ be rational function an element **a** in the domain of f is called a zero of f, if and only if N(a)=0 and D(a) ≠ 0

Illustrative Example

Find the zeros of each of the following rational function

a) $f(x) = \dfrac{x^2-4}{x^2-9}$ c) $f(x) = \dfrac{x+4}{x-3}$

b) $f(x) = \dfrac{x^2+5x+6}{x^2+6x+8}$ d) $f(x) = \dfrac{x+3}{x^2-9}$

Solution:

a) $\dfrac{x^2-4}{x^2-9} \Rightarrow x^2-4=0 \Rightarrow (x-2)(x+2)=0$

$\therefore$ The zeros are x = 2 and x = -2

b) First we factorize both numerator and denominator

$f(x) = \dfrac{(x+2)(x+3)}{(x+2)(x+4)} = \dfrac{x+3}{x+4}$, domain = R\{-2, -4}

$\therefore$ The zeros of f are only x = -3, but not x = -2

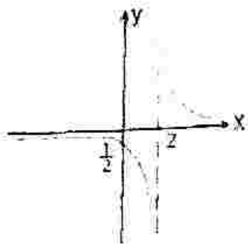
c) $f(x) = 0 \Leftrightarrow \dfrac{x+4}{x+3} = 0 \Rightarrow x = -4 \leftarrow$ the zeros of f

d) $\dfrac{x+3}{(x-3)(x+3)} = \dfrac{1}{x-3}$

$\therefore$ No zeros, that makes f(x) = 0.

Sketching the graph of rational function

Sketch the graph of the following function



$f(x)=\frac{1}{x-2}$

A) Let $f(x) = \dfrac{1}{x-2}$, then

Find a) x and y-intercept

b) domain and range

c) asymptote

d) how behaves f as $x \to 2^+$ and $x \to 2^-$

e) how behaves f(x) as $x \to \infty$ and $x \to -\infty$

f) for what values of x

i) $f(x) > 0$ ii) $f(x) < 0$ iii) $f(x) > 3$

Solution: a) No, x-intercept

y –intercept $\left(0, -\frac{1}{2}\right)$

b) Domain = R{2}

Range = R\{0}

c) V.A, x = 2, H.A, y = 0

d) as $x \to 2^+$, $f(x) \to \infty \leftarrow$ f(x) increase with out bound

as $x \to 2^-$, $f(x) \to -\infty \leftarrow$ f(x) decrease with out bound

e) as $x \to \infty$, $f(x) \to 0^+$

as $x \to -\infty$, $f(x) \to 0^-$

f) i) $f(x) < 0$ when $x < 2$

ii) $f(x) > 0$ when $x > 2$

iii) $f(x) \geq 3$ when $2 < x \leq \frac{7}{3}$

- f is symmetric with respect to (2, 0)
- parity, neither even nor odd.

B) Let $f(x) = \frac{x+1}{x+3}$, Investigate the behavior. Sketch the graph

Solution: i) The x-intercept, (-1, 0)

The y-intercept $\left(0, \frac{1}{3}\right)$

ii) Domain = R\{-3}

Range = R\{1}

iii) V.A, x = -3

H.A, y = 1

iv) as $x \to -3^+$, $f(x) \to -\infty \longleftarrow$ f(x) decrease with out bound

as $x \to -3^-$, $f(x) \to \infty \longleftarrow$ f(x) increase with out bound

v) as $x \to \infty$, $f(x) \to 1 \longleftarrow$ f(x) approaches to 1 as x goes

as $x \to -\infty$, $f(x) \to 1$ very large number.

Explanation

as $x \to -3^+$

$$\frac{-3^+ + 1}{-3^+ + 3} = \frac{-2}{0^+} \to -\infty$$

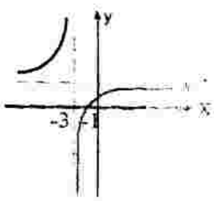
as $x \to -3^-$

$$\frac{-3^- + 1}{-3^- + 3} = \frac{-2}{0^-} \to \infty$$

From the graph, we can observe

vi) • $f(x) < 1 \Rightarrow \frac{x+1}{x+3} < 1$, when $x > -3$

• $f(x) > 1 \Rightarrow \frac{x+1}{x+3} > 1$, when $x < -3$



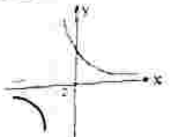
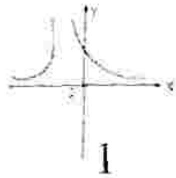
- $f(x) < 0 \Rightarrow \dfrac{x+1}{x+3} \leq 0$ when $-3 < x \leq -1$

$x \leq -1$ or $x \leq 3$

- $f(x) \geq 0 \Rightarrow \dfrac{x+1}{x+3} \geq 0$ when $x \in (-\infty, -3] \cup [-1, \infty)$

$x \geq -1$ $x \geq -3$

c) Quick sketch of the following table

i)	ii)	iii)
 $f(x) = \dfrac{1}{x+2}$ as $x \to -2^{+}$, $f(x) \to \infty$ as $x \to -2^{-}$, $f(x) \to -\infty$ as $x \to \infty$, $f(x) \to 0$	 $f(x) = \dfrac{1}{(x+2)^{2n}}$ as $x \to -2$, $f(x) \to \infty$ as $x \to \infty$, $f(x) \to 0$	 $f(x) = \dfrac{x}{x+2}$ as $x \to -2^{-}$, $f(x) \to -\infty$ as $x \to -2^{-}$, $f(x) \to \infty$ as $x \to \infty$, $f(x) \to 1$

D) Let $f(x) = \dfrac{x^2 - 3x - 4}{x^2 - 6x + 8}$

i) Investigate all the behavior

ii) sketch the graph iii) for what value of x

a) $f(x) \geq 0$ b) $f(x) < 0$ c) $f(x) \geq 2$ d) $f(x) < 1$

Solution: First we factorize the numerator and denominator and simplify $f(x) = \dfrac{x^2 - 3x - 4}{x^2 - 6x + 8} = \dfrac{(x-4)(x+1)}{(x-4)(x-2)} = \dfrac{x+1}{x-2}, x \neq 4, 2$

Hence,

i) • the x-intercept by setting $f(x) = 0$, we get $(-1, 0)$

- the y-intercept by setting, $x = 0$, we get

$$\left(0, -\frac{1}{2}\right) \rightarrow f(0) = -\frac{1}{2}$$

- the vertical asymptote, $x = 2 \leftarrow$ the zero of denominator
- the graph has hole at $x = 4$
- the horizontal asymptote, $y = 1$
- the behavior near the V.A, $x = 2$

as $x \to -2^{+}, f(x) \to \infty$

as $x \to -2^{-}, f(x) \to -\infty$

- what happens $f(x)$ as x goes to very large number as $x \to \infty, f(x) \to 1^+$

as $x \to -\infty, f(x) \to 1^-$

iii) a) $f(x) \geq 0$ when $x \in (\infty, -1] \cup (2, 4) \cup (4, \infty)$

b) $f(x) < 0$ when $x \in (-1, 2)$

c) $f(x) \geq 2$ when $x \in (2, 4) \cup (4, 5]$ because 4 is not in domain

d) $f(x) < 1$ when $x \in (-\infty, 2)$

iv) f is neither even nor odd

E) Let $f(x) = \dfrac{x-1}{x^2 - x - 6}$

i) Discuss all the behavior.

ii) sketch the graph iii) For what value of x

a) $\dfrac{x-1}{x^2-x-6} > 0$ b) $\dfrac{x-1}{x^2-x-6} < 0$

Solution: we have $\dfrac{x-1}{x^2-x-6} = \dfrac{x-1}{(x-3)(x+2)}$

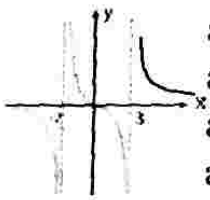
i) * The x-int, $(1, 0) \longrightarrow f(x) = 0$

* The y-int $\left(0, \dfrac{1}{6}\right) \longleftarrow f(0) = \dfrac{1}{6}$

* V.A, $x = -2$ and $x = 3 \longleftarrow$ the zero of denominator

* H.A, $y = 0$

The behavior near the vertical asymptote



as $x \to -2^+$, $f(x) \to \infty$, the graph goes up

as $x \to -2^-$, $f(x) \to -\infty$, the graph goes down

as $x \to 3^+$, $f(x) \to \infty$

as $x \to 3^-$, $f(x) \to -\infty$

- The behavior as x goes to very large number

as $x \to \infty$, $f(x) \to 0^+$

as $x \to -\infty$, $f(x) \to 0^-$

From the graph we observe that

iii) $\dfrac{x-1}{x^2-x-6} > 0$, when $x \in (-2, 1) \cup (3, \infty) \leftarrow$ The graph lies above the x-axis

b) $\frac{x-1}{x^2-x-6}<0$, when x $\in(-\infty, -2) \cup (1, 3)$ ← The graph lies below the x-axis

iv) parity, neither even nor odd

F) Let $f(x)=\frac{x^2+1}{x^2+x-2}$

i) Discuss all the behavior

ii) sketch the graph iii) for what values of x

a) $\frac{x^2+1}{x^2+x-2}<0$ b) $\frac{x^2+1}{x^2+x-2}\geq 0$

Solution: Here, $\frac{x^2+1}{x^2+x-2}=\frac{x^2+1}{(x+2)(x-1)}$

i) No, x-intercept

- y-intercept $\left(0,-\frac{1}{2}\right)$
- vertical asymptote, x = -2 and x = 1 ← the zero of denominator
- Horizontal asymptote, y = 1

The behavior near the V.A

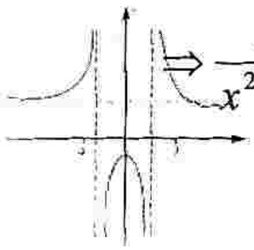
- as $x \to -2^+, f(x) \to -\infty$, the graph goes down
- as $x \to -2^-, f(x) \to \infty$, the graph goes up
- as $x \to 1^+, f(x) \to \infty$ • as $x \to 1^-, f(x) \to -\infty$

The behavior as x goes to very large number

- $x \to \infty, f(x) \to 1$ ← the graph approaches to 1
- $x \to -\infty, f(x) \to 1$

The graph crosses the horizontal asymptote at y = 1

This can be determined by solving $f(x) = 1$



$$\Rightarrow \frac{x^2+1}{x^2+x-2}=1 \Rightarrow x^2+1=x^2+x-2$$

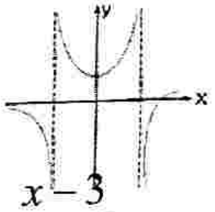
$$\Rightarrow x-2=1$$

$$\therefore x=3$$

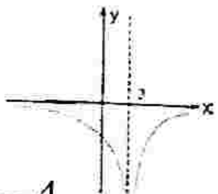
iii) $\dfrac{x^2+1}{x^2+x-2} < 0$, when $x \in (-2, 1) \leftarrow$ The interval on which the graph lies below the x-axis

b) $\dfrac{x^2+1}{x^2+x-2} \geq 1$ When $x \in (-\infty, 2) \cup (1,3]$

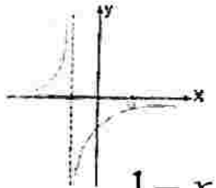
G. Quick sketch of the graph



$\dfrac{x-3}{x^2-1}$



$\dfrac{-4}{(x-1)^2}$



$f(x) = \dfrac{1-x}{x^2-1}$

Oblique Asymptote

Let $f(x) = \dfrac{a_1x^n + \ldots + a_0}{bx^m + \ldots + b_0}$ f has oblique asymptote

If and only if $n = m + 1$ and we can find by long division.

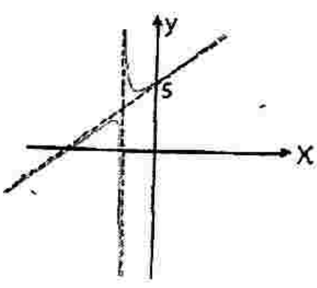
Illustrative Example

Find the oblique asymptote and sketch the graph of f.

a) $f(x) = \dfrac{x^2+6x+8}{x+1}$ b) $f(x) = \dfrac{x^2+2x-3}{(x-2)}$

Solution: a) First write f(x) in two different ways. Factoring the numerator

$$f(x) = \frac{x^2+6x+8}{x+1} = \frac{(x+2)(x+4)}{x+1}$$



* The x-intercepts: (-2, 0), (-4, 0) ← The graph meets the x-axis
* The y-intercepts: (0, 8) ← $f(0) = 8$
* V.A, x = -1 ⟶ the zeros of denominator
* $f(x) = \dfrac{x^2+6x+8}{x+1} = x + 5 + \dfrac{3}{x+1}$

$\therefore$ y = x + 5 ⟵ oblique asymptote

The behavior near V.A, x = -1

$$\text{As } x \to -1^{+} = \frac{(-1+2)(-1+4)}{-1^{+}+1} = \frac{3}{0^{+}} = \infty$$

$\therefore$ as $x \to -1^{+}, f(x) \to \infty$

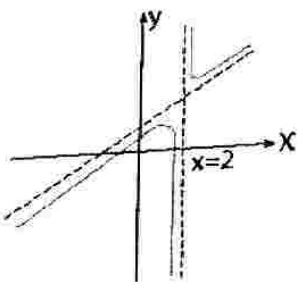
$x \to -1^{-}, f(x) \to -\infty$

Note: From the graph we can observe that

* $\frac{x^2+6x+8}{x+1} \geq 0$ when $x \in (-1, \infty) \cup [-4, -2)$
* $\frac{x^2+6x+8}{x+1} \leq 0$ when $x \in [-2, -1) \cup (-\infty, -4]$

Parity, neither even nor odd

b) $f(x) = \frac{x^2+2x-3}{x-2} = \frac{(x-1)(x+3)}{x-2}$



* x-intercepts: (1, 0) and (-3, 0)
* y-intercepts: $\left(0, \frac{3}{2}\right) \to f(0) = \frac{3}{2}$
* V.A, $x = 2 \longleftarrow$ the zeros of denominator
* $\frac{x^2+2x-3}{x-2} = x + 4 + \frac{5}{x-2}$

$\therefore y = x + 4 \longleftarrow$ oblique asymptote

The behavior near V.A, x = 2

as $x \to 2^{+}, f(x) \to \infty$

$x \to 2^{-}, f(x) \to -\infty$

Note: From the graph we can observe that

* $\frac{x^2+2x-3}{x-2} \leq 0$ when $x \in (-\infty, -3] \cup [1, 2)$
* $\frac{x^2+2x-3}{x-2} \geq 0$ when $x \in [-3, 1) \cup [2, \infty)$

Parity, neither even nor odd

Solved Problem

1. Perform the indicated operation and simplify

a) $\dfrac{x^2-6x+9}{x^2-1}\cdot\dfrac{2x-2}{x-3}$ c) $\dfrac{\frac{x}{y}-\frac{y}{x}}{\frac{1}{x}-\frac{1}{y}}$

b) $\dfrac{x+2}{2x-3}\div\dfrac{x^2-4}{2x^2-3x}$

Solution:

a) $\dfrac{x^2-6+9}{x^2-1}\cdot\dfrac{2x-2}{x-3}=\dfrac{(x-3)^2\cdot 2(x-1)}{(x-1)(x+1)(x-3)}=\dfrac{2(x-3)}{x+1}$,

if x ≠3, x≠-1

b) $\dfrac{x+2}{2x-3}\div\dfrac{x^2-4}{2x^2-3x}=$

$\dfrac{x+2}{2x-3}\cdot\dfrac{2x^2-3x}{x^2-4}=\dfrac{x+2}{2x-3}\cdot\dfrac{(x)(2x-3)}{(x+2)(x-2)}$

$=\dfrac{x}{x-2},\ x\neq -2,\ \dfrac{3}{2}$

c) $\dfrac{\frac{x}{y}-\frac{y}{x}}{\frac{1}{x}-\frac{1}{y}}=\dfrac{\frac{x^2-y^2}{xy}}{\frac{y-x}{xy}}=\dfrac{(x-y)(x+y)}{xy}\cdot\dfrac{xy}{y-x}=-(x+y)$

2. The value (S) of x where the graph of $y=\dfrac{x^2-4x+3}{x^3+1}$ crosses its horizontal asymptote, is (are) …. EHEECE

A. x = -2 C. x = 0

B. x = 3 and x = 1 D. $x=\sqrt{2}\ and\ x=1$

Solution: The degree of numerator is less than the degree of denominator by one

∴ H.A, y = 0

$$\Rightarrow y = \frac{x^2 - 4x + 3}{x^3 + 1} = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-3)(x-1) = 0$$

$$\Rightarrow x = 3 \text{ and } x = 1$$

Answer: B

3. Which one of the following represent the *Solution* set for the equation $\frac{1}{x-2} = \frac{3}{x+2} - \frac{6x}{x^2-4}$ ……. EHEECE

A. $\left\{-\frac{1}{2}, \frac{1}{2}\right\}$ B. {-4, 2} C. {-2} D. ∅

Solution: Restriction, $x \neq 2$, $x \neq -2$

LCM of $(x-2)(x+2)(x^2-4) = (x-2)(x+2)$

Multiply both side by LCM$(x-2)(x+2)$, we get

$\Rightarrow x + 2 = 3(x-2) - 6x$

$\Rightarrow x + 2 = 3x - 6 - 6x \Rightarrow 4x = -6 - 2 \Rightarrow 4x = -8$

$x = -2 \longleftarrow$ not in the domain

$\therefore$ S.S $= \varnothing$,

Answer: D

4. For what values of x, the following inequality true

a) $\frac{-1}{x^2+1} < 0$ b) $\frac{-1}{x^2+1} > 0$

c) $\frac{4x^2}{(x-3)(x+2)} < 0$ d) $\frac{4x^2}{(x-3)(x+2)} > 0$

e) $\frac{x-3}{x^2-9} > 0$ f) $\frac{x-3}{x^2-9} < 0$

g) $\frac{-x^2-3}{x(x+2)} < 0$

Solution: a) $\frac{-1}{x^2+1} < 0$, for all real number the graph lies below the x-axis

$$\therefore \frac{-1}{x^2+1} < 0 \text{ on } (-\infty, \infty)$$

b) $\frac{-1}{x^2+1} > 0$, is ∅.

No *Solution*, for all x, the graph lies below the x-axis

c) $\dfrac{4x^2}{(x-3)(x+2)} < 0 \Leftrightarrow (x - 3)(x+2) < 0 \Rightarrow \{-2 < x < 3\}$.

because $4x^2 > 0$ for all x.

d) $\dfrac{4x^2}{(x-3)(x+2)} > 0 \Leftrightarrow (x - 3)(x+2) > 0 \Rightarrow x< -2$ or $x >3$

because $4x^2 > 0$ for all x.

e) $\dfrac{x-3}{x^2-9} > 0 \;\Leftrightarrow \dfrac{1}{x+3} > 0$ on $(-3, 3) \cup (3, \infty)$

f) $\dfrac{x-3}{x^2-9} < 0 \Leftrightarrow \dfrac{1}{x+3} < 0$ on $(-\infty, -3)$

g) $\dfrac{-x^2-3}{x(x+2)} < 0 \;\Leftrightarrow \dfrac{x^2+3}{x(x+2)} > 0 \Leftrightarrow x(x + 2) > 0$ because $x^2 + 3 > 0$ for all x

$\therefore \dfrac{-x^2-3}{x(x+2)} < 0$ on $(-\infty, -2) \cup (0\ \infty)$

$\therefore \dfrac{-x^2-3}{x(x+2)} > 0$ on $(-2, 0)$

5. Simplified form of $\dfrac{4-(a-2)^2}{a^4+2a^2+1} \div \dfrac{1-\dfrac{1}{a^2+1}}{a}$ ….. UEE 2004/12

A. $\dfrac{4-a}{a}$ C. $-\dfrac{1}{a^2(a^2+1)}$

B. $\dfrac{a}{a^2+1}$ D. $\dfrac{4-a}{a^2+1}$ **Answer: D**

6. If $f(x) = \dfrac{x-1}{(x-2)^2(x+1)}$ which of the following is true about f …UEE 2004/12.

A. It has an oblique asymptote

B. The graph does not meet its asymptote

C. As $\to -1^+$, $f(x) \to -\infty$

D. As $\to 2^-$, $f(x) \to -\infty$

Solution: • It has no oblique asymptote

• The graph meets the horizontal asymptote at x = 1

• As $x \to -1^+$, $f(x) = \dfrac{-1^+ - 1}{(-1^+ - 2)^2(-1^+ + 1)} = \dfrac{-}{(+)(0^+)} \to -\infty$

• As $x \to 2^-$, $f(x) \to \infty$ ∴ Answer C

7. If $P(x) = 4x^3$ and $q(x) = 3x^2 - 3x$, then what is the solution set of $\dfrac{P(x)}{4xq(x)} + \dfrac{1}{x} = \dfrac{1}{q(x)}$?

A. {1, 4} B. {1} C. {4} D. {-1, 4}

Explanation:

Multiplying both side by q (x), we get

$\Rightarrow \dfrac{P(x)}{4x} + \dfrac{q(x)}{x} = 1 \Leftrightarrow \dfrac{4x^3}{4x} + \dfrac{3x^2 - 3x}{x} = 1$

$\Rightarrow x^2 + 3x - 3 = 1 \Leftrightarrow x^2 + 3x - 4 = 0$

$\Rightarrow (x - 1)(x + 4) = 0 \Rightarrow x = 1$ and $x = -4$

But, $q(1) = 0$, so reject $x = 1$

∴ The solution set is {4}.

Answer: C

8. What is the solution set of $\dfrac{2 - \frac{1}{x}}{1 - \frac{1}{x^2}} = 4x^2 - \dfrac{2x}{2 + \frac{1}{x}}$?

A. $\left\{1, -\dfrac{1}{4}\right\}$ B. $\left\{\dfrac{1}{4}\right\}$ C. $\left\{-1, \dfrac{1}{4}\right\}$ D. $\left\{-\dfrac{1}{4}\right\}$

Solution:

Simplify both side;

$$\frac{2 - \frac{1}{x}}{4 - \frac{1}{x^2}} = \frac{2 - \frac{1}{x}}{\left(2 - \frac{1}{x}\right)\left(2 + \frac{1}{x}\right)} = \frac{1}{2 + \frac{1}{x}} = \frac{x}{2x + 1} \text{......(*)}$$

and $4x^2 - \dfrac{2x}{2 + \frac{1}{x}} = 4x^2 - \dfrac{2x^2}{2x + 1} = x^2\left(4 - \dfrac{2}{2x + 1}\right)$..........(**)

From (*) and (**) we have $\dfrac{x}{2x + 1} = x^2\left(4 - \dfrac{2}{2x + 1}\right)$

$$\Rightarrow \frac{1}{2x+1} = 4x - \frac{2x}{2x+1} \Leftrightarrow 4x = \frac{1}{2x+1} + \frac{2x}{2x+1}$$

$$\Rightarrow 4x = \frac{2x+1}{2x+1} = 1 \Rightarrow 4x = 1 \Leftrightarrow x = \frac{1}{4}$$

Answer: B

9. What are the respective values of A, B and C, so that

$\frac{x-1}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$? .... UEE 2003/11

A. -1, 1, 1 B. -1, 1, 2 C. 1, -2, 0 D. 3, 1, 2

Solution: $\frac{x-1}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$?

$\Leftrightarrow x - 1 = A(x^2 + 1) + (Bx + C)x$

$\Rightarrow x - 1 = Ax^2 + A + Bx^2 + Cx \Rightarrow x - 1 = (A + B)x^2 + Cx + A$

$\Rightarrow 0x^2 + x - 1 = (A + B)x^2 + Cx + A \Rightarrow A + B = 0$

$\Rightarrow Cx = x \Rightarrow C = 1 \Rightarrow A = -1 \quad \therefore B = 1$

Hence $\frac{x-1}{x^3+x} = \frac{-1}{x} + \frac{x+1}{x^2+1}$

Answer: A

10. What is the solution set of $1 - \frac{5}{x^2+4} = \frac{x-1}{x^2-x} - \frac{1}{x}$ ....UEE 2003/11

A. $\{1, -1\}$ B. $\{2\}$ C. $\{-1\}$ D. $\{1, 2, -1\}$

Solution: Multiply both sides by $x(x - 1)(x^2 + 4)$

$\Rightarrow x(x - 1)(x^2 + 4) - 5x(x - 1) = (x-1)(x^2 + 4) - (x - 1)(x^2 + 4)$

$\Rightarrow (x^2 - x)(x^2 + 4) - 5x^2 + 5x = 0$

$\Rightarrow x^2(x^2 + 4) - x(x^2 + 4) - 5x^2 + 5x = 0$

$\Rightarrow x^4 + 4x^2 - x^3 - 4x - 5x^2 + 5x = 0$

$\Rightarrow x^4 - x^3 - x^2 - 4x + 5x = 0$

$\Rightarrow x^4 - x^3 - x^2 + x = 0 \Rightarrow x(x^3 - x^2 - x + 1) = 0$

$\Rightarrow x(x + 1)(x - 1)^2 = 0$

$\therefore x = 0, x = -1, \text{and } x = 1$

But, $x = 0$, and $x = 1$ are not in the domain

$\therefore x = 0$ and $x = 1$ are

$$= \frac{x^2+2x-x^2-8}{(x+2)(x-4)} = \frac{2(x-4)}{(x+2)(x-4)} = \frac{2}{x+2}$$ not in

the solution set

$\therefore$ Solution set is $\{-1\}$ only

Answer: C

11. Which of the following is not true about the graph of $g(x) = \frac{x^2-1}{x^2+1}$ ....UEE 2003/11

A. The range of g is $(-\infty, 1)$
B. g is even
C. $y = 1$ is a horizontal asymptote
D. As $x \to -\infty$, $g(x) \to 1$

Solution: To find the range, solve for x

$$\Rightarrow y = \frac{x^2-1}{x^2+1} \Leftrightarrow yx^2 + y = x^2 - 1 \Rightarrow yx^2 - x^2 = -1 - y$$

$$\Rightarrow x^2(y-1) = -1 - y \qquad \therefore x = \pm\sqrt{\frac{-1-y}{y-1}}$$

$$\therefore \text{range} = \left\{y : \frac{-1-y}{y-1} > 0\right\} = \{y : -1 \le y < 1\}$$

B. $g(x) = g(-x)$ $\therefore$ g is even
C. $y = 1$ is a horizontal asymptote
D. As $x \to -\infty$, $g(x) \to 1$ $\therefore$ Answer A

Review – Exercise on Unit 2

I. Find a. the vertical
b. the horizontal if any
c. oblique asymptotes if any
d. the intercepts if any of each rational function

1. $R(x) = \frac{3x+3}{2x+4}$ 2. $R(x) = \frac{3x+1}{x^2-4}$ 3. $R(x) = \frac{x^2}{x^2+x-6}$

4. $R(x) = \frac{x^3+2}{x^2-4}$ 5. $R(x) = \frac{2x^3-3}{x^2-9}$ 6. $R(x) = \frac{x^2+x-6}{x^2+7x+12}$

II. State the domain and solve each pf the rational equation

7. $\frac{x^2}{x+3} + \frac{3x}{x^2+5x+6} = \frac{2x}{x+3}$ 8. $\frac{2}{x-3} + \frac{3}{x+3} = \frac{5x-3}{x^2-9}$

9. $\frac{5x}{x-2} = 7 + \frac{10}{x-2}$ 10. $\frac{7}{x-4} - \frac{3}{x+4} = \frac{13-5x}{x^2-16}$

III. Simplify

11. $x-\dfrac{x}{1-\dfrac{x}{1-x}}$

12. $\left(\dfrac{y}{y+2}-\dfrac{y}{y-2}\right)\div\left(\dfrac{y}{y+2}+\dfrac{y}{y-2}\right)$

ANSWER TO REVIEW EXERCISE

1. a. V.A: $x = -2$ b. H.A: $y=\frac{3}{2}$; not intersected

d. x – intercept: - 1; no y – intercept.

2. a. vertical asymptote; $x = 2, x = -2$

b. horizontal asymptote: y = 0, intersected at $\left(0,-\frac{1}{3}\right)$

d. x – intercept: $-\frac{1}{3}$, y - intercept, $-\frac{1}{4}$

3. a. vertical asymptote: $x = 2, x = -3$

b. horizontal asymptote: y = 1 intersected at (6,1)

c. intercept: (0, 0)

4. a. vertical asymptote: $x = 2, x = -2$ b. No. H. Asy

b. oblique asymptote: $y = x$, intersected at $\left(-\frac{1}{2},-\frac{1}{2}\right)$

d. x – intercept, $-\sqrt[3]{2}$, y- intercept: $-\frac{1}{2}$

5. a. vertical asymptote: $x = 3, x = -3$ b. No, H.A

b. oblique asymptote: $y = 2x$, intersected at $\left(\frac{1}{6},\frac{1}{3}\right)$

6. a. vertical asymptote: $x = -4$,

b. horizontal asymptote: y = 1, intersected at (-3, 1).

7. $\{-1, 0, 1\}$ 8. $\mathbb{R}\setminus\{3, -3\}$ 9. Ø 10. {-3}

3. Find the equation of the line with slope m and y-intercept b

a) m = -2, b = 1 b) $m = \frac{\pi}{2}$, b = 4, c) m = -3, b = 0

Solution: use y = mx + b

Thus a) y = -2x + 1 b) $y = \frac{\pi}{2}x + 4$, c) y = -3x

4. a) Find the equation of straight line parallel to the x-axis at distance
i) 4 unit above it ii) 5 unit below it
Solution: i) y = 4 ii) y = -5

5. Find the equation of straight line with x-intercept 3 y-intercept 2

Solution: use $\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow \frac{x}{3} + \frac{y}{2} = 1 \Leftrightarrow 2x + 3y = 6$

6. Find the equation of straight line through (2, 5) and making equal intercept of opposite sign on x-axis

Solution: use $\frac{x}{-a} + \frac{y}{a} = 1$

The line passes through (2, 5) $\Rightarrow \frac{2}{-a} + \frac{5}{a} = 1 \Leftrightarrow 2 - 5 = -a \Rightarrow a = 3$

$\therefore$ The equation of straight line $\frac{x}{-3} + \frac{y}{3} = 1 \Leftrightarrow x - y = -3$

Let ℓ_1 and ℓ_2 be two non-vertical line with respective slope m_1 and m_2, then

i) If $\ell_1 \perp \ell_2$ then $m_1 \cdot m_2 = -1$
ii) If $\ell_1 \parallel \ell_2$ then $m_1 = m_2$

Illustrative Example

7. Determine which of the following pairs if lines are perpendicular, parallel, intersecting (but not perpendicular)

a) ℓ_1: 3x – y + 5 = 0 and ℓ_2: x + 3y – 1 = 0
b) ℓ_1: 3x – 4y + 1 = 0 and ℓ_2: 4x - 3y + 1 = 0
c) ℓ_1: 3x + 2y = 6 and ℓ_2: 6x + 4y = 10
d) ℓ_1: 2x + 3y = 5 and ℓ_2: 4x + 6y = 10

Solution: Here

a) slope of ℓ_1: $3x - y + 5 = 0$ is $\dfrac{-3}{-1} = 3 = m_1$

slope of ℓ_2: $x + 3y - 1 = 0$ is $\dfrac{-1}{3} = m_2$ $-3y = x - 1$ $y =$

$\Rightarrow m_1 \cdot m_2 = (3)\left(\dfrac{-1}{3}\right) = -1 \longleftarrow$ intersecting perpendicular

$\therefore \ell_1$ is perpendicular to ℓ_2

b) slope of ℓ_1: $3x - 4y + 1 = 0$ is $m_1 = \dfrac{3}{4}$

slope of ℓ_2: $4x - 3y + 1 = 0$ is $m_2 = \dfrac{4}{3}$

$\therefore$ neither parallel nor perpendicular
$\therefore$ It is intersecting line

c) slope of ℓ_1: $3x + 2y = 6$ is $m_1 = \dfrac{-3}{2}$

slope of ℓ_2: $6x + 4y = 10$ is $m_2 = \dfrac{-6}{4} = \dfrac{-3}{2}$

$\therefore m_1 = m_2$
$\therefore \ell_1$ is parallel to ℓ_2

d) exercise left for student.

3. Find an equation of the line through $p(5, -7)$ that is parallel to the line $6x + 3y = 4$

Solution: slope of $6x + 3y = 4$ is $m_1 = \dfrac{-6}{3} = -2$

Since parallel lines have the same slop $m_1 = m_2$
$\therefore$ slope of the required line has slope $m_2 = -2$
Now using $y - y_1 = m_2(x - x_1)$ in $(x_1, y_1) = p(5, -7)$
$\Rightarrow y + 7 = -2(x - 5) \Rightarrow y = -2x + 3 \longleftarrow$ the required parallel line
$\Leftrightarrow y + 2x = 3$

Method II

It should be noted that any line parallel to $6x + 3y = 4$ is of the form $6x + 3y = k$ where k is constant

If $6x + 3y = k$ passes through $(5, -7)$
$\Rightarrow 6(5) + 3(-7) = k \Rightarrow k = 9$

$\therefore$ the required equation $6x + 3y = 9 \Leftrightarrow 2x$ ...

9. Find an equation of the line through (2, -6) ...

Solution: Any line parallel to $5x + y = 8$ is ...

If $5x + y = k$ passes through (2, -6)

$\Rightarrow 5(2) + -6 = k \Rightarrow k = 4$

$\therefore 5x + y = 4 \longleftarrow$ the equation of the required line

10. Find the equation of the line that passes through the point (2, -1) perpendicular to the line $2x - 3y - 4 = 0$

Solution: slope of $2x - 3y = 4$ is $m_1 = \frac{2}{3}$

Slope of perpendicular line $m_2 = \frac{-3}{2}$

$\therefore y + 1 = \frac{-3}{2}(x-2) \Leftrightarrow 2y + 3x = 4 \leftarrow$ the equation of required line

11. Let ℓ be the line with equation $4x - 3y = 8$. Find the equation of the line that passes through the point p(1, 2) that is

a) parallel to ℓ b) perpendicular to ℓ

Solution: Exercise left for the student

Answer: a) $4x - 3y = -2$ b) $3x + 4y = 11$

12. Find k so that the line $4x + ky - 12 = 0$ will be

a) parallel to the line with equation $x - 3y = 8$

b) perpendicular to the line with equation $x - 3y = 8$

Solution: we have

a) slope of $4x + ky - 12 = 0$ is $\frac{-4}{k}$

slope of $x - 3y = 8$ is $\frac{1}{3}$

since they are parallel $\Rightarrow \frac{-4}{k} = \frac{1}{3} \Leftrightarrow k = -12$

b) since they are perpendicular

$\Rightarrow \left(\frac{-4}{k}\right)\left(\frac{1}{3}\right) = -1 \Rightarrow \frac{-4}{3k} = 1 \Leftrightarrow 3k = 4 \quad \therefore k = \frac{4}{3}$

13. Given lines ℓ_1: $(6-k)x + y = -1$ and ℓ_2: $8x + ky = 7$
Find the value of k so that

a) $\ell_1 \parallel \ell_2$ b) $\ell_1 \perp \ell_2$

Solution:

a) slope of ℓ_1 is $\dfrac{-(6-k)}{1} = k-6$ and slope of ℓ_2 is $\dfrac{-8}{k}$

since $\ell_1 \parallel \ell_2 \Rightarrow k-6 = \dfrac{-8}{k} \Leftrightarrow k^2 - 6k + 8 = 0$

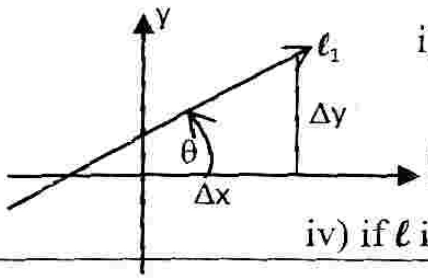
$\Rightarrow (k-2)(k-4) = 0 \Rightarrow k = 2$ or $k = 4$

b) $\ell_1 \perp \ell_2 \Rightarrow (k-6)\left(\dfrac{-8}{k}\right) = -1 \Leftrightarrow -8k + 48 = -k$

$\Rightarrow 7k = -48 \Rightarrow k = \dfrac{48}{7}$

Angle of inclination and slope

The angle measured from positive x-axis to a line in counter clock wise direction is called angle of inclination



i) $\tan\theta = \dfrac{\Delta y}{\Delta x}$ ii) slope $m = \dfrac{\Delta y}{\Delta x}$

iii) $\tan\theta = slope\ of\ \ell = \dfrac{\Delta y}{\Delta x}$

iv) if ℓ is horizontal, then $\theta = 0$

Illustrative Example

14. Find the slope of a line ℓ if its inclination is

a) 45° b) 150° c) 60°

Solution: a) slope of $\ell = \tan 45^0 = 1$

b) slope of $\ell = \tan 150^0 = -\tan 30^0 = \dfrac{-1}{\sqrt{3}}$

c) slope of $\ell = \tan 60^0 = \sqrt{3}$

15. Find the equation of straight line passing through (2, 3) and making with the x-axis an angle of

a) 45° b) 60° c) 90°

Solution: a) $m = \tan 45^0 = 1$, using $y - y_1 = m(x - x_1)$

$\Rightarrow y - 3 = 1(x - 2) \Leftrightarrow y = x + 1$

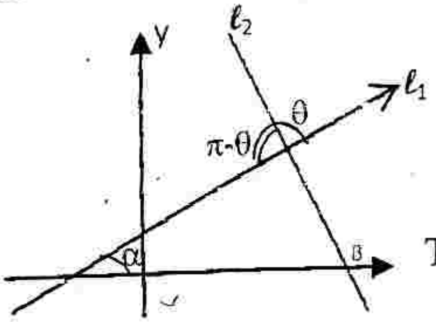
b) $m = \tan 60^0 = \sqrt{3} \Leftrightarrow y - 3 = \sqrt{3}(x - 2)$

$\Rightarrow y = \sqrt{3}x - 2\sqrt{3} + 3$

c) when $\theta = 90^0$, the line is vertical $\therefore x = 2$

Angle Between two lines

The angle between two intersecting lines ℓ_1 and ℓ_2 is angle θ which is measured counter clock wise from ℓ_1 to ℓ_2 about the point of intersection in order to concide with ℓ_2.



- slope of $\ell_1 = m_1 = \tan\alpha$
- slope of $\ell_2 = m_2 = \tan\beta$

You know that $\alpha + \theta = \beta$

$\Rightarrow \theta = \beta - \alpha$

Thus, $\tan\theta = \tan(\beta - \alpha) = \dfrac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$

Hence angle between ℓ_1 and ℓ_2 is given by

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1} \text{ if } m_1m_2 \neq -1$$

i) If $1 + m_2m_1 = 0 \Rightarrow m_1 \cdot m_2 = -1$, then $\theta = 90^0$

ii) If $m_2 - m_1 = 0 \Rightarrow m_1 = m_2$, then $\theta = 0^0$

Illustrative Example

16. Find the tangent of the angle of intersection of the lines with slope.

a) $m_1 = 2$ and $m_2 = 3$

b) $m_1 = \dfrac{-2}{3}$ and $m_2 = -5$

c) $m_1 = \dfrac{-1}{3}$ and $m_2 = -2$

d) $m_1 = 0$ and $m_2 = 1$

e) $m_1 = 1$ and $m_2 = -1$

Solution: we use $\tan\theta = \dfrac{m_2 - m_1}{1 + m_2m_1}$

a) $\tan\theta = \dfrac{3 - 2}{1 + (3)(2)} = \dfrac{1}{7} \longleftarrow$ the acute angle between the lines

$\tan\theta = \frac{-1}{7} \longleftarrow$ the obtuse angle between the line

b) $\tan\theta = \frac{-5+\frac{2}{3}}{1+(5)\frac{2}{3}} = \frac{\frac{-13}{3}}{\frac{13}{3}} = -1$

$\tan\theta = -1 \longleftarrow$ the obtuse angle between the line

$\Rightarrow \theta = 135^0$

$\tan\theta = 1 \longleftarrow$ the acute angle between the line

$\Rightarrow \theta = 45^0$

e) $\theta = 90^0$

c and d exercise

17. Find the acute angle of intersection of the lines with slopes $\frac{1-m}{1+m}$ and $-m$, where $m \neq -1$

Solution: we have $m_1 = \frac{1-m}{1+m}$ and $m_2 = -m$

Use $\tan\theta = \left|\frac{m_2 - m_1}{1+m_2m_1}\right|$ for acute angle between thee two lines

$$\therefore \tan\theta = \left|\frac{-m-\left(\frac{1-m}{1+m}\right)}{1+(-m)\left(\frac{1-m}{1+m}\right)}\right| = \left|\frac{m^2+1}{m^2+1}\right| = 1 \Rightarrow \theta = 45^0$$

18. Find the acute angle of intersection of the lines with slopes

and -3

Solution: $\tan\theta = \left|\frac{-\frac{1}{2}+3}{1+\left(\frac{-1}{2}\right)(-3)}\right| = \frac{\frac{5}{2}}{\frac{5}{2}} = 1 \Rightarrow \theta = 45^0$

19. If a line L_1 passes through (1, -1) and (3, 1) and a line L_2 contains points (4, 5) and (6, 9) then find the tangent of the angle between ℓ_1 and ℓ_2

Solution: First find the slope

Slope of ℓ_1: $m_1 = \dfrac{1-(-1)}{3-1} = 1$

Slope of ℓ_1: $m_2 = \dfrac{9-5}{6-4} = \dfrac{4}{2} = 2$

If θ is the angle between ℓ_1 and ℓ_2 then

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{2-1}{1+(2)(1)} = \frac{1}{3}$$

20. The angle between two lines is 45^0 and the slope of one of the line is $\frac{3}{5}$. Find the slope of the other line

Solution: **case 1**: Let $m_1 = \dfrac{3}{5}$ and $m_2 = ?$

$$\Rightarrow \tan 45^0 = \frac{m_2 - \frac{3}{5}}{1 + \frac{3}{5}m_2} = 1 \Rightarrow m_2 - \frac{3}{5} = 1 + \frac{3}{5}m_2$$

$$\Rightarrow m_2 - \frac{3}{5}m_2 = 1 + \frac{3}{5} \Rightarrow \frac{2m_2}{5} = \frac{8}{5} \Leftrightarrow 2m_2 = 8 \Rightarrow m_2 = 4$$

Case II Let $m_2 = \dfrac{3}{5}$ *and* $m_1 = ?$

$$\Rightarrow \tan 45^0 = \frac{\frac{3}{5} - m_1}{1 + \frac{3}{5}m_1} = 1 \Leftrightarrow \frac{3}{5} - m_1 = 1 + \frac{3}{5}m_1$$

$$\Rightarrow \frac{3}{5} - 1 = \frac{3}{5}m_1 + m_1 \Rightarrow \frac{-2}{5} = \frac{8m_1}{5} \Leftrightarrow m_1 = -\frac{1}{4}$$

21. What is the angle between the line through (-1, 3) and (2, 3) and the line with equation $\sqrt{3}y - x + 7 = 0$

A. $120°$ B. $60°$ C. $30°$ D. $15°$

Solution: slope of ℓ_1: $m_1 = \frac{3-3}{2+1} = 0$

Slope of ℓ_2: $m_2 = \frac{1}{\sqrt{3}}$

$$\Rightarrow \tan\theta = \frac{\frac{1}{\sqrt{3}} - 0}{1+0} = \frac{1}{\sqrt{3}}$$

$\therefore \theta = 30^0$

22. Find the equation of the line that

a) passes through (2,-1) and the angle from the line $6x+5y-1=0$ to the line 45^0

b) have y-intercept 4 and is inclined at an angle of 135^0 to the line $2x + 3y - 7 = 0$

Solution: a) Let the slope of $\ell_1 = 6x + 5y - 1 = 0$ is $m_1 = \frac{-6}{5}$

Let the slope of required line ℓ_2 is m_2 the angle between the two line can be obtained by

$$\Rightarrow \tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1} \Rightarrow \tan 45^0 = \frac{m_2 + \frac{6}{5}}{1 - \frac{6}{5}m_2}$$

$$\Rightarrow m_2 + \frac{6}{5}m_2 = 1 - \frac{6}{5} \Rightarrow \frac{11m_2}{5} = \frac{-1}{5}$$

$$\therefore m_2 = \frac{-1}{11}$$

Thus, the equation of the line passing through (2, -1) have slope

$m = \frac{-1}{11}$

Given by $y+1=\frac{-1}{11}(x-2)$

$\Rightarrow 11y + 11 = -x + 2$

$\Rightarrow x + 11y = -9$

Case II the angle between the given straight line and the required straight line is 135^0

Hence, $\tan 135^0 = \dfrac{m+\frac{6}{5}}{1-\frac{6}{5}m}$

$\Rightarrow -1=\dfrac{5m+6}{5-6m} \Leftrightarrow -5+6m=5m+6 \Rightarrow m=11 \Leftrightarrow y-1=11(x-2)$

$\therefore$ the required straight lines are $x + 11y + 9 = 0$ or $11x - y - 23 = 0$

b) Let m be the slope of the required lines.

Slope of $2x + 3y - 7 = 0$ is $\dfrac{-2}{3}$

Now the angle between the given straight line and the required straight line is 45^0 or 135^0

$\therefore \tan 45^0 = \dfrac{m+\frac{2}{3}}{1-\frac{2}{3}m}$ $\qquad$ $\therefore \tan 135^0 = \dfrac{m+\frac{2}{3}}{1-\frac{2}{3}m}$

$\Rightarrow 1=\dfrac{m+\frac{2}{3}}{1-\frac{2}{3}m}$ $\qquad$ $\Rightarrow -1=\dfrac{m+\frac{2}{3}}{1-\frac{2}{3}m}$

$\Rightarrow 1-\frac{2}{3}m=m+\frac{2}{3}$ $\qquad$ $\Rightarrow -1+\frac{2}{3}m=m+\frac{2}{3}$

$\Rightarrow 3 - 2m = 3m + 2$ $\qquad$ $\Rightarrow -3 + 2m = 3m + 2$

$\Rightarrow 1 = 5m$ $\qquad$ $\Rightarrow -5 = m$

$\therefore m = \dfrac{1}{5}$

$\Rightarrow$ Equation of ℓ is $y = \frac{1}{5}x + 4$ or y = -5x + 4

Thus the equation of the straight lines are, for b = 4

$y = \frac{1}{5}x + 4$ or $y = -5x + 4$

23. Find the tangent of the angle between the given lines
   a) ℓ_1: y = -4x + 2; ℓ_2: y = -2x
   b) ℓ_1: y = 3x - 2; ℓ_2: y = 4x – 6
   c) ℓ_1: -3x + 4y – 15 = 0; ℓ_2: 4x + 3y – 30 = 0
   d) ℓ_1: 5x – 3y = 15; ℓ_2: x + 2y – 4 = 0
   e) ℓ_1: y = 2; ℓ_2: x + 2y = 8

Solution: Here we use $\tan\theta = \dfrac{m_2 - m_1}{1 + m_2 m_1}$

a) the slope of ℓ_1: $m_1 = -4$
   the slope of ℓ_2: $m_2 = -2$

$$\therefore \tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{-2-(-4)}{1+(-2)(-4)} = \frac{2}{9} \longleftarrow \text{the angle between from } \ell_1 \text{ to } \ell_2$$

b) $m_1 = 3$, $m_2 = 4$

$$\Rightarrow \tan\theta = \frac{4-3}{1+(4)(3)} = \frac{1}{13} \longleftarrow \text{the angle between from } \ell_1 \text{ to } \ell_2$$

c) $m_1 = \frac{3}{4}, m_2 = -\frac{4}{3}$

$$\Rightarrow \tan\theta = \frac{\frac{-4}{3} - \frac{3}{4}}{1 + \left(\frac{-4}{3}\right)\left(\frac{3}{4}\right)} = \frac{\frac{-25}{12}}{\frac{12-12}{12}} = \frac{-25}{0}$$

Since ℓ_1 is perpendicular to ℓ_2

$\therefore$ The angle between the two line is $\theta = 90^0$

d) $m_1 = \frac{5}{3}, m_2 = -\frac{1}{2}$

$$\Rightarrow \tan\theta = \frac{\frac{1}{2}-\frac{5}{3}}{1+\left(-\frac{1}{2}\right)\left(\frac{5}{3}\right)} = \frac{\frac{-3-10}{6}}{\frac{6-5}{6}} = -13 \leftarrow \text{the angle from } \ell_1 \text{ to } \ell_2$$

e) $m_1 = 0, m_2 = -\frac{1}{2} \Rightarrow \tan\theta = \frac{-\frac{1}{2}}{1} = -\frac{1}{2} \leftarrow$ the angle from ℓ_1 to ℓ_2

24. Find the tangent of the acute angle between the line

a) $3x - y + 5 = 0$
$2x + y + 7 = 0$

b) $\sqrt{3}x + \sqrt{2}y - 2 = 0$
$\sqrt{6}x - 3y + 3 = 0$

c) $6x + 5y - 1 = 0$
$11x - y - 23 = 0$

Solution: To find the acute angle between the line we use

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_2 m_1}\right|$$

Note, slope of $Ax + By + C = 0$ is $\left(\frac{-A}{B}\right)$

a) $m_1 = 3$, $m_2 = -2$

$$\tan\theta = \left|\frac{-2-3}{1-6}\right| = \left|\frac{-5}{-5}\right| = 1$$

$\therefore \theta = 45^0$

b) $m_1 = \frac{-\sqrt{3}}{\sqrt{2}}$, $m_2 = \frac{\sqrt{6}}{3}$, *Note,* $\frac{-\sqrt{3}}{\sqrt{2}} = \frac{-\sqrt{6}}{2} = \frac{-3}{\sqrt{6}}$

$$\tan\theta = \left|\frac{\frac{\sqrt{6}}{3} + \frac{\sqrt{3}}{\sqrt{2}}}{1 + \left(\frac{\sqrt{6}}{3}\right)\left(\frac{-\sqrt{3}}{\sqrt{2}}\right)}\right| = \left|\frac{\frac{\sqrt{6}}{3} + \frac{3}{\sqrt{6}}}{1-1}\right| \Rightarrow \theta = 90^0$$

c) $m_1 = \frac{-6}{5}$, $m_2 = 11$

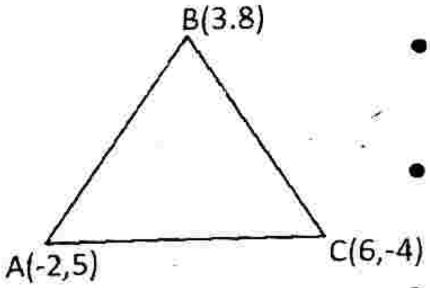
$$\tan\theta = \left|\frac{11+\frac{6}{5}}{1-\frac{66}{5}}\right| = \left|\frac{\frac{61}{5}}{\frac{-61}{5}}\right| \Rightarrow |-1| = 1$$

$\therefore \theta = 45^0$

25. Let A(-2, 5), B(3, 8) and C(6, -4) and vertices of a triangle (ΔABC) then, Find the tangent angle

a) at A b) at B c) at C

Solution: Consider the triangle, then find pair of slopes.



- slope of $\overline{AB} = \frac{8-5}{3-(-2)} = \frac{3}{5}$
- slope of $\overline{AC} = \frac{-4-5}{6-(-2)} = \frac{-9}{8}$
- slope of $\overline{BC} = \frac{8-(-4)}{3-6} = -4$

Therefore, tangent angle at

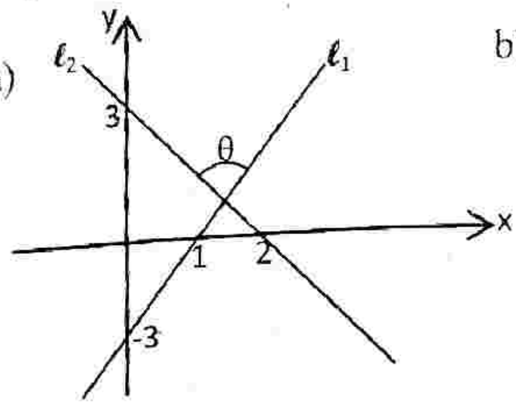
a) A is $\tan A = \frac{\frac{3}{5}-\left(\frac{-9}{8}\right)}{1+\left(\frac{3}{5}\right)\left(\frac{-9}{8}\right)} = \frac{69}{13}$

b) at B, $\tan\hat{B} = \frac{-4-\frac{3}{5}}{1+(-4)\left(\frac{3}{5}\right)} = \frac{23}{7}$

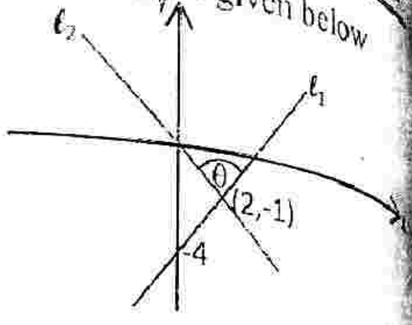
c) at C,. $\tan\hat{C} = \frac{\frac{-9}{8}-(-4)}{1+\left(\frac{-9}{8}\right)(-4)} = \frac{23}{44}$

26. Find the angle between ℓ_1 and ℓ_2 whose graph is given below

a)



b)



Solution: To find the angle between ℓ_1 and ℓ_2, first determine slope,

a) slope of ℓ_1: $m_1 = 3$ and slope of $\ell_2 = -\frac{3}{2} = m_2$

$$\therefore \tan\theta = \frac{\frac{-3}{2} - 3}{1 + \left(\frac{-3}{2}\right)(3)} = \frac{9}{7} \leftarrow \text{angle from } \ell_1 \text{ to } \ell_2$$

b) since the point (0, -4) and (2, -1) on ℓ_1, and (0, 0) and (2, -1) on ℓ_2, so the angle between ℓ_1 and ℓ_2 will be

$$\therefore m_1 = \frac{3}{2} \text{ and } m_2 = -\frac{1}{2}$$

$$\therefore \tan\theta = \frac{\frac{-1}{2} - \frac{3}{2}}{1 + \left(\frac{3}{2}\right)\left(\frac{-1}{2}\right)} = -8$$

27. Find the tangent angle between
a) the x-axis and the line: y = x – 4
b) the x-axis and the line y + x = 6
c) the line y = 3 and the line 3x – 2y = 6

Solution: (a) Now, ℓ_1 as x-axis (i.e. y = 0)

ℓ_2 as y = x – 4 $\therefore m_1 = 0$ *and* $m_2 = 1$

$$\therefore \tan\theta = \frac{1-0}{1+(1)(0)} = 1 \Rightarrow \theta = 45^0$$

(b) $m_1 = 0$ *and* $m_2 = -1$

$$\therefore \tan\theta = \frac{-1-0}{1+0} = -1 \ \therefore \theta = 135^0$$

(c) since the slope of horizontal line $m_1 = 0$

$$\Rightarrow m_1 = 0 \ \text{and} \ m_2 = \frac{3}{2} \ \therefore \tan\theta = \frac{\frac{3}{2}}{1+0} = \frac{3}{2}$$

Distance between two point

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ are coordinate points in a plane, then the distance d between P and Q is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_2)^2}$$

Illustrative Example

28. Find the distance between the point of each of the following

a)	(2, 5) and (5, 9)	c)	(0, a) and (a, 0)
b)	(a, -b) and (-a, b)	d)	(x, -y) and (-y, x)

Solution: Let distance between be d

a) $d = \sqrt{(5-2)^2 + (9-5)^2} = \sqrt{9+16} = 5unit$

b) $d = \sqrt{(a-(-a))^2 + (-b-b)^2} = \sqrt{4a^2 + 4b^2} = 2\sqrt{a^2 + b^2}$

c) $d = \sqrt{(0-a)^2 + (a-0)^2} = \sqrt{a^2 + a^2} = \sqrt{2}\,|a|$

d) $d = \sqrt{(x+y)^2 + (x+y)^2} = \sqrt{2(x+y)^2} = \sqrt{2}\,|x+y|$

29. Find all points on the x-axis that are 10 units from (4, 6)

Solution: all points on the x-axis are (x, 0)

$$(x,0)\xrightarrow{d = 10 \ unit}(4,6)$$

$$\Leftrightarrow \sqrt{(x-4)^2 + (0-6)^2} = 10 \Rightarrow \sqrt{x^2 - 8x + 52} = 10$$

$$\Rightarrow x^2 - 8x + 52 = 100 \longleftarrow \text{squaring both side}$$

$$\Rightarrow x^2 - 8x - 48 = 0 \Rightarrow (x-12)(x+4) = 0$$

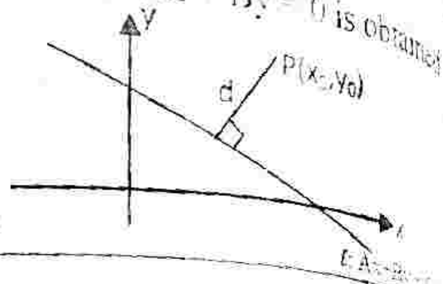
$\therefore$ x = 12 or x = -4

$\therefore$ The possible points are (12, 0) or (-4, 0)

Distance between a point $P(x_0, y_0)$ and a line ℓ: $Ax + By + C = 0$ on the coordinate plane.

The distance from a point $P(x_0, y_0)$ to the line ℓ: $Ax + By + C = 0$ is obtained by the formula

$$d = \left|\frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}}\right|$$



Illustrative Example

30. Find the distance from each line to the indicated point.

a) $x - 2y + 5 = 0$, P(0, 0) c) $3x - 5y = 0$, P(4, -2)

b) $2x - 3y + 6 = 0$, P(4, 5) d) $5x + 2y - 1 = 0$, P(4, 1)

Solution: a) The distance from the origin to the line $x - 2y + 5 = 0$

$$\Rightarrow d = \left|\frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}}\right| \Leftrightarrow d = \left|\frac{1\cdot 0 - 2\cdot 0 + 5}{\sqrt{1^2 + 2^2}}\right| = \frac{5}{\sqrt{5}} = \sqrt{5}$$

$$\text{b) } d = \left|\frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}}\right| \Leftrightarrow \left|\frac{2(3) - 3(5) + 6}{\sqrt{2^2 + 3^2}}\right| = \frac{1}{\sqrt{13}} = \frac{\sqrt{13}}{13}$$

$$\text{c) } d = \left|\frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}}\right| \Leftrightarrow \left|\frac{3(4) - 5(-2) + 0}{\sqrt{3^2 + 5^2}}\right| = \frac{22}{\sqrt{34}}$$

$$\text{d) } d = \left|\frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}}\right| \Leftrightarrow \left|\frac{5(4) + 2(1) - 1}{\sqrt{5^2 + 2^2}}\right| = \frac{21}{\sqrt{29}}$$

31. Find the point on the line y = x- 2 at a distance of $3\sqrt{2}$ units from (1, -1)

Solution: If y = x – 2, then (x, y) = (x, x – 2)

$$\therefore (x, x-2) \xrightarrow{d = 3\sqrt{2}} (1, -1)$$

Distance between point

$\Rightarrow \sqrt{(x-1)^2 + (x-2+1)^2} = 3\sqrt{2} \longleftarrow$ squaring both side

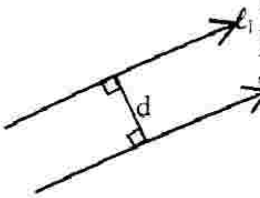
$\Rightarrow 2(x - 1)^2 = 18 \Leftrightarrow (x - 1)^2 = 9$

$\Rightarrow x - 1 = \pm 3 \Rightarrow x - 1 = 3$ or $x - 1 = -3$

$\therefore x = 4$ or $x = -3 + 1 = -2$

$\therefore$ Point on the line $y = x - 2$ are $(4, 2)$ or $(-2, -4)$

The distance between two parallel lines



If ℓ_1 is parallel to ℓ_2

Let take point $P(h, k)$ on ℓ_1.

Let the equation of ℓ_2: $Ax + By + C = 0$

Then the distance between ℓ_1 and ℓ_2 is

obtained by $d = \dfrac{|Ah + Bk + C|}{\sqrt{A^2 + B^2}}$.

Illustrative Example

32. Find the distance between the pair of parallel lines whose equations are given below.

$3x - 10$

$3(x - \frac{10}{3})$

a) ℓ_1: $3x + 4y - 10 = 0$ and ℓ_2: $6x + 8y + 3 = 0$

b) ℓ_1: $2x - 5y + 1 = 0$ and ℓ_2: $2x - 5y - 3 = 0$

c) ℓ_1: $y = -2x + 3$ and ℓ_2: $3y = -6x + 1$

Solution: a) First take point on $\ell_1(h, k) = (2, 1)$ on ℓ_1: $3x + 4y - 10$ then the distance between the point $P(2, 1)$ and the line ℓ_2, $6x + 8y + 3 = 0$

$$\Rightarrow d = \frac{|6(2) + 4(1) + 3|}{\sqrt{6^2 + 8^2}} = \frac{23}{\sqrt{100}} = \frac{23}{10} = 2.3unit$$

b) Let $P(h, k) = P\left(3, \frac{7}{5}\right)$ on ℓ_1 i.e. when $x = 3$, $y = \frac{7}{5}$ then

distance between the point $P\left(3, \frac{7}{5}\right)$ and the line

$2x - 5y - 3 = 0$ will be

$$\Rightarrow d = \frac{\left|2(3) + 5\left(\frac{7}{5}\right) - 3\right|}{\sqrt{2^2 + 5^2}} = \frac{4}{\sqrt{29}} unit \leftarrow \text{Distance between } \ell_1 \text{ and } \ell_2$$

c) Let $P(h, k) = (-1, 5)$ on $y = -2x + 3$

$$\Rightarrow d = \frac{|6(-1) + 3(5) - 1|}{\sqrt{36 + 9}} = \frac{8}{\sqrt{45}} = \frac{8}{3\sqrt{5}} \leftarrow \text{Distance between}$$

ℓ_1 and ℓ_2

Locus

The locus of a point is the path traced out by the point as it moves under certain condition.

- All points satisfying the given condition lie on the locus.
- We define a locus as a set of points satisfying a given condition.

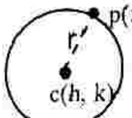
Circle

Definition: A circle is the locus of a point P which moves in plane with fixed distance from fixed point.

- The fixed distance is called radius
- The fixed point is called centre of circle.

Equation of a circle

Let P(x, y) be a point on circle with center C(h, k) and the distance PC = r, now using distance between two point is given by



$$(x-h)^2+(y-k)^2=r^2$$

Illustrative example

33. Find the equation of a circle with the following centers and radii

a) C(0, 0), r = 4 c) C(2, -1), r = $\sqrt{5}$

b) C(-1, -1), r = 3 d) C(3, 2), r = 2

Solution: use $(x-h)^2+(y-k)^2=r^2$, for c(h, k)

a) $(x-0)^2+(y-0)^2=4^2 \Leftrightarrow x^2+y^2=16$

b) $(x+1)^2+(y+1)^2=3^2 \Rightarrow (x+1)^2+(y+1)^2=9$

c) $(x-2)^2+(y+1)^2=5$

d) $(x-3)^2+(y-2)^2=4$

34. Find the equation of the circle of center C(2, -1) which passes

i) through the point (3, 4)

ii) through the point (4, 1)

Solution: Center (h, k) = (2, -1)

i) The equation of a circle is $(x-2)^2+(y+1)^2=r^2$ since the point (3, 4) on the circle

$\therefore (3-2)^2+(4+1)^2=r^2$

$\therefore r^2=26$

$\therefore$ The equation of a circle $(x-2)^2+(y+1)^2=26$

ii) Since the point (4, 1) lies on the circle

$\therefore (4-2)^2+(1+1)^2=r^2$

$\therefore r^2=8$

$\therefore$ The equation of circle is $(x-2)^2+(y+1)^2=8$

35. In each of the following find the equation of the circle with
   a) center (-1, -3) passes through (-4, -2)
   b) center (-4, 5) tangent to the x-axis
   c) center (1, -7) tangent to the y-axis
   d) end points of diameter are (-2, -3) and (4, 5)

Solution: use $(x-h)^2+(y-k)^2=r^2$, for center (h, k)

a) $(x+1)^2+(y+3)^2=r^2$, since the point (-4, -2) on the circle

$\therefore (-4+1)^2+(-2+3)^2=r^2$

$\therefore r^2=10$ and $r=\sqrt{10}$

$\therefore$ The equation of circle is $(x+1)^2+(y+3)^2=10$

Note: for center (h, k), if the circle is tangent to x-axis, then the radius, r = k for center (h, k) if te circle is tangent to y-axis, then the radius, r = h

$(x+4)^2+(y-5)^2=25$

c) $r=1$

$\therefore (x-1)^2+(y+7)^2=1$

d) center $=\left(\dfrac{-4+-2}{2},\dfrac{5+-3}{2}\right)=(-3,1)$

1 4 + 4 + 4

148 = r^2

$\therefore$ the equation of circle $(x+3)^2+(y-1)^2=r^2$

Since (-2, -3) on the circle

$(4+3)^2+(5-1)^2=r^2$

$\therefore r^2=(-2+3)^2+(-3-1)^2=17$

$(12)^2+(4)^2$

$\therefore$ the equation of circle $(x+3)^2+(y-1)^2=17$

36. Find the equation of a circle containing the point P(0, -3) and Q(4, 0) and whose center is on the line $x+y=0$

Solution: Let C(h, k) the center, $\rightarrow \overline{PC}=\overline{QC}=r$

$\Rightarrow (h-0)^2+(k+3)^2=(h-4)^2+(k-0)^2$

$\Rightarrow h^2+k^2+6k+9=h^2-8h+16+k^2$

$\Rightarrow 6k+8h=7$, since (h, k) on the line $x+y=0 \Rightarrow h+k=0$

$\Rightarrow h=-k \Rightarrow 6k+8(-k)=7 \Rightarrow k=\dfrac{-7}{2},\quad h=\dfrac{7}{2}$

$\therefore$ center $(h, k) = \left(\frac{7}{2}, \frac{-7}{2}\right)$

Hence the equation of the circle $\left(x - \frac{7}{2}\right)^2 + \left(y + \frac{7}{2}\right)^2 = r^2$

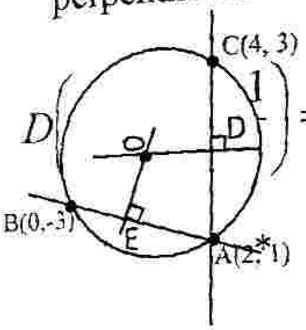
Since (0, -3) on the circle

$$r^2 = \left(0 - \frac{7}{2}\right)^2 + \left(-3 + \frac{7}{2}\right)^2 = \frac{49}{4} + \frac{1}{4} = \frac{50}{4} = \frac{25}{2}$$

$\therefore$ the equation of circle $\left(x - \frac{7}{2}\right)^2 + \left(y + \frac{7}{2}\right)^2 = \frac{25}{2}$

37. Find the equation of the circle which passes through the points A(2, 1), B(0, -3) and C(4, 3)

Solution; The center is the point of intersection of the perpendicular bisectors of any two chords



* mid-point of AC = D( ) = (3,2)

mid-point of AB = $E\left(\frac{0+2}{2}, \frac{-3+1}{2}\right) = (1,-1)$

Since slope of AC $= \frac{3-1}{4-2} = 1$, therefore the slope of

OD = -1, because OD $\perp$ AC

$\therefore$ equation of OD, is $y - 2 = -1(x - 3) \Rightarrow y + x = 5 = 5$

And slope of AB $= \frac{1+3}{2-0} = 2$

$\Rightarrow$ slope of OE $= -\frac{1}{2}$, because $\overline{AB} \perp \overline{OE}$

$\therefore$ the equation of OE is $y + 1 = -\frac{1}{2}(x - 1) \Rightarrow 2y + x = -1$

From (*) and (**) solving together

$$\begin{cases} y+x=5 \\ 2y+x=-1 \end{cases} \Rightarrow (11,-6) \longleftarrow \text{the center}$$

$$\text{radius} = \sqrt{(11-0)^2+(-6+3)^2} = \sqrt{130}$$

Hence the equation of circle is $(x-11)^2+(y+6)^2=130$

Note: The equation of circle with center (h, k) and tangent to the line ℓ: $Ax+By+C=0$ is given by

$$(x-h)^2+(y-k)^2=r^2 \text{ where } r=\frac{|Ah+Bk+C|}{\sqrt{A^2+B^2}}$$

Illustrative Example

38. In each of the following, find the equation of a circle with
   a) center (4, 3) and tangent to $2x-3y+6=0$
   b) center (-2, 1) and tangent to $3x-4y-1=0$
   c) center (0, 0) and tangent to $3x+4y-10=0$

Solution:

a) $(x-4)^2+(y-3)^2=r^2$ and $r=\frac{|2(4)-3(3)+6|}{\sqrt{2^2+3^2}}=\frac{5}{\sqrt{13}}$

$$\therefore (x-4)^2+(y-3)^2=\frac{25}{13}$$

b) $(x-2)^2+(y-1)^2=r^2$ where $r=\frac{|3(-2)-4(1)-1|}{\sqrt{3^2+4^2}}=\frac{11}{5}$

$$\therefore (x+2)^2+(y-1)^2=\frac{121}{25}$$

c) $(x-0)^2+(y-0)^2=r^2$, where

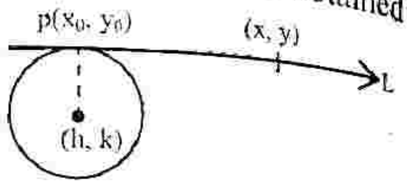
$$r=\frac{|0+4(0)-10|}{\sqrt{3^2+4^2}}=\frac{10}{5}=2$$

$$\therefore x^2+y^2=2^2 \Leftrightarrow x^2+y^2=4$$

Circle and tangent line

Note:
Let a line L tangent to a circle $(x-h)^2+(y-k)^2=r^2$ at a point $P(x_0, y_0)$ then the equation of tangent line L can be obtained by

$$\frac{y-y_0}{x-x_0}=-\left(\frac{x_0-h}{y_0-k}\right)$$



Illustrative Example

39. Find the equation of the tangent line to each circle at the indicated point
 a) $(x-1)^2+(y+2)^2=40$, P(3, 4)
 b) $(x+3)^2+(y-1)^2=29$, P(2, 3)
 c) $x^2+y^2=2$; +(1, -1)
 d) $(x-3)^2+y^2=8$, P(1, 2)

Solution: a) $P(x_0, y_0)=(3, 4)$, $C(h, k)=(1, -2)$

$$\Rightarrow \frac{y-4}{x-3}=-\left(\frac{3-1}{4+2}\right) \Leftrightarrow y-4=-\frac{1}{3}(x-3)$$

$\therefore$ the equation of the tangent line L: $x+3y-15=0$

b) $$\frac{y-3}{x-2}=-\left(\frac{2+3}{3-1}\right) \Leftrightarrow y-3=-\frac{5}{2}(x-2)$$

$\therefore$ the equation of the tangent line L: $2y+5x=16$

c) $$\frac{y+1}{x-1}=-\left(\frac{1-0}{-1-0}\right) \Leftrightarrow y+1=x-1$$

$\therefore$ The equation of the tangent line L: $y-x+2=0$

Line and Circle

Definition: If the perpendicular distance from center of circle to the line ℓ: $Ax+By+C=0$ is

a) greater than radius of r, then line does not intersect the circle

i.e. $$\left|\frac{Ah+Bk+C}{\sqrt{A^2+B^2}}\right|>r$$

b) Smaller than radius or r, then the line intersect **at two point** which is called secant line.

c) equal to the radius of r, then the line intersects **at one point** only, the line called tangent line

$$\text{i.e. } r = \left|\frac{Ah + Bk + C}{\sqrt{A^2 + B^2}}\right|$$

Illustrative Example

40. Find the perpendicular distance from center of the circle with equation $(x-1)^2 + (y+4)^2 = 25$ and identify the line is intersect:

* at one point * at two point

* does not intersect to each of the following line with equations

a) $3x + 4y - 12 = 0$ b) $3x + 4y + 5 = 0$

b) $3x + 4y - 13 = 0$ d) $3x + 4y + 4 = 0$

Solution: Let the perpendicular distance from center of circle (1, -4) to line = D

Let be r radius, so that $r^2 = 25$ $\therefore r = 5$

a) $D = \dfrac{|3(1) + 4(-4) - 12|}{\sqrt{3^2 + 4^2}} = \dfrac{25}{5} = 5 \longleftarrow$ perpendicular distance

Since D = r = 5 $\therefore$ It intersect at one point

$\therefore$ the line $3x + 4y - 12 = 0$ is tangent to $(x-1)^2+(y+4)^2= 25$

b) $D = \dfrac{|3(1) + 4(-4) + 5|}{\sqrt{3^2 + 4^2}} = \dfrac{8}{5} \longleftarrow$ perpendicular distance

since $D < r \Rightarrow \dfrac{8}{5} < 5$ $\therefore$ It intersect at two point

$\therefore$ $3x + 4y - 12$ is secant line to circle $(x-1)^2+(y+4)^2 = 25$

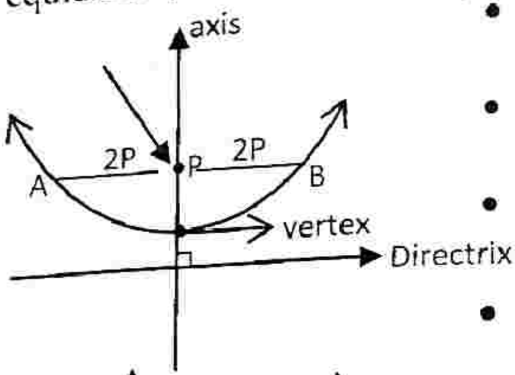
c) $D = \dfrac{|3(1) + 4(-4) + -13|}{\sqrt{3^2 + 4^2}} = \dfrac{26}{5} \longleftarrow$ perpendicular

distance since $D > r \Rightarrow \dfrac{26}{5} > 5$

$\therefore$ The line does not intersect the circle.

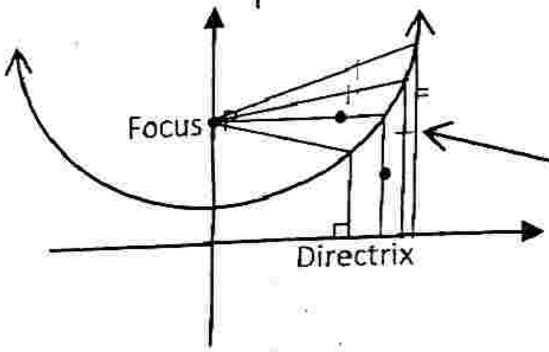
Parabola

Definition: A parabola is the set of points in the plane that are equidistant from fixed line (directrix) and fixed point (the focus).



- The distance between the focus and the vertex is P
- The vertex is equidistant from thee focus and the directrix
- The distance between the focus and directrix is 2P
- The length $\overline{AB}$ that pass through focus (P) is |4P|

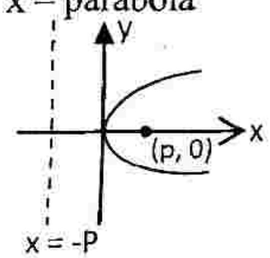
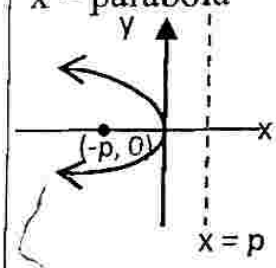
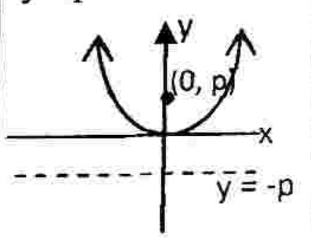
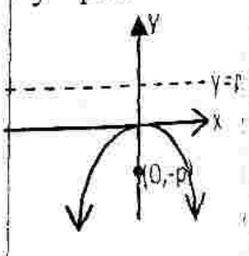
(i.e. $\overline{AB}$ = |4P|) It is called Latus rectum.



All points on the parabola are equidistant from the focus and directrix

The equation of a parabola is simplest if the vertex is the origin and the axis of symmetry is along the x-axis or y-axis. These are called the standard equation.

Parabola in standard Equation

x – parabola 	x – parabola 	y– parabola 	y– parabola 
-Equation: $y^2=4Px$	$y^2 = -4Px$	$x^2 = 4Py$	$x^2 = -4Py$
- Focus: (P, O)	(-P, 0)	(0, P)	(0, -P)
- Vertex: (0, 0)	(0, 0)	(0, 0)	(0, 0)
- latus rectum:(4P)	|4P|	|4P|	|4P|
- directrix x = -P	x = p	y = -p	y = p

Illustrative example

8. Find the focus, vertex, latus rectum, directrix.
a) $y^2 = -8x$, b) $x^2 = 12y$ c) $y = x^2$ d) $y = -\frac{1}{2}x^2$

Solution: a) i) latus rectum $4P = |-8| \Leftrightarrow 4P = 8 \Rightarrow P = 2$
ii) Focus, $(-2, 0)$ iii) direct rix, $x = 2$

b) i) Latusrectum, $4P = 12 \Leftrightarrow P = \frac{12}{4} = 3$
ii) focus, $(0, 3)$ iii) directrix, $y = 3$ iv) vertex, $(0, 0)$

c) i) latus rectum $4P = 1 \Leftrightarrow P = \frac{1}{4}$ ii) focus, $(0, \frac{1}{4})$
iii) directrix, $y = -\frac{1}{4}$ iv) vertex, $(0, 0)$

d) i) latus rectum $4P = 2 \Leftrightarrow P = \frac{1}{2}$
ii) focus $\left(0, -\frac{1}{2}\right)$ iii) directrix, $y = \frac{1}{2}$

9. Find the equation of the parabola with the given properties
a) Focus $(-2, 0)$ and directrix, $x = 2$
b) Focus $(0, -6)$, and directrix, is $y = 6$
c) Symmertry, y, axis, vertex $(0, 0)$, and passes through $(5, 2)$

Solution: a) since It is x – parabola, and $P = -2$
$\therefore 4P \Rightarrow 4(2) = 8$ $\therefore y^2 = 4Px \Leftrightarrow y^2 = -8x$

b) It is y-parabola and $P = -6$
$\therefore x^2 = 4Py \Leftrightarrow x^2 = 4(-6)y \Rightarrow x^2 = -24y$

c) since the parabola is symmetric about the y-axis
$\therefore x^2 = 4Py$ or $x^2 = -4Py$, since $(5, 2)$ on parabola
$\therefore 25 = 4P(2) \Leftrightarrow 4P = \frac{25}{2}$ $\therefore x^2 = \frac{25}{2}y$

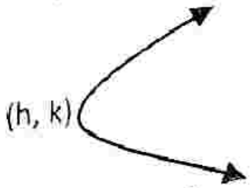
Parabolas with vertex at (h, k)

Equation of parabola that are translated from their standard position can be obtained by replacing for $x \longrightarrow x - h$ and for $y \longrightarrow y - k$ in their standard equations. So that

Vertex $(0, 0) \longrightarrow (h, k)$

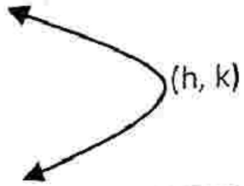
i) Parabola with vertex (h, k) and axis parallel to x-axis

a) $(y-k)^2 = 4P(x-h)$, [opens to right]



- latus rectum 4P
- Focus, $x - h = P \Leftrightarrow x = h + P$ and $y = k$
- Directrix, $x - h = -P \Leftrightarrow x = h - P$
- axis, $y - k = 0 \Leftrightarrow y = k$

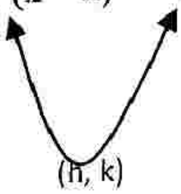
b) $(y-k)^2 = -4P(x-h)$, (opens to left)



- Latus rectum $|4P|$
- Focus, $x - h = -P$ and $y - k = 0$ i.e. $(h - p, k)$
- directrix, $x - h = P \Leftrightarrow x = h + P$
- axis, $y - k = 0 \Leftrightarrow y = k$

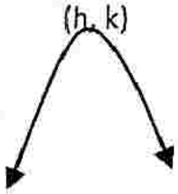
ii) Parabola with vertex (h, k) and axis parallel to y - axis

c) $(x-h)^2 = 4P(y-k)$ (opens up)



- Latus rectum $|4P|$
- Focus, $y - k = P$ and $x - h = 0$ $(h, p + k)$
- directrix, $y - k = -p \Rightarrow y = k - p$
- axis, $x - h = 0 \Rightarrow x = h$

d) $(x-h)^2 = -4P(y-k)$, opens down ward



- Latus rectum, $|4P|$
- Focus, $y - k = -P$ and $x - h = 0$ i.e. $(h, k - p)$
- directrix $y - k = P \Rightarrow y = k + P$
- axis $x - h = 0 \Leftrightarrow x = h$

10. Find the latus rectum, focus, directrix, axis and vertex of parabola given by each of the following

a) $y = 4(x-1)^2 + 2$ c) $3x - 2 = (y+3)^2$

b) $3x + 2y^2 + 8y - 4 = 0$ d) $2x^2 - 8x - 4y + 3 = 0$

Solution: $y = 4(x-1)^2 + 2 \Leftrightarrow y - 2 = 4(x-1)^2$

$\Rightarrow (x-1)^2 = \frac{1}{4}(y-2)$

i) latus rectum $4P = \frac{1}{4}$. Hence $P = \dfrac{1}{16}$

ii) Focus, $y - 2 = \dfrac{1}{16}$ and $x - 1 = 0$ $\therefore \left(1, \dfrac{33}{16}\right)$

iii) directrix, $y - 2 = -\dfrac{1}{16} \Leftrightarrow y = 2 - \dfrac{1}{16} = \dfrac{31}{16}$

iv) axis, $x - 1 = 0 \Leftrightarrow x = 1$

v) vertex (1, 2)

b) $2y^2 + 8y + 3x - 4 = 0 \Leftrightarrow 2y^2 + 8y = -3x + 4$

$\Rightarrow 2(y^2 + 4y + 4) = -3x + 4 + 8$

$\Rightarrow 2(y + 2)^2 = -3(x - 4)$

h+P = x

$\therefore (y + 2)^2 = \dfrac{-3}{2}(x - 4)$

i) latus rectum $4P = \dfrac{3}{2}$, hence $P = \dfrac{3}{8}$

ii) focus, $x - 4 = \dfrac{-3}{8}$ and $y + 2 = 0$

$\therefore$ Focus $\left(\dfrac{29}{8}, -2\right)$

iii) directrix, $x - 4 = \dfrac{3}{8} \Rightarrow x = \dfrac{35}{8}$

iv) axis, $y + 2 = 0 \Rightarrow y = -2$

v) vertex (4, -2)

c) $3x - 2 = (y + 3)^2 \Leftrightarrow (y + 3)^2 = 3(x - \frac{2}{3})$

i) latus rectum $4P = 3$, Hence focal length $P = \dfrac{3}{4}$

ii) Focus, $x - \dfrac{2}{3} = \dfrac{3}{4}$, and $y + 3 = 0 \therefore$ Focus $\left(\dfrac{17}{12}, -3\right)$

iii) Directrix, $x - \dfrac{2}{3} = -\dfrac{3}{4} \Leftrightarrow x = \dfrac{2}{3} - \dfrac{3}{4} = \dfrac{-1}{12}$

iv) axis, $y + 3 = 0 \Leftrightarrow y = -3$

v) vertex, $\left(\dfrac{2}{3}, -3\right)$

x - h = -P

h = -5

d) *Exercise left for you.*

11. Find an equation of parabola with the given properties

a) Focus (-5, -2); directrix, $x = 3$

b) Focus (3, 3), vertex (3, 2)

c) Vertex (6, 3), directrix, $x = 4$

d) Vertex (3, -2), passes through (1, 6), axis parallel to y axis

Solution: a) the distance between the focus and directrix is 2P

$\therefore$ 2P = 3 + 5 = 8 $\Rightarrow$ P = 4

$\therefore$ vertex = (-5 + 4, -2) = (-1, -2)

$\therefore$ the equation $(y + 2)^2 = -16(x + 1)$

b) The distance between focus and vertex is P

$\therefore P = 3 - 2 = 1 \qquad \Rightarrow 4P = 4$

$\therefore$ the equation of parabola: $(x - 3)^2 = 4(y - 2)$

c) The distance between directrix and vertex is P

$P = 6 - 4 = 2 \qquad \therefore 4P = 8$

$\therefore$ the equation of parabola: $(y - 3)^2 = 8(x - 6)$

d) The equation of parabola for the axis parallel to y axis is given by

$(x - h)^2 = 4P(y - k)$ or $(x - h)^2 = -4P(y - k)$

$\therefore (x - 3)^2 = 4P(y + 2)$

$$\Leftrightarrow (1 - 3)^2 = 4P(6 + 2) \Leftrightarrow 4P = \frac{4}{8} = \frac{1}{2}$$

$\therefore (x - 3)^2 = \frac{1}{2}(y + 2)$

12. Let $y = ax^2 + bx + c$

Find a) Latus rectum b) focal length c) vertex d) focus

Solution: $y = ax^2 + bx + c$

$$\Leftrightarrow y = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

$$\Leftrightarrow y = a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right]$$

$$\Leftrightarrow y = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

$$\Leftrightarrow y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{1}{a}\left(y + \frac{b^2 - 4ac}{4a}\right)$$

a) Latus rectum, $4P = \frac{1}{a}$ c) Vertex $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$

b) Focal length, $P = \frac{1}{4a}$

13) a) A parabolic reflector is to be constructed with diameter 12 meter and its depth 2m

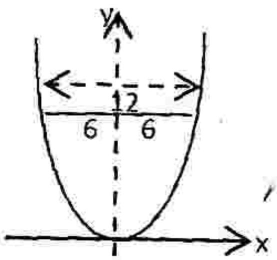
Find a) focus b) the equation

c) how wide is at the focus

d) how wide is at 4m from the vertex.

Solution: Assume the vertex origin (0,0)

$\Rightarrow X^2 = 4Py$



- Diameter 12m $\Rightarrow$ radius 6m

$\Rightarrow$ (6,2) on parabola

$\Rightarrow 36 = 4p(2) \Rightarrow P = \frac{36}{8} = \frac{9}{2} \leftarrow$ focal length

- Focus $\left(0, \frac{9}{2}\right)$

b) $x^2 = 4\left(\frac{9}{2}\right)y \Rightarrow x^2 = 18y$

c) $4p = 18m \leftarrow$ The opening at the focus (length of lotus rectum)

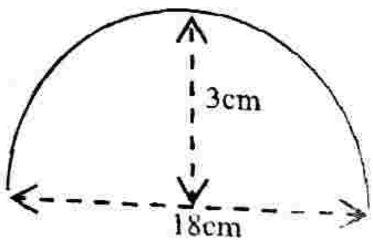
d) when $y = 4 \Rightarrow x^2 = (18)(4) = 72 \Rightarrow x = 6\sqrt{2} \Rightarrow 12\sqrt{2} \leftarrow$ is wider.

II) The dimension of a parabolic head light lamp reflector are 18cm wide and 3cm deep

a) Find a) focal length

b) The equation c) How wide is at the focus

Solution: vertex origin (0,0)



- Use $x^2 = 4py$
- Diameter = 18 $\Rightarrow$ radius = 9
- (9, − 3) is point on parabola

$\Rightarrow 9^2 = 4P(-3) \Rightarrow -12P = 81 \Rightarrow P = \frac{-27}{4}$

$\Rightarrow$ Focus $\left(0, \frac{-27}{4}\right)$

b) $x^2 = 4Py \Rightarrow x^2 = 4\left(\frac{-27}{4}\right)y$

$\therefore x^2 = -27y \leftarrow$ the equation.

c) $|4P| = 27 \leftarrow$ the width at the focus (length of latus rectum)

C) Find the equation of parabola:

i) vertex (5,7); focus (2,7) ii) vertex (-4 2); focus (1,2)

iii) vertex (1,3); focus (1,5) iv) vertex (– 2, 9); focus (– 2, 7)

Solution: If y component value is the same then it is horizontal parabola, use $(y - k)^2 = 4P(x - h)$, and $P = F - V$ of x components.

- If X component value is the same then it is vertical parabola, use $(x - h)^2 = 4P(y - k)$ and $P = F - V$ of y, components.

i) $h = 5, k = 7$ and $P = 2 - 5 = -3$,
$\therefore (y - 7)^2 = -12(x - 5) \leftarrow$ the same y component; horizontal parabola

ii) $h = -4, k = 2, p = 1 - (-4) = 5$, and the same y components $\therefore (y - 2)^2 = 20(x + 4)$

iii) $h = 1, k = 3, P = F - v = 5 - 3 = 2$, the same x components
$\therefore (x - 1)^2 = 8(y - 3)$

iv) $h = -2, k = 9, P = 7 - 9 = -2$
$\therefore (x + 2)^2 = -8(y - 9)$

d) Find the equation of parabola

i) vertex is at (– 3,5), axis parallel to the x – axis and passing through the point (5,9).

ii) vertex is at (2,3); passes through (4,6), vertical axis.

iii) vertex (– 5, –3); axis, y = – 3, P = 2

iv) focus (3,1); P = 5, vertical axis

Solution: Here

i) $(y - 5)^2 = 4P(x+3)$, since (5,9) on the parabola

thus $(9 - 5)^2 = 4P(5+3) \Rightarrow 16 = 32P \Rightarrow P = \frac{1}{2}$

$\therefore (y-5)^2 = 2 \qquad (x+3) \leftarrow$ equation of parabola

ii) $(x-2)^2 = 4P(y-3)$, since (4,6) on parabola

thus, $(4-2)^2 = 4P(6-3) \Rightarrow 4 = 12P \Longrightarrow P = \frac{1}{3}$ $\quad x = h + P$

$\therefore (x-2)^2 = \frac{4}{3}(y-3) \leftarrow$ Equation of vertical of vertical $\quad P = x - h$

parabola. $\quad P = h + \quad P = 2 - 5$

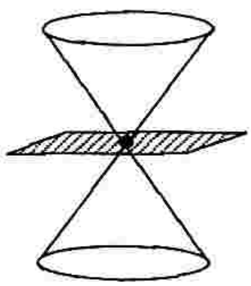
iii) $(y+3)^2 = 4P(x+5)$, since $P = 2$

$\therefore (y+3)^2 = 8(x+5)$

iv) Since it is vertical parabola, thus $V(3, 1-5) = (3, -4)$

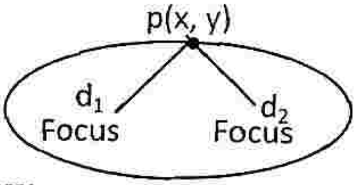
$\therefore (x-3)^2 = 20(y+4)$

The Ellipse



An ellipse is another of conic sections formed when plane intersects aright circular cone.

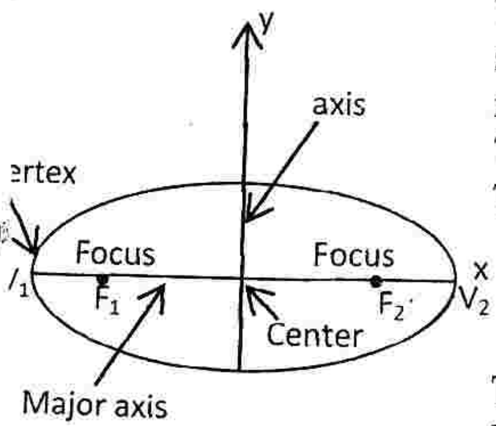
If the plane intersects the cone at the vertex of the cone so that the resulting figure is a point, the point is degenerate ellipse.



Definition

An ellipse is the set of all points p(x, y) in the plane the sum of whose distances from two fixed point (foci) is a positive constant.

Ellipses with center at (0, 0)



The graph of an ellipse has two axes of symmetry. The longer axis is called the major axis.

The foci of the ellipse are on major axis

The shorter axis is called the minor axis.

The length of major axis = 2a

The length of minor axis = 2b

The distance between foci = 2c

The mid point of major axis is the centre

The end point of major axis is the vertices

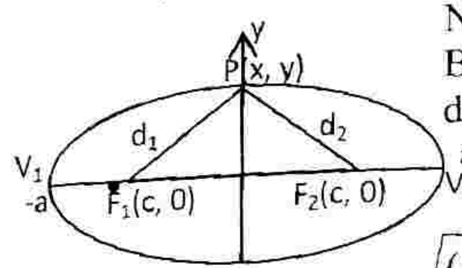
Ve not that the sum of distance from a point on the ellipse to each focus s constant

Now by hypothesis

$d_1 + d_2 = v_2F_1 + v_2F_2$

$$\sqrt{c^2+b^2}+\sqrt{c^2+b^2}=(a+c)+(a-c)$$

$$2\sqrt{c^2+b^2}=2a \Rightarrow a^2=b^2+c^2 \Rightarrow c^2=a^2-b^2$$



Now let P(x, y) be any point on the ellipse. By using definition of ellipse we have $d(PF_1)+d(PF_2)=2a$

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

$\Rightarrow \left[\sqrt{(x+c)^2+y^2}\right]=\left[2a-\sqrt{(x-c)^2+y^2}\right]^2$ .... subtracting and squaring both side.

$\Rightarrow (x+c)^2+y^2=4a^2=4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$

$\Rightarrow 4cx-4a^2=-4a\sqrt{(x-c)^2+y^2}$

$\Rightarrow [-cx+a^2]^2=\left[a\sqrt{(x-c)^2+y^2}\right]^2$ .... Divide by (-4) and then square each side.

$\Rightarrow c^2x^2-2cxa^2+a^4=a^2x^2-2cxa^2+a^2c^2+a^2y^2$

$\Rightarrow -a^2x^2+c^2x^2-a^2y^2=-a^4+a^2c^2$ .... Rewrite with x and y terms on the left side. $\Rightarrow -(a^2-c^2)x^2-a^2y^2=-a^2(a^2-c^2)$

From hypothesis, $b^2=a^2-c^2$

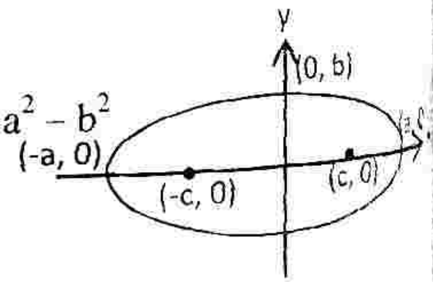
$\Rightarrow -b^2x^2-a^2y^2=-a^2b^2$ .... Divide each side by $-a^2b^2$

$\Rightarrow \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ ...... An equation of an ellipse with center at (0, 0)

Standard equation of an ellipse with unter (0, 0)

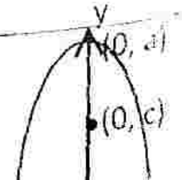
i) Major axis on the x-axis

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, for $a \geq b > 0$, where $c^2=a^2-b^2$



- Focas (c, 0) and (-c, 0)
- vertices are (-a, 0) and (a, 0)
- The length of latus rectum $\dfrac{2b^2}{a}$

ii) Major Axis on the y-axis (vertical ellipse)



$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \text{ for } a \geq b > 0$$

- Foci are (0, c) and (0, -c)
- vertices are (0, a) and (0, -a)
- length of latus rectum $\frac{2b^2}{a}$

Illustrative example

13. Find the vertices and foci of the ellipse given by

a) $\frac{x^2}{25}+\frac{y^2}{49}=1$ b) $x^2+4y^2=8$ c) $3x^2+y^2=3$

Solution: a) Because the y^2 term has the larger denominator, the major axis is on the y-axis

$a^2 = 49$, $b^2 = 25$, $c^2 = a^2 - b^2 = 49 - 25 = 24$

$a = 7$ $b = 5$ $c = \sqrt{24} = 2\sqrt{6}$

- The vertices are (0, 7) and (0, -7)
- the foci are $(0,\ 2\sqrt{6})$ and $(0,\ -2\sqrt{6})$

b) Divide both sides of the equation by 8, we obtain

$$x^2+4y^2=8 \Leftrightarrow \frac{x^2}{8}+\frac{y^2}{2}=1$$

Because the x^2 term has the larger denominator the major axis is on the x-axis.

Thus $a^2 = 8 \Rightarrow a = \sqrt{8} = 2\sqrt{2}$

$b^2 = 2 \Rightarrow b = \sqrt{2}$

Therefore the foci lie on the x-axis and

$$c=\sqrt{a^2-b^2}=\sqrt{8-2}=\sqrt{6}$$

- Consequently the foci are the points $(-\sqrt{6},\ 0)$ and $(\sqrt{6},\ 0)$
- The vertices are $(-2\sqrt{2},\ 0)$ and $(2\sqrt{2},\ 0)$

c) $3x^2+y^2=3$ write as $\frac{3x^2}{3}+\frac{y^2}{3}=\frac{3}{3} \Rightarrow x^2+\frac{y^2}{3}=1$

Since y^2 term has larger denominator the major axis is
axis.

Therefore, $a^2 = 3 \Rightarrow a = \sqrt{3}$

$b^2 = 1 \Rightarrow b = 1$

Also, $c = \sqrt{a^2 - b^2} = \sqrt{3 - 1} = \sqrt{2}$

Thus foci $(0, \sqrt{2})$ and $(0, -\sqrt{2})$

Vertices $(0, \sqrt{3})$ and $(0, -\sqrt{3})$

14. Find the equation of the ellipse having center at the origin
   a) a focus at (2, 0) and the vertex (-3, 0)
   b) a focus at (4, 0) and the length of major axis is 10
   c) a focus at (0, 4) and the length of minor axis is 6

Solution

a) $c = 2$, and $a = 3$ and $b^2 = a^2 - c^2 = b^2 = 9 - 4 = 5$

Thus $\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1$

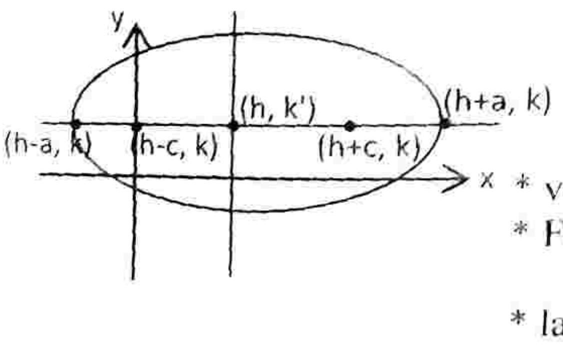
b) $c = 4,\ 2a = 10 \Rightarrow a = 5$ and $b^2 = a^2 - c^2 = 25 - 16 = 9$

Thus $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

c) $c = 4,\ 2b = 6 \Rightarrow b = 3$ and $b^2 = a^2 - c^2 = 9 = a^2 - 16 \Rightarrow$

Thus $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$

Standard equation of the ellipse with center at (h, k)



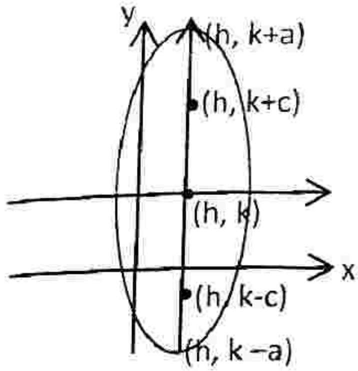
i) Major axis is parallel to the x center (h, k)

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$c^2 = a^2 - b^2$ and $a > b$

* vertices are $(h - a, k)$ and $(h + a, k)$
* Foci are $(h - c, k)$ and $(h + c, k)$
* latus rectum $\dfrac{2b^2}{a}$

ii) Major axis parallel to the center (h, k)



$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1,\ a>b,\ c^2=a^2-b^2$$

7/25

* vertices are (h, k+a) and (h, k-a)
* foci are (h, k+c) and (h, k-c)
* length of latus rectom $\frac{2b^2}{a}$
* eccentercity, $e=\frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}$

15. Find the equation of the ellipse having the given properties
    a) The foci are (2, 3) and (2, -3) and the length of the minor axis is 10
    b) The foci are (2, 1) and (2, -1) and the length of the major axis is 4.
    c) Center at (1, 2), one focus at (-4, 2) and passing through the point (4, 6)

Solution: a) center $=\left(\frac{2+2}{2},\frac{3+(-3)}{2}\right)=(2,0)$

$$b=5,\ 2c=6\Rightarrow c=3$$

Thus $b^2=a^2-c^2=25=a^2-9\Rightarrow a^2=34$

Therefore, $\frac{(x-2)^2}{25}+\frac{(y-0)^2}{34}=1$

b) $2c=2\Rightarrow c=1$ and $2a=4\Rightarrow a=2$
Therefore, $b^2=a^2-c^2\Rightarrow b^2=4-1=3$
Center (h, k) = (2, 0)

Thus, $\frac{(x-2)^2}{3}+\frac{y^2}{4}=1$

c) Let (m, n) be other side of the focus. Consequently follows;

$$\frac{-4+m}{2}=1,\text{ and }\frac{2+n}{2}=2$$

$m=6$ and $n=2$

Thus $2c=6-(-4)=10\Rightarrow c=5$

$$\Rightarrow \frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1$$

$$\frac{(4-1)^2}{a^2} + \frac{(6-2)^2}{b^2} = 1 \Rightarrow \frac{9}{a^2} + \frac{16}{b^2} = 1 \text{ and } b^2 = a^2 - c^2$$

$$\Rightarrow b^2 = a^2 - 25$$

$$\Rightarrow \frac{9}{a^2} + \frac{16}{a^2-25} = 1 \Rightarrow 9a^2 - 225 + 16a^2 = a^2(a^2 - 25)$$

$\Rightarrow 25a^2 - 225 = a^4 - 25a^2 \Rightarrow 50a^2 - 225 = a^4 \Rightarrow a^4 - 50a^2 +$
$= 0$

$\Rightarrow (a^2 - 45)(a^2 + 5) = 0 \Rightarrow a^2 = 45$ and $a^2 = -5$

We take $a^2 = 45$

Thus $\dfrac{(x-1)^2}{45} + \dfrac{(y-2)^2}{20} = 1$

16. Find the vertices, foci and latus rectum of the ellipse
   a) $3x^2 + y^2 - 6x + 6y + 13 = 0$
   b) $4x^2 + 9y^2 - 8x + 36y + 4 = 0$

Solution: $3x^2 + y^2 - 6x + 6y + 13 = 0$

$3x^2 - 6x + y^2 + 6y + 13 = 0$

$3(x^2 - 2x + 1) + y^2 + 6y + 9 + 13 - 9 - 3 = 0$

$3(x-1)^2 + (y+3)^2 = -1$

Therefore, the equation does not represent an ellipse.

b) $4x^2 + 9y^2 - 8x + 36y + 4 = 0$

$4x^2 - 8x + 9y^2 + 36y + 4 = 0$

$4(x^2 - 2x + 1) + 9(y^2 + 4y + 4) + 4 - 4 - 36 = 0$

$4(x-1)^2 + 9(y+2)^2 = 36 \Rightarrow \dfrac{(x-1)^2}{9} + \dfrac{(y+2)^2}{4} = 1$

- center (1, -2), $a^2 = 9$. Thus a = 3, thus the vertices
  3, -2) and (1-3, -2) = (4, -2) and (-2, -2)
- foci are $(1 + \sqrt{5}, -2)$ and $(1 - \sqrt{5}, -3)$
- latus rectum $\dfrac{2b^2}{a} = \dfrac{2(4)}{3} = \dfrac{8}{3}$

17. Find the coordinates for the center, vertices, end point of minor axis, length of latus rectum and foci of the ellipse given $2x^2 + 3y^2 + 8x - 18y - 1 = 0$

$$\Rightarrow \frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1$$

$$\frac{(4-1)^2}{a^2} + \frac{(6-2)^2}{b^2} = 1 \Rightarrow \frac{9}{a^2} + \frac{16}{b^2} = 1 \text{ and } b^2 = a^2 - c^2$$

$$\Rightarrow b^2 = a^2 - 25$$

$$\Rightarrow \frac{9}{a^2} + \frac{16}{a^2 - 25} = 1 \Rightarrow 9a^2 - 225 + 16a^2 = a^2(a^2 - 25)$$

$\Rightarrow 25a^2 - 225 = a^4 - 25a^2 \Rightarrow 50a^2 - 225 = a^4 \Rightarrow a^4 - 50a^2 + 225 = 0$

$\Rightarrow (a^2 - 45)(a^2 + 5) = 0 \Rightarrow a^2 = 45$ and $a^2 = -5$

We take $a^2 = 45$

Thus $\dfrac{(x-1)^2}{45} + \dfrac{(y-2)^2}{20} = 1$

16. Find the vertices, foci and latus rectum of the ellipse
a) $3x^2 + y^2 - 6x + 6y + 13 = 0$
b) $4x^2 + 9y^2 - 8x + 36y + 4 = 0$

Solution: $3x^2 + y^2 - 6x + 6y + 13 = 0$

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$3(x^2 - 2x + 1) + y^2 + 6y + 9 + 13 - 9 - 3 = 0$

$3(x-1)^2 + (y+3)^2 = -1$

Therefore, the equation does not represent an ellipse.

b) $4x^2 + 9y^2 - 8x + 36y + 4 = 0$

$4x^2 - 8x + 9y^2 + 36y + 4 = 0$

$4(x^2 - 2x + 1) + 9(y^2 + 4y + 4) + 4 - 4 - 36 = 0$

$4(x-1)^2 + 9(y+2)^2 = 36 \Rightarrow \dfrac{(x-1)^2}{9} + \dfrac{(y+2)^2}{4} = 1$

- center (1, -2), $a^2 = 9$. Thus a = 3, thus the vertices are (1 + 3, -2) and (1-3, -2) = (4, -2) and (-2, -2)
- foci are (1 + $\sqrt{5}$, -2) and (1 - $\sqrt{5}$, -3)
- latus rectum $\dfrac{2b^2}{a} = \dfrac{2(4)}{3} = \dfrac{8}{3}$

17. Find the coordinates for the center, vertices, end point of the minor axis, length of latus rectum and foci of the ellipse given by $2x^2 + 3y^2 + 8x - 18y - 1 = 0$

Solution: First, we begin by arranging the equation in standard form and completing the square.

$(2x^2 + 8x) + (3y^2 - 18y) = 1$

$2(x^2 + 4x + 4)+3(y^2-6y + 9) = 1 + 8 + 27$

$2(x + 2)^2 + 3(y - 3)^2 = 36$

$$\frac{(x+2)^2}{18}+\frac{(y-3)^2}{12}=1$$

$2(x^2+4x)+3(y^2-6y)=1$

$16x-9y=1$

Now, $a^2 = 18$ and $b^2 = 12$

$a = \sqrt{18}$ and $b = \sqrt{12}$

$a = 3\sqrt{2}$ and $b = 2\sqrt{3}$

also, $c = \sqrt{18-12} = \sqrt{6}$

Thus center (-2, 3)

- vertices $(-2 - 3\sqrt{2}, -3 + 3\sqrt{2})$
- endpoint of minor axis $(-2,3 + 2\sqrt{3})$ and $(-2,3-2\sqrt{3})$
- foci: $(-2 - \sqrt{6}, 3)$ and $(-2 + \sqrt{6}, 3)$
- length of latus rectum $\frac{2b^2}{a}=\frac{2(12)}{3\sqrt{2}}=\frac{8}{\sqrt{2}}$
- eccentercity, $e = \frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}=\frac{\sqrt{18-12}}{\sqrt{18}}=\sqrt{\frac{1}{3}}$

Eccentricity of an Ellipse

The graph of an ellipse can be very long and thin or it can be much like a circle.

The eccentricity of an ellipse is a measure of its "roundness."

Definition: The eccentricity of an ellipse is denoted bye and is defined as

$$e=\frac{distance\ between\ the foci}{length\ of\ the\ major\ axis}$$

That is $e=\frac{2c}{2a}=\frac{c}{a}$

Note, $0 < e < 1$

When $e \approx 0$, the graph is circle

When e ≈ 1, the graph is long and thin.

Example

18. Find the eccentricity of the ellipse given by $8x^2 + 9y^2 = 18$

Solution: First, write the equation of the ellipse in standard form.
Divide each side of the equation by 18

$$\frac{8x^2}{18} + \frac{9y^2}{18} = 1 \Leftrightarrow \frac{4x^2}{9} + \frac{y^2}{2} = 1 \Rightarrow \frac{x^2}{\frac{9}{4}} + \frac{y^2}{2} = 1$$

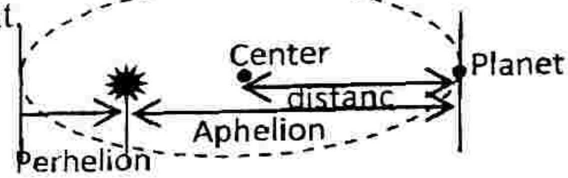
$$\Rightarrow a^2 = \frac{9}{4} \text{ and } a = \frac{3}{2}$$

Use the definition $c^2 = a^2 - b^2$ to find c

$$c^2 = \frac{9}{4} - 2 = \tfrac{1}{4} \text{ and } c = \sqrt{\frac{1}{4}} = \frac{1}{2} \Rightarrow e = \frac{c}{a} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$$

Application

The planets travel around the sun in elliptical orbit. The sun is located at a focus of the orbit.



The terms per in e lion and a phelion are used denote the position of planet in ts orbit around the sun.
Note, the length of semi major axis of a planet's elliptical orbit is called the mean distance of the planet from the sum

Illustrative example

19. Earth has mean distance of 93 million miles and perihelion distance of 91.5 million miles. Find an equation of Earth's orbit

Solution: Semi major axis = a = 93
Aphelion distance = 2(93) – 91.5 = 94.5
c = aphelion distance – a
c = aphelion distance -93 = 94.5 – 93 = 1.5 million mil

The length of b of the semi minor axis of the orbit is

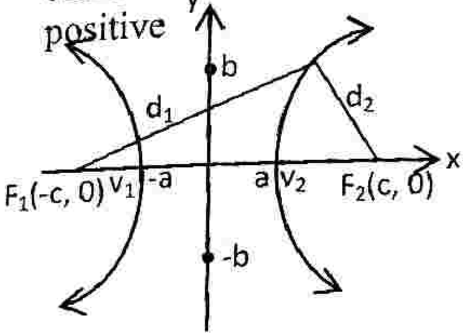
$$b = \sqrt{a^2 - c^2} = \sqrt{93^2 - 1.5^2} = \sqrt{8646.75}$$

Therefore an equation of Earth's orbit is $\dfrac{x^2}{93^2} + \dfrac{y^2}{8646.75} = 1$

3. Hyperbolas

Hyperbola is formed when plane intersects right circular cone parallel to the axis of the cone.

Definition: A hyperbola is the set of all points P(x, y) in the plane, the **difference between** whose distance from two fixed points (foci) is positive



- $|d_1 - d_2| = 2a$
- $\overline{v_1 v_2}$. *The transverse axis = 2a*
- *length of conjugate axis = 2b*
- $\overline{F_1 F_2}$, *distance between foci = 2c*

Find $|v_2F_1 - v_2F_2|$

Solution: $\overline{v_2 F_1} = a + c$ and $v_2F_2 = c - a$

Thus, $|v_2F_1 - v_2F_2| = |a + c - (c - a)| = 2a$

- The line through the two foci is an axis of symmetry

Note: i) If the axis of the hyperbola is parallel to the x-axis, then it is called an x-hyperbola or horizontal hyperbola.

ii) If the axis of the hyperbola is parallel to the y-axis, then it is called y-hyperbola or vertical hypherbola.

To find the standard equation using distance formula

For any point P(x, y) on the hyperbola, let

$d_1 = pF_1$, and $PF_2 = d_2$

$d_1 = \sqrt{(x+c)^2 + y^2}$ and $d_2 = \sqrt{(x-c)^2 + y^2}$

since $|PF_1 - PF_2| = |d_1 - d_2| = 2a$, we deduce that

$$\left|\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}\right| = 2a$$

$$\Leftrightarrow \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a$$

$$\Leftrightarrow \sqrt{(x+c)^2+y^2} \pm 2a + \sqrt{(x-c)^2+y^2}$$

Squaring both side, simplifying, we get

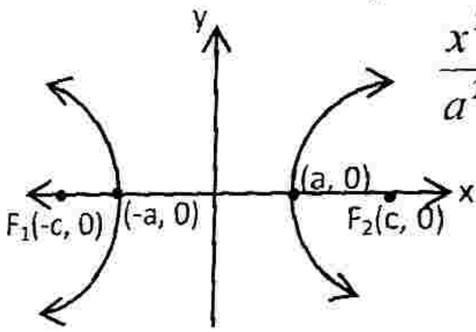
$$cx^2 - a^2 = \pm a\sqrt{(x-c)^2+y^2} \Leftrightarrow (cx - a^2)^2 = a^2((x-c)^2+y^2)$$

Expanding and collecting like terms, we have
$(c^2 - a^2)x^2 - a^2y^2 = a^2)c^2 - a^2)$
Since $c > a$ and letting $c^2 - a^2 = b^2$, $b > 0 \Rightarrow b^2x^2 - a^2y^2 = a^2b^2$
Dividing both side by a^2b^2, we get the coordinate of every point on the hyperbola satisfies the equation

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

Which is called standard equation of x-hyperbola with center origin.
Standard forms of the equation of hyperbola with center at the origin

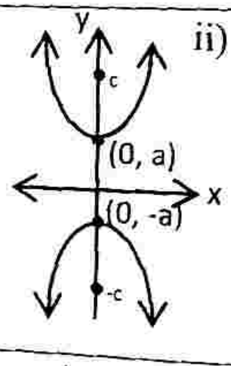
i) Transverse axis on the x-axis



$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ - vertice are (a, 0) and (-a, 0)

- Foci are (c, 0) and (-c, 0)
- $c^2 = a^2 + b^2$

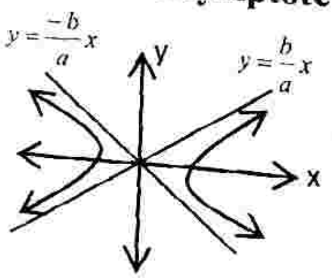
ii) Transverse axis on the y-axis



$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

- vertices are (0, a) and (0. -a)
- foci are (0, c) and (0, -c)
- $c^2 = a^2 + b^2$

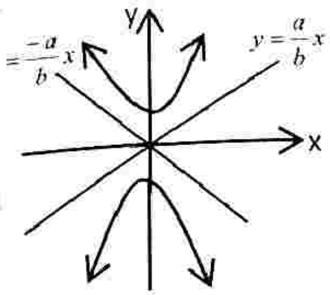
Asymptote of a hyperbola with center at origin



- Each hyperbola has two asymptote that pass through the center of hyperbola
- *The asymptotes of the hyperbola defined by*

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are given by the equation

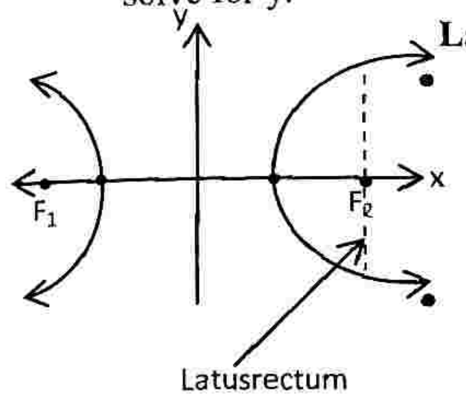
$$y=\frac{b}{a}x \text{ and } y=-\frac{b}{a}x$$



- *The asymptotes of the hyperbola defined by* $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ are given by the equation

$$y=\frac{a}{b}x \text{ and } y=-\frac{a}{b}x$$

Note:- To find asymptote equation let 0 by 1 or replace 1 by 0 and then solve for y.



Latus rectum

- The line segment that pass through a focus of a hyperbola, is perpendicular to the principal axis, and has its end points on the hyperbola is called a latus rectum.
- Length of latus rectum for hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{2b^2}{a}$

Illustrative Example

20. Find the vertices, foci, and asymptote, and latus rectum of each equation

a) $\frac{x^2}{1}-\frac{y^2}{4}=1$ b) $\frac{y^2}{9}-\frac{x^2}{4}=1$

Solution: a) Because the x^2 term is positive, the trans verse axis on x –axis, we know

$a^2 = 1$; thus $a = 1$

$b^2 = 4$, thus, $b = 2$, and $c^2 = a^2 + b^2 = 1 + 4 = 5$

vertices are (1, 0) and (-1, 0)

$\Rightarrow$ foci: ($\sqrt{5}$, 0) and (-$\sqrt{5}$, 0)

More over, the asymptote are the line y = 2x and y = -2x

- length of latus rectum = $\frac{2(4)}{1}=8$

b) Because the y^2 terms is positive, the transverse axis on y-axis, and $a^2=9$, thus a = 3, and $b^2=4$ thus b=2

$* \ c^2 = a^2 + b^2 = 9 + 4 = 13 \Rightarrow c = \sqrt{13}$

Vertice (0, 3) and (0, -3)

Foci are $(0, \sqrt{13})$ and $(0, -\sqrt{13})$.

Asymptote $y = \frac{a}{b}x \ and \ y = \frac{-a}{b}x$

$$\Rightarrow y = \frac{3}{2}x \ and \ y = \frac{-3}{2}x$$

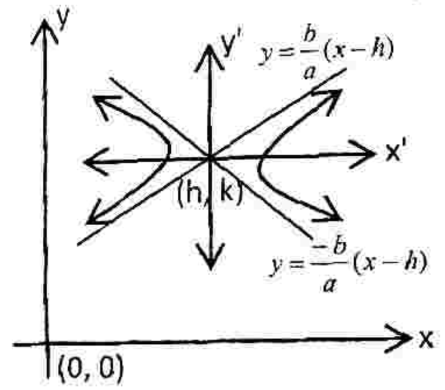
Length of latus rectum is $\frac{2b^2}{a} = \frac{2(4)}{3} = \frac{8}{3}$

Note: Ayperbola for which a = b is called equilateral

Ayperbola is equilateral if and only if its asymptote are perpendicular.

Hyperbola with center at (h, k)

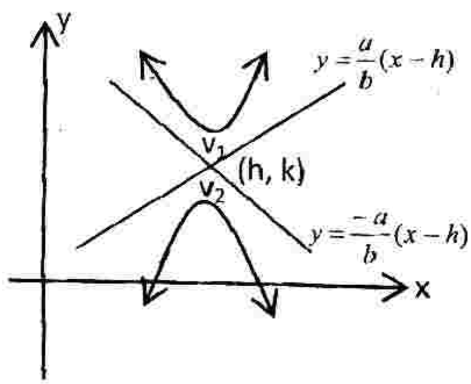
- Standard forms of the equation of hyperbola with center at (h, k)



i) Transverse axis parallel to the x-axis

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

- vertices are $v_1(h + a, k)$ and $v_2(h-a, k)$
- foci are $F_1(h - c, k)$ and $F_2(h - c, k)$
- $c^2 = a^2 + b^2$
- asymptote are $y - k = \pm\frac{b}{a}(x-h)$
- length of latus rectum $\frac{2b^2}{a}$
- eccentricity $e = \frac{c}{a} = \frac{\sqrt{a^2+b^2}}{a}$



ii) transverse axis parallel to y-axis

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Solution: $m_1 = \frac{4}{3}$, $m_2 = \frac{3}{2}$, angle between L_1 and L_2 from l_1 to l_2

$$\tan\alpha = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{\frac{3}{2} - \frac{4}{3}}{1 + \frac{3}{2}\cdot\frac{4}{3}} = \frac{1}{18}$$

65) The floor of conference hall is made in a shape of ellipse. If the largest chord has length 20 meters and the smallest is 16m what is the length between the foci?

A. 24m B. 12m C. $9\sqrt{2}$ m D. 6m

Solution: $2a = 20 \Rightarrow a = 10 \Rightarrow a^2 = 100$

$2b = 16 \Rightarrow b = 8 \Rightarrow b^2 = 64$

- $c^2 = a^2 - b^2 = 100 - 64 = 36 \Rightarrow c = 6 \Rightarrow 2c = 12$ **Answer: B**

66) What is an asymptote of the hyperbola $4x^2 - y^2 + 2y = 5$ ……………. EHEECE.

A. $y = -2x + 1$ C. $y = \frac{-1}{2}x + 1$

B. $y = 2x - 1$ D. $y = \frac{1}{2}x - 1$

Solution: $4x^2 - y^2 + 2y = 5 \Rightarrow 4x^2 - (y^2 - 2y + 1)$

$$\Rightarrow 4x^2 - (y-1)^2 = 4 \Rightarrow x^2 - \frac{(y-1)^2}{4} = 1$$

$\Rightarrow$ Asymptote: $y - 1 = \pm 2x \Rightarrow y = 2x+1$ and $y = -2x+1$

Answer:

67) For what value of K is $x^2 + y^2 + 6x - 4y + k = 0$ represent a circle …EHEECE.

A. For all $k \geq 0$ C. For all $K < 13$

B. For all $K \geq 10$ D. for $K = 16$

Solution: $x^2 + 6x + y^2 - 4y = -K$

$\Rightarrow (x+3)^2 + (y-2)^2 = -k + 9 + 4 \Rightarrow$ circle if $13 - k > 0 \Rightarrow$ for $k < 13$, **Answer:**

68) What is the distance between the two foci of the hyperbola $4(y-1)^2 - 16(x+3)^2 = 1$….(EHEEC

A. $3\sqrt{2}$ B. $\frac{\sqrt{5}}{2}$ C. $\frac{1}{4}$ D. $\frac{1}{\sqrt{10}}$

Solution: $\Leftrightarrow \frac{(y-1)^2}{\frac{1}{4}} - \frac{(x+3)^2}{\frac{1}{16}} = 1 \Rightarrow a^2 = \frac{1}{4}, b^2 = \frac{1}{16}$

$\Rightarrow c^2 = a^2 + b^2 = \frac{1}{4} + \frac{1}{16} = \frac{5}{16} \Rightarrow c = \frac{\sqrt{5}}{4}$

$\Rightarrow 2c = (2)\left(\frac{\sqrt{5}}{4}\right) = \frac{\sqrt{5}}{2}$ .... **Answer: B**

69) Let l be line containing (1,5) and its angle of inclination be $\theta = 135^0$. What is the distance from the line ℓ to (– 1,1) … EHEECE

A. $2\sqrt{2}$ B. $3\sqrt{2}$ C. $4\sqrt{2}$ D. $6\sqrt{2}$

Solution: Slope of the line, $\tan\theta = \tan 135^0 = -1$

- equation of the line ℓ : $\frac{y-5}{x-1} = -1 \Rightarrow y - 5 = -x + 1$

$\Leftrightarrow y - 5 = -x + 1 \Leftrightarrow x + y = 0$

- Distance, from $(-1,1) + 0x + y - 6 = 0$ is

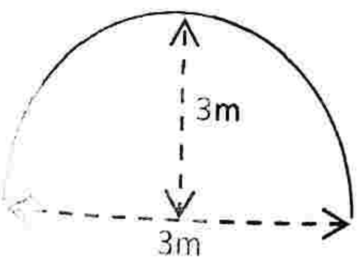
$$\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}} = \frac{|(-1)(1) + (1)(1) - 6|}{\sqrt{1+1}} = \frac{6}{\sqrt{2}} = 3\sqrt{2}$$

70) Consider a gate whose shape is a parabolic arch of height 3m and base width 3m If the origin of the coordinate system is at the vertex, the equation of the edge of the gate is ….EHEECE 2002/2010

A. $3x^2 + 3y = 0$ C. $y = -3x^2 + 1$

B. $4x^2 + 3y = 0$ D. $3y^2 - 4x = 0$

Solution: Vertex (0,0), the point $\left(\frac{3}{2}, -3\right)$ on parabola.



$\Rightarrow x^2 = 4py \Rightarrow \frac{9}{4} = 4P(-3)$

$\Rightarrow P = \frac{-3}{16}$

$$\Rightarrow x^2 - 4py = 0 \Rightarrow x^2 - 4\left(\frac{-3}{16}\right)y = 0$$

$$\Rightarrow x^2 + \frac{3}{4}y \Leftrightarrow 4x^2 + 3y = 0$$

Answer

71) The equation $x^2 - 4y^2 = -1$ represents… EHEECE 2002/10

Solution: $x^2 - 4y^2 = -1 \Rightarrow -x^2 + 4y^2 = 1$

$\Rightarrow 4y^2 - x^2 = 1 \leftarrow$ hyperbola. $\Rightarrow a^2 \frac{1}{4} \Rightarrow a = \frac{1}{2}, b^2 = 1 \Rightarrow b=$

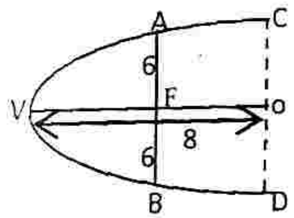
$$\Rightarrow c^2 = a^2 + b^2 = \frac{1}{4} + 1 = \frac{5}{4} \Rightarrow c = \frac{\sqrt{5}}{2}$$

$$\therefore \text{Vertice } \left(0, \pm\frac{1}{2}\right) \textit{ and foci } \left(0, \pm\frac{\sqrt{5}}{2}\right)$$

72) A cross – section of parabolic reflector is shown below. The opening at the focus $\overline{AB}$ is 12cm. What is the diameter of the opening $\overline{CD}$, 8cm from the vertex. . .UEE 2003/11.

A. $8\sqrt{6}$ cm B. 12cm C. 9cm D. $6\sqrt{8}$ cm

Solution:



- $4P = 12 \Rightarrow P = 3$, $\overline{VF} = P = 3$
- Assume the vertex origin.

$\Rightarrow y^2 = 4Px \Rightarrow y^2 = 12x$

When $x = 8 \Rightarrow y^2 = (12)(8) = 96$

$\Rightarrow y = \overline{OC} = 8\sqrt{6}$

Answer

Supplementary exercise

1. Find the distance between the points whose coordinates are given.
   a) $(\sqrt{3}, \sqrt{8}), (\sqrt{12}, \sqrt{27})$ b) (a, b), (-a, -b) c) (x, 4x), (-2x,
2. Find all points on the x-axis that are 10 units from (4, 6)
3. Find all points on the y-axis that are 12 units from (5, -3)
4. Find the equation of a line which passes through the point (and makes the intercepts on the x-axis equal in magnitude opposite in sign.

5. The vertices of a triangle are (2, 0), (0, 2) and (4, 6). Find the median through the first vertex.
6. Find the ratio in which the join of (-1, 0) and (-2, 3) is divided by the line $x + 2y = 3$
7. Find the acute angle between the lines:
   a) $3x + 2y = 11$ and $2x + y = -12$
   b) $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{a'} + \frac{y}{b'} = 1$
   c) $2x + y = 3$ and $3x - 2y = 5$
8. A circle passes through the points (-3, 1) and (-1, 5). Its centre lies on the x-axis

Find a) the coordinate of the center b) the equation of the circle

9. Find the coordinates of the circum center of a triangle whose coordinates are (7, -1),(5, 1) and (-3, -7). Find the circum center
10. If (-3, 2); (1, -2) and (5, 6) are the mid-points of the sides of a triangle, find the coordinates of the vertices of the triangle.
11. The points $\left(2, \frac{3}{2}\right), \left(-3, \frac{-7}{2}\right), \left(k, \frac{9}{2}\right)$ are collinear, find k
12. Find the equation of the two lines through the point (3, -2) and inclined at 60^0 to the straight line $\sqrt{3}x + y = 1$
13. If $x + y^2 + kxy + 8x - 6y + 9 = 0$ represents a circle state the value of k. substituting this value of k in the equation, find the center and radius of the circle.
14. Find the equation of the circle
    a) whose center lies on the x-axis and which passes through the points (-1, 0) and (5, 0)
    b) whose center lies on the y-axis and which passes through the points (0, 3) and (0, -7)
15. Find the equations of the circles passing though the following sets of points A(-2, 2), B(2, 2) and C(-2, -4)
16. Which of the points A(2, 5), B(1, -3) and C(1, 0) lies inside or outside the circle $x^2 + y^2 - 3x + 2y - 12 = 0$
17. Find the length of a tangent from (-3, 2) to the circle $x^2+y^2 = 3+2x -2y$
18. Find how the circle $x^2 + y^2 = 1$ and $x^2 + y^2 - 6x + 5 = 0$ touch each other, and find the coordinates of the point of contact.
19. Find the equation of the tangents to the circles

a) $x^2 + y^2 + 4x - 10y - 12 = 0$ at $(3, 1)$
b) $x^2 + y^2 - 6x - 3y - 16 = 0$ at $(-2, 0)$

20. Find the equation of the circle whose center lies on the line $x = 2y$, passes through the point $(2, 5)$, and for which the length of the tangent from the origin is $\sqrt{7}$

21. Find the equation of the parabola with vertex at the origin having its axis along the x-axis and passing through the point $(2, 3)$

22. Find latus rectum, focus, vertex, and directrix of each of the following

a)	$y^2 = 1x$	d)	$(x + 2)^2 = -(y + 2)$
b)	$x^2 = -9y$	e)	$x^2 = 4x + 2y - 1 = 0$
c)	$(y - 3)^2 = 6x - 12$	f)	$x = y^2 - 4y + 2$

23. Find an equation for the parabola that satisfies the given condition

A. vertex $(0, 0)$; focus $(3, 0)$
B. vertex $(0, 0)$, directrix $x = 7$
C. focus $(0, -3)$, directerix, $y = 3$
D. vertex $(1, 1)$, directerix, $y = -2$
E. vertex $(-1, 2)$, focus, $(-1, 3)$
F. vertex $(-4, 1)$, has its axis of symmetry parallel to the y-axis, and passes through the point $(-2, 2)$
G. focus $(3, -3)$ and directrix $y = -5$

24. Find the vertices and foci of the ellipse given

a)	$\frac{x^2}{16} + \frac{y^2}{25} = 1$	f)	$\frac{(x-1)^2}{21} + \frac{(y-3)^2}{4} = 1$
b)	$4x^2 + 9y^2 = 36$	g)	$\frac{9(x-1)^2}{16} + \frac{(y+1)^2}{9} = 1$
c)	$9x^2 + y^2 = 9$	h)	$3x^2 + 4y^2 = 12$
d)	$\frac{(x-3)^2}{9} + \frac{(y+2)^2}{16} = 1$	i)	$4x^2 + y^2 - 24x - 8y + 48 = 0$
e)	$\frac{(x+2)^2}{9} + \frac{y^2}{25} = 1$	j)	$x^2 + 9y^2 + 2x - 18y + 1 = 0$
k)	$4x^2 + y^2 + 8x - 10y + 13 = 0$		

25. Find an equation for the ellipse that satisfies the given condition.

a) Ends of major axis $(\pm 3, 0)$; end of minor axis $(0, \pm 2)$
b) length of major axis 26; foci $(\pm 5, 0)$

c) Foci (± 1, 0); $b = \sqrt{2}$

d) vertices (6, 0), (-6, 0); ellipse passes through (0, -4) and (0, 4)

e) major axis of length 12 on the x-axis center (0, 0), ellipse passes through (2, -3)

f) center (-2, 4), vertices (-6, 4) and (2, 4), foci at (-5, 4) and (1, 4)

g) center (2, 4) major axis parallel to the y-axis and length 10; ellipse passes through the point (3, 3)

h) Eccentricity $\frac{2}{5}$ major axis on the x-axis and of length 10 center at (0,0)

i) eccentricity $\frac{2}{3}$, foci (0, -4) and (0, 4)

j) eccentricity $\frac{2}{5}$, foci at (-1, 3) and (3, 3)

26. Find an equation for a hyperbola that satisfies the given conditions.

a) vertices (± 2, 0); foci (± 3, 0)

b) vertices (± 1, 0); asymptote $y = \pm 2x$

c) asymptote $y = \pm \frac{3}{2}x$; $b = 4$

d) foci (0, ± 5); asymptote $y = \pm 2x$

e) vertices (2, 4) and (10, 4); foci 10 units apart

f) asymptote $y = 2x+1$ and $y = -2x+3$; passes through the origin.

g) asymptote $y = ½x$ and $y = -\frac{1}{2}d$, vertices (0, 4) and (0, -4)

h) foci (1, -2) and (7, -2), slope of an asymptote $\frac{5}{4}$

i) Passing through (9, 4), slope of asymptote $\frac{1}{2}$, center (7, 2), transverse axis parallel to the y-axis

7. Use the eccentricity to find the equation in standard form of each hyperbola

vertices (1, 6) and (1, 8) hyperbola.

a) Eccentricity 2, foci (4, 0) and (-4, 0)

b) center (4, 1), conjugate axis of length 4 eccentricity

c) $\frac{4}{3}$ (there are two answer)

8. Use the definition of a hyperbola to find the equation of the hyperbola in standard form.

a) foci (2, 0) and (-2, 0); passes through the point (2, 3)

b) foci (0, 4) and (0, -4); passes through the point $\left(\frac{7}{3}, 4\right)$

9. Find the center, vertices, foci and asymptotes for the hyperbola given by each equation.

a) $\frac{x^2}{16} - \frac{y^2}{25} = 1$ d) $\frac{(x-3)^2}{16} - \frac{(y+4)^2}{9} = 1$

b) $\frac{y^2}{4} - \frac{x^2}{25} = 1$ e) $\frac{(y+2)^2}{4} - \frac{(x-1)^2}{16} = 1$

c) $\frac{4x^2}{9} - \frac{y^2}{16} = 1$ f) $\frac{(x+2)^2}{9} - \frac{y^2}{25} = 1$

g) $\frac{9(x-1)^2}{16} - \frac{(y+1)^2}{9} = 1$ h) $x^2 - y^2 = 9$

i) $x^2 - y^2 - 6x + 8x - 3 = 0$ j) $9x^2 - 4y + 36x - 8y + 68$

- $c^2 = a^2 + b^2 = 16 + 9 = 25 \Rightarrow c = 5$
- Distance c = 5

Answer: A

9) One of the asymptote of the hyperbola: $y^2 - x^2 + 4x - 5 = 0$
A. $y = x + 2$ B. $y = -x + 2$ C. $y = -x - 2$ D. $y = x - 4$
Solution: $y^2 - (x^2 - 4x + 4) = 5 - 4 = 1$

$$\Rightarrow \frac{y^2}{1} - \frac{(x-2)^2}{1} = 1 \Rightarrow a^2 = 1, b^2 = 1$$

- Asymptote: $y = \pm(x - 2) \Rightarrow y = -x + 2$ and $y = x - 2$

Answer: B

1) The equation of the circle with center at $(-3,1)$ and tangent to the circle $(x-1)^2 + (y+2)^2 = 1$ is . . . EHEECE
A. $(x + 3)^2 + (y - 1)^2 = 4$ C. $(x - 3)^2 + (y + 1)^2 = 16$
B. $(x+3)^2 + (y - 1)^2 = 25$ D. $(x + 3)^2 + (y - 1)^2 = 16$
Solution: Distance between $(-3,1)$ and $(1, -2)$

$$\Rightarrow r_1 + r_2 = \sqrt{(1+3)^2 + (-2-1)^2} = 5$$

$$r_2 = 5 - r_1 = 5 - 1 = 4$$

$\therefore$ the equation of circle $(x + 3) + (y - 1)^2 = 16$

2) An iron wire bent in shape of parabola has latus rectum of length 60cm. what is its focal length.
A. 60cm B. 30cm C. 15cm D. 7.5cm
Solution: length of latus rectum, $4P = 60 \Rightarrow P = 15$

3) Length of the major axis of the ellipse $x^2+9y^2-2x+18y + 1 = 0$ is
A. 3 B. $\sqrt{10}$ C. 6 D. 9
Solution:
$x^2 - 2x + 9y^2 + 18y = -1 \Rightarrow (x - 1)^2 + 9(y+1)^2 = -1 + 1 + 9$

$$\Rightarrow \frac{(x-1)^2}{9} + \frac{(y+1)^2}{1} = 1 \Rightarrow a^2 = 9 \Rightarrow a = 3 \Rightarrow 2a = 6$$

Answer: C

4) Let L_1,: $3y - 9 = 4x$ and L_2: $2y - 3x + 2 = 0$ represent two intersecting lines If α is the angle between L_1 and L_2. then what is $\tan\alpha$? . . . (EHEECE).
A. $\frac{3}{2}$ B. $\frac{4}{3}$ C. $\frac{1}{6}$ D. $\frac{1}{18}$

54) Find all values of F such that the graph of the equation $x^2 + y^2 -$ $6x - 10y + F = 0$ represent.

a) circle b) point circle c) no locus

Solution: $x^2 + y^2 - 6x - 10y + 25 = -F + 9 + 25$

$\Rightarrow x^2 - 6x + 9 + y^2 - 10y + 25 = -F + 9 + 25$

$\Rightarrow (x-3)^2 + (y-5)^2 = 34 - F$

a) circle, if $34 - F > 0 \Rightarrow F < 34$

b) point circle if $34 - F = 0 \Rightarrow F = 34$

c) no circle if $34 - F < 0 \Rightarrow F > 34$

55) Find the equation of circle whose center is (4,5) and which passes through the center of the circle $x^2 + y^2 + 4x + 6y - 12 = 0$

Solution: $x^2 + y^2 + 4x - 6y - 12 = 0 \Rightarrow (x+2)^2 + (y+3)^2 = 25$

$\Rightarrow$ center $(-2,-3) \Rightarrow (x-4)^2 + (y-5)^2 = r^2$, since $(-2,-3)$ is on

$\Rightarrow (-2-4)^2 + (-3-5)^2 = r^2 \Rightarrow r^2 = 100 \Rightarrow (x-4)^2 + (y-5)^2 = 100$

56) What is the equation of circle passing through (3,1), its center on the x − axis and radius, $r = \sqrt{5}$

Solution: Let (a,0) be center on x − axis $(x-a)^2 + (y-0)^2 = 5$

$\Rightarrow (3-a)^2 + 1 = 5 \Rightarrow 3 - a = \pm 2 \Rightarrow a = 1$ or $a = 5$

$\Rightarrow (x-1)^2 + y^2 = 5$ or $(x-5)^2 + y^2 = 5$

57) The foci of the conic section defined by the equation $4x^2 + 9y^2 - 24x + 18y + 9 = 0$

A. $(3-\sqrt{5},-1)$ *and* $(3+\sqrt{5},-1)$ C. $(3, -1)$

B. $(3,-1-\sqrt{5})$ *and* $(3,-1+\sqrt{5})$ D) $(3-\sqrt{13},-1)$ *and* $(3+\sqrt{13},-1)$

Solution: $4x^2 + 9y^2 - 24x + 18y + 9 = 0 \Rightarrow 4x^2 - 24x + 9y^2 + 18y = -9$

$\Rightarrow (4x^2 - 6x + 9) + 9(y^2 + 2y + 1) = -9 + 36 + 9 = 36$

$$\Rightarrow \frac{(x-3)^3}{9} + \frac{(y+1)^2}{4} = 1, \text{ center } (3,-1)$$

$\Rightarrow a^2 = 9$, and $b^2 = 4 \Rightarrow c^2 = a^2 - b^2 = 9 - 4 = 5$

It is x − ellipse.

$\therefore$ foci $(3-\sqrt{5},-1)$ *and* $(3+\sqrt{5},-1)$

58) What is the distance from center to one of the foci of the hyperbola $\dfrac{(y-3)^2}{16} - \dfrac{(x+1)^2}{9} = 1$

A. 5 B. 10 C. 6

Solution: center $(-1,3)$, $a^2 = 16$, $b^2 = 9$

- Slope of the line $y = x$ is 1
- Slope of the line passing through h(2,2) and (a, 0) is -1

  $\Rightarrow \dfrac{2-0}{2-a} = -1 \Rightarrow a - 2 = 2 \Rightarrow a = 4$

  $\therefore$ the center of circle is (4,0)
- the radius, $r = \sqrt{(2-0)^2 + (2-4)^2} = \sqrt{8} \Rightarrow r^2 = 8$

  $\Rightarrow$ Equation of circle $(x-4)^2 + y^2 = 8$

Answer: C

52) What is the vertex and the equation of the directrix, resp. of the parabola $x + y^2 + 2y + 1 = 0$

A. $(0, -1)$, $x = \dfrac{-1}{4}$ C. $(0, -1)$, $x = \dfrac{1}{4}$

B. $(-1, 0)$, $y = \dfrac{-1}{4}$ D. $(-1, 0)$, $y = \dfrac{1}{4}$

Solution: $x + y^2 + 2y + 1 = 0 \Rightarrow (y+1)^2 = -x$

- vertex: $(0, -1)$, $4P = 1 \Rightarrow p = \dfrac{1}{4}$
- directrix, $x = \dfrac{-1}{4}$, **Answer: A**

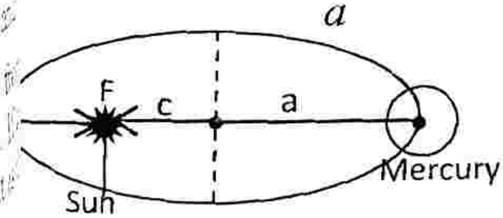
3) The or bit of Mercury around the sun forms an ellipse with eccentricity, $e = 0.206$, length of the major axis 1.16×10^8km and sun at one focus. What is the best approximation of the maximum distance from Mercury to the sun? . . . UEE 2004/2012.

A. 7.596×10^7 km C. 8.695×10^7 km

B. 5.695×10^7 km D. 6.995×10^7 k

Solution: $2a = 1.16\times 10^8\text{km} \Rightarrow a = 5.8\times 10^7$ km

$$e = \frac{c}{a} \Rightarrow c = ea = (0.206)(5.8\times10^7)$$



$\Rightarrow c = 1.195 \times 10^7$

maximum distance;

$\Rightarrow a+c=(1.195+5.8)\times10^7 \simeq 6.995\times10^7$ km **Answer: D**

Minimum distance, $a - c = (5.8 - 1.195) \times 10^7 = 4.61 \times 10^7$ km.

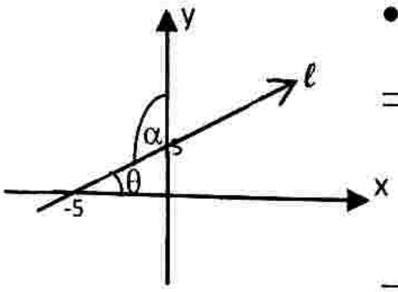
- foci, $F\left(1,-3+\dfrac{5\sqrt{13}}{6}\right), F^{1}\left(1,-3-\dfrac{5\sqrt{13}}{6}\right)$
- vertice $\left(1,-3+\dfrac{5}{2}\right) and \left(-3-\dfrac{5}{2}\right)=\left(1,\dfrac{-1}{2}\right),\left(1,\dfrac{-11}{2}\right)$
- $B\left(1,-\dfrac{5}{3},-3\right) and\ B^{1}\left(1+\dfrac{5}{3},-3\right)=\left(\dfrac{-2}{3},-3\right),\left(\dfrac{8}{3},-3\right)$
- Eccentricity, $e=\dfrac{c}{a}=\dfrac{5\sqrt{13}}{6}\cdot\dfrac{2}{5}=\dfrac{\sqrt{13}}{3}$
- asymptote, $y+3=\pm\dfrac{2}{3}(x-1)\Rightarrow y=\pm\dfrac{2}{3}(x-1)-3$

Solved Problem

50) Line l passes through (0,5) and (-5, 0) what is the angle between the y – axis and l in radian measure . . . UEE 2004/2012.

A) $\dfrac{\pi}{4}$ B) $\dfrac{\pi}{3}$ C) $\dfrac{\pi}{2}$ D) $\dfrac{3}{4}\pi$

Solution: Slope of $\ell=\dfrac{5-0}{0-(-5)}=\dfrac{5}{5}=1$



- Angle of inclination, $\tan\theta=1$

$\Rightarrow\theta=\dfrac{\pi}{4}\leftarrow$ the angle measured from positive x – axis to line ℓ

$\Rightarrow\alpha=\pi-\dfrac{\pi}{4}=\dfrac{3\pi}{4}\leftarrow$ the angle between y – axis and ℓ measured C.C.W from y to l

51) Consider a circle whose center is on the x – axis. If a line given by y = x is tangent to the circle at point (2,2), what is the equation of the circle . . . UEE 2004/2012.

A) $x^2+y^2=8$
B) $(x-2)^2+y^2=4$
C) $(x-4)^2+y^2=8$
D) $(x-1)^2+y^2=5$

Solution: Let the center (a,0)

- eccentricity, $e = \frac{c}{a} = \frac{\sqrt{10}}{\sqrt{2}} = \sqrt{5}$
- asymptote, $y = \pm 2x$

1/4
16

c) $8y^2 - 2x^2 = 16 \Rightarrow \frac{y^2}{2} - \frac{x^2}{8} = 1$

- $a^2 = 2 \Rightarrow a = \sqrt{2}, b^2 = 8 \Rightarrow b = \sqrt{8}$
- $c^2 = a^2 + b^2 = 2 + 8 = 10 \Rightarrow c = \sqrt{10}$
- center (0,0), foci $(0, \pm\sqrt{10})$, $V(0, \pm\sqrt{2})$
  $B(0, \pm 2\sqrt{2}) \leftarrow$ conjugate vertex.
- eccentricity, $e = \frac{c}{a} = \frac{\sqrt{10}}{\sqrt{2}} = \sqrt{5}$
- asymptote, $y = \pm \frac{1}{2}x$

d) $\frac{(x-2)^2}{4} - \frac{(y+5)^2}{9} = 1 \Rightarrow a^2 = 4, b^2 = 9$

- $c^2 = a^2 + b^2 \Rightarrow c^2 = 4 + 9 = 13 \Rightarrow c = \sqrt{13}$
- center (2, – 5), foci $F(2 - \sqrt{13}, -5)$ and $F^1(2 + \sqrt{13}, -5)$
- vertice, $V(4, -5)$, $V^1(0, -5)$ and $B(2, -2), B^1(2, -8)$
- eccentricity, $e = \frac{c}{a} = \frac{\sqrt{13}}{2}$
- asymptote, $y + 5 = \pm \frac{3}{2}(x-2) \Rightarrow y = \pm \frac{3}{2}(x-2) - 5$

e) $4(y+3)^2 - 9(x-1) = 25 \Rightarrow \frac{4}{25}(y+3)^2 - \frac{9}{25}(x-1)^2 = 1$

- $a^2 = \frac{25}{4} \Rightarrow a = \frac{5}{2}$ *and* $b^2 = \frac{25}{9}, b = \frac{5}{3}$
- $c^2 = a^2 + b^2 \Rightarrow c^2 = \frac{25}{4} + \frac{25}{9} \Rightarrow c = \frac{5}{6}\sqrt{13}$
- center, (1, - 3),

$$\Rightarrow \frac{y^2}{36} - \frac{x^2}{64} = 1$$

j) $V(4,1)$ and $V^1(-2,1)$, $\Rightarrow 2a = 4-(-2) = 6 \Rightarrow a = 3, a^2 = 9$

- $e = \frac{c}{a} \Rightarrow \frac{4}{3} = \frac{c}{3} \Rightarrow c = 4$
- $c^2 = a^2 + b^2 \Rightarrow 16 = 9 + b^2 \Rightarrow b^2 = 16 - 9 = 7$
- It is horizontal hyperbola with center $(1,1)$

$$\Rightarrow \frac{(x-1)^2}{9} - \frac{(y-1)^2}{7} = 1$$

49) Find center, foci, vertice, eccentricity and equations of the asymptote.

a) $4x^2 - y^2 = 16$

b) $8x^2 - 2y^2 = 16$

c) $8y^2 - 2x^2 = 16$

d) $\frac{(x-2)^2}{4} - \frac{(y+5)^2}{9} = 1$

e) $4(y+3)^2 - 9(x-1)^2 = 25$

Solution: a) $4x^2 - y^2 = 16 \Rightarrow \frac{4x^2}{4} - \frac{y^2}{16} = 1 \Rightarrow \frac{x^2}{1} - \frac{y^2}{16} = 1$

- $a^2 = 1 \Rightarrow a = 1,\ b^2 = 16 \Rightarrow b = 4$
- $c^2 = a^2 + b^2 = 1 + 16 = 17 \Rightarrow c = \sqrt{17}$
- center $(0,0)$, foci, $F(-\sqrt{17}, 0)$ and $F^1(-\sqrt{17}, 0)$
- vertice, $V(-1, 0)$ and $V^1(1,0)$, $B(0, \pm 4)$
- eccentricity, $e = \frac{c}{a} = \frac{\sqrt{17}}{1} = \sqrt{17}$
- asymptote, $y = \pm 4x$

b) $8x^2 - 2y^2 = 16 \Rightarrow \frac{x^2}{2} - \frac{y^2}{8} = 1 \Rightarrow a^2 = 2, b^2 = 8$

- $c^2 = a^2 + b^2 = 2 + 8 = 10 \Rightarrow c = \sqrt{10}$
- center $(0,0)$, foci
  $\left(\left(\pm\sqrt{10},0\right), V\left(\pm\sqrt{2},0\right), B\left(\pm\sqrt{8},0\right)\right)$

f) Center (− 3, 6), V(0,6), ⇒ a = 0 − (− 3) = 3 ⇒ $a^2 = 9$

- It is horizontal (x − hyperbola)

$$\Rightarrow \Rightarrow \frac{(x+3)^2}{a^2} - \frac{(y-6)^2}{b^2} = 1 \text{, since (1,1) on graph}$$

$$\Rightarrow \frac{(1+3)^2}{9} - \frac{(1-6)^2}{b^2} = 1 \Rightarrow \frac{16}{9} - \frac{25}{b^2} = 1$$

$$\Rightarrow \Rightarrow b^2 = \frac{225}{7} \Rightarrow \frac{(x+3)^2}{9} - \frac{7(y-6)^2}{225} = 1$$

g) V(− 4, 2), V^1(8,2), F(− 7,2), ⇒ 2a = 8 − (− 4) = 12 ⇒ a = 6

- Center $\left(\frac{8-4}{2}, \frac{2+2}{2}\right) = (2,2), \Rightarrow C = 2-(-7) = 9$
- $c^2 = a^2 + b^2 \Rightarrow 81 = 36 + b^2 \Rightarrow b^2 = 81 - 36 = 45$
- it is horizontal (x − hyperbola)$\Rightarrow \frac{(x-2)^2}{36} - \frac{(y-2)^2}{45} = 1$

h) V(3,0) and V^1(−3,0), 2a 3 − (−3) = 6 ⇒ a = 3, $a^2 = 9$

- It is horizontal hyperbola with center

$$\left(\frac{3+-3}{2}, \frac{0+0}{2}\right) = (0,0)$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \text{asymptote, } y = \pm 3x$$

$$\Rightarrow \frac{b}{a} = \frac{b}{3} = 3 \Rightarrow b = 9 = b^2 = 81 \Rightarrow \frac{x^2}{9} - \frac{y^2}{81} = 1$$

i) F(0,10) and F^1 (0, - 10) ⇒ 2c = 10 − (−10) = 20, c = 10

- It is vertical hyperbola with center (0,0)

$$\Rightarrow \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \Rightarrow \text{asymptote, } y = \pm\frac{a}{b}x = \pm\frac{3}{4}x$$

$$\Rightarrow \frac{a}{b} = \frac{3}{4} \Rightarrow 4a = 3b \Rightarrow b = \frac{4a}{3} \Rightarrow b = 9 = b^2 = 81$$

- $c^2 = a^2 + b^2 \Rightarrow 10^2 = a^2 + \frac{(16a^2)}{9} \Rightarrow a^2 = 36 \text{ and } b^2$

$= 8^2 = 64$

axis, $\Rightarrow \frac{(x-2)^2}{a^2} - \frac{(y-1)^2}{b^2} = 1, a = 6, b = 4$

$$\Rightarrow \frac{(x-2)^2}{36} - \frac{(y-1)^2}{16} = 1$$

b) $F(-2,-1), F^1(-2,9), 2a = 6 \Rightarrow a = 3$

$\Rightarrow 2c = 9 - (-1) = 10 \Rightarrow C = 5$

- $C^2 = a^2 + b^2 \Rightarrow 5^2 = 3^2 + b^2 \Rightarrow b^2 = 25 - 9 = 16$
- It is y – hyperbola (vertical) with center

$$\left(\frac{-2+-2}{2}, \frac{-1+9}{2}\right) = (-2,4) \Rightarrow \frac{(y-4)^2}{9} - \frac{(x+2)^2}{16} = 1$$

c) $V(1,2)$ and $V^1(-1,2)$, $2b = 8 \Rightarrow b = 4, b^2 = 16$

$2a = 1 - (-1) = 2 \Rightarrow a = 1 \Rightarrow a^2 = 1$

- it is x – hyperbola with center (0,2)

$$\Rightarrow \frac{(x-0)^2}{1} - \frac{(y-2)^2}{16} = 1$$

d) center $(3,-1)$, $F(5,-1)$, $V(4,-1)$

- Distance between center $(3,-1)$ and $V(4,-1)$ is $a = 4 - 3 = 1 \Rightarrow a^2 = 1$
- Distance between center $(3,-1)$ and $G(5,-1)$: $C = 5 - 3 = 2 \Rightarrow C^2 = 4$
- $C^2 = a^2 + b^2 \Rightarrow 4 = 1 + b^2 \Rightarrow b^2 = 4 - 1 = 3$
- It is horizontal (x – hyperbola)

$$\Rightarrow \frac{(x-3)^2}{1} - \frac{(y+1)^2}{3} = 1$$

e) Center (2,3), V(2,8); $a = 8 - 3 = 5 \Rightarrow a^2 = 25$

- Asymptote, $5y - 4x = 7 \Rightarrow y = \frac{4}{5}x + \frac{7}{5}$
- It is vertical (y – hyperbola)
- From asymptote, we have $\frac{a}{b} = \frac{4}{5} \Rightarrow \frac{5}{b} = \frac{4}{5}$

$$\Rightarrow b = \frac{25}{4} \Rightarrow b^2 = \frac{625}{16} \Rightarrow \frac{(y-3)^2}{25} - \frac{4(x-2)^2}{625} = 1$$

- vertices $v_1(h, k + a)$ and $v_2(h, k - a)$
- Foci are $F_1(h, k + c)$ and $F_2(h, k - c)$
- asymptote $y - k = \pm\frac{a}{b}(x - h)$
- eccentricity, $e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}$

Eccentricity of Hyperbola

The graph of a hyperbola can be very wide or very narrow. The eccentricity of a hyperbola measure of its "wideness."

$$eccentricity(e) = \frac{distance\ between\ foci}{distance\ between\ vertices} = \frac{2c}{2a} = \frac{c}{a}$$

Note, i) $e > 1$ for hyperbola ii) if $e = 0$ circle iii) if $0 < e < 1$, Ellipse

48. Find the equation of each hyperbola that satisfies the states condition:
   a) center $(2,1)$, $a = 6$, $b = 4$ with horizontal transverse axis
   b) foci, $F(-2, -1)$ and $F^1(-2, 9)$ and transverse axis has length 6
   c) vertice $V(1,2)$ and $V^1(-1, 2)$, length of conjugate axes are equal to 8
   d) center $(3, -1)$, focus $F(5, -1)$; vertex $V(4, -1)$.
   e) center $C(2,3)$, vertex $V(2,8)$; equation of the asymptote. $5y - 4x = 7$.
   f) center $(-3,6)$, vertex $V(0,6)$ passing through $P(1,1)$
   g) vertices at $(-4, 2)$ and $V1(8,2)$ one focus at $F(-7, 2)$
   h) vertices $V(3,0)$ and $V1(-3,0)$, asymptote: $y = \pm 3x$
   i) focus, $F(0,10)$ and $F1(0, -10)$, asymptote: $y = \pm\frac{3}{4}x$
   j) vertices, $V(4,1)$ and $V^1(-2, 1)$, eccentricity, $e = \frac{4}{3}$

Solution:

Note: • Distance between vertices = 2a, (length of transverse)

- Distance between foci = 2C
- Center is mid – point of vertices or foci.
- Distance between center and vertice = a

a) center $(2,1)$ with horizontal transverse

Note: Quadratic function $f(x) = ax^2 + bx + c$ is one to one and invertible in $x \geq -\frac{b}{2a}$ or $x \leq -\frac{b}{2a}$.

Illustrative Example

56. Determine whether the following functions are invertible if so, find their respective inverses.

a) $f: \mathbb{R} \longrightarrow [1, \infty)$ be defined by $f(x) = (x-2)^2 + 1$

b) $f: [2, \infty) \longrightarrow [1, \infty)$ be defined by $f(x) = (x-2)^2 + 1$

c) $f: (-\infty, -3] \longrightarrow [0, \infty)$ be defined by $f(x) = (x+3)^2$

d) $f: [-\infty, \infty) \longrightarrow [3, \infty)$ be defined by $f(x) = |x+2| + 3$

e) $f: [-2, \infty) \longrightarrow [3, \infty)$ be defined by $f(x) = |x+2| + 3$

Solution:

a) This function is not one to one in the given domain $(-\infty, \infty)$. For example, $(-1, 10)$ and $(5, 10)$ have the same y – coordinate. $\therefore f$ is not invertible.

b) This function is one – to – one in the domain $[2, \infty)$, so it is invertible.

To find the inverse: Let $y = f(x) \Rightarrow y = (x-2)^2 + 1$, for $x \geq 2$

$$\Rightarrow x = (y-2)^2 + 1 \Leftrightarrow x - 1 = (y-2)^2, \text{ for } y \geq 2$$

$$\Rightarrow y - 2 = \sqrt{x-1} \Rightarrow y = 2 + \sqrt{x-1}$$

$$\therefore f^{-1}(x) = 2 + \sqrt{x-1}$$

Domain of $f^{-1} = \{x: x \geq 1\}$ and range of $f^{-1} = \{y: y \geq 2\}$.

c) This function is one – to – one in the domain $(-\infty, -3]$, so it is invertible.

To find the inverse for $x \leq -3$

Let $y = f(x) = (x+3)^2$ then $x = (y+3)^2$, for $y \leq -3$

$\Rightarrow y + 3 = \pm\sqrt{x} \Rightarrow y = -3 - \sqrt{x}$, for $y \leq -3$

$\therefore f^{-1}(x) = -3 - \sqrt{x}$

Domain of $f^{-1} = \{x: x \geq 0\}$ and range of $f^{-1} = \{y: y \leq -3\}$.

d) This function is not one – to – one in the domain $(-\infty, \infty)$ for example $(1, 6)$ and $(-5, 6)$ have the same y – coordinate. $\therefore f(x) = |x+2| + 3$ is not invertible in the domain $(-\infty, \infty)$.

e) This function is one – to – one in the domain $[-2, \infty)$, so it has inverse.

To find the inverse for $x \geq -2$

Unit Four

Mathematical Reasoning

- Mathematical logic is the art of reasoning.
- Mathematical reasoning is tool for organizing evidence in a systematic way through mathematical logic.
- Its reasoning is based on the form of the sequence of statements.

Statement and open statement

- **Statement (Proposition)**: is an assertive or declarative sentence which is either **true (T) or false (F) but not both.**
- **Open proposition** (open statement) is not a statement but is sentence which contains a variable and which can be converted into a statement by replacing the variable by the name of specific number or object.
- **Truth value**: The true and false value of the statement is truth value.

Note: Hope, wish, interest, beauty, intellegency command exclamatory interrogative and imperative sentences are not **logical statement** (not proposition) in mathematics.

Illustrative example

Identify whether proposition, an open proposition or neither

a) 4 is even number b) 3 is less than 2

c) Long live Ethiopia

d) I wish all Ethiopian love their country

e) $4 + 10 = 12$ f) For some x, $x^2 - 4 = 0$

g) For all x, $x^2 + 1 > 0$

h) There exist x that satisfies: $x^2 + 5 = 0$

i) What do you feel about your Nationality.

j) Every student has the right to be engineer.

k) X is a city in Ethiopia.

l) $2x + 5 = 9$ m) You are good at mathematics

Solution:

i) a, b, e, f, g j and h are proposition (statement)
(It can be said either true or false)

ii) c, d and m neither proposition nor open proposition (it can't be said either true or false)

iii) K and l are open proposition.

Fundamental logical connectives

As operating number by $(+,-,\times,\div)$ we operate statement by $(\neg,\vee,\wedge,\Rightarrow,\Leftrightarrow)$ which are called logical connectives (operator)

Name	Symbol	Meaning	How to write
• Negation	$\neg$	Not (It is false that)	$\neg P$
• Disjunction	$\vee$	Or, either	$P\vee q$
• Conjunction	$\wedge$	And, yet, But	$P\wedge q$
• Implication	$\Rightarrow$	If . . . then	$P\Rightarrow q$
• Bi-implication	$\Leftrightarrow$	If and only if	$P\Leftrightarrow q$

Rule. (Let P and q be a proposition)

i) **Rule of negation ($\neg P$):**

$\neg P$ is true if and only if P is false

2) Find the negation of each of the following statement.

a) p: All rectangles are square b) P: $5 + 7 \leq 8$

Solution: a) $\neg P \equiv$ Note all rectangles are square

$\equiv$ It is false that all rectangle are square

$\equiv$ some rectangle are not square

b) $\neg P \equiv 5 + 7 > 8$

ii) **Rule of disjunction ($P\vee q$)**

Disjunction is the joining of two statement p,q by connective or denoted by $P\vee q$

- $(P\vee q)$ is false if and only if both p and q are false, but all other cases true.

i.e., $F\vee F \equiv F$, a but $T\vee F \equiv T$, $F\vee T \equiv T$, $T\vee T = T$

iii) **Rule of conjunction ($P\wedge q$)**

- $(P\wedge q)$ is true (T) if and only if both p and q true, but all other cases false.

i.e. $T\wedge T \equiv T$, $F\wedge T \equiv F$

$T\wedge F \equiv F$ $F\wedge F \equiv F$

iv) **Rule of implication ($P\Rightarrow q$)**

- $P\Rightarrow q$ is false if and only if p is true and q is false, but all other case true.

i.e., • $T \Rightarrow F \equiv F$, • $F \Rightarrow F \equiv T$

• $F\Rightarrow T \equiv T$ • $T \Rightarrow T \equiv T$

V) **Rule of bi-implication ($P \Leftrightarrow q$)**

- $P \Leftrightarrow q$ is true if and only if both p and q have the same truth value.

i.e. • $F \Leftrightarrow F \equiv T$, • $F \Leftrightarrow T \equiv F$

• $T \Leftrightarrow T \equiv T$ • $T \Leftrightarrow F \equiv F$

3) Find the truth value of each of the following

a) $5 + 3 = 10$ or $2 + 9 = 11$
b) $6 - 4 = 8$ or $1 + 6 = 5$
c) $5 + 3 = 10$ and $2 + 9 = 11$
d) $6 - 4 = 8$ and $1 + 6 = 5$
e) If $5 + 3 = 10$ then $2 + 9 = 11$
f) If $6 - 4 = 8$ then $1 + 6 = 5$
g) $5 + 3 = 10$ if and only if $2 + 9 = 11$
h) $6 - 4 = 10$ if and only if $1 + 6 = 5$

Solution:

a) $F \vee T \equiv T$ e) $F \Rightarrow T \equiv T$
b) $F \vee F \equiv F$ f) $F \Rightarrow F \equiv T$
c) $F \wedge T \equiv F$ g) $F \Leftrightarrow T \equiv F$
d) $F \wedge F \equiv F$ h) $F \Leftrightarrow F \equiv T$

4) Let $P \equiv 4 < 3$, $q \equiv 5$ is an odd number $r \equiv 13$ is not prime number, then which of the following statements has truth value "**True**".

A) $q \Rightarrow (p \vee r)$ C) $\neg(P \vee r) \Rightarrow q$
B) $\neg r \Leftrightarrow (p \wedge q)$ D) $\neg P \wedge (q \Rightarrow r)$

Solution: the truth value of $P \equiv F$, $q \equiv T$ and $r \equiv F$

A) $q \Rightarrow (p \vee r)$
$T \Rightarrow (F \vee F)$
$T \Rightarrow F \equiv F$

C) $\neg(P \vee r) \Rightarrow q$
$\neg(F \vee F) \Rightarrow T$
$T \Rightarrow T \equiv T$

B) $\neg r \Leftrightarrow (p \wedge q)$
$T \Leftrightarrow (F \wedge T)$
$T \Leftrightarrow F \equiv F$

D) $\neg P \wedge (q \Rightarrow r)$
$T \wedge (T \Rightarrow F)$
$T \wedge F \equiv F$

Answer: C

5) Let $P \equiv \sqrt{2}$ is an irrational number $q \equiv 2$ is an odd number then which of the following compound statement has the truth value "True"

A) $\neg p \vee q$ B) $p \Rightarrow q$ C) $\neg q \Leftrightarrow p$ D) $p \wedge q$

Solution: The truth value of $P \equiv T$, $q \equiv F$, $\neg p \equiv F$, $\neg q \equiv T$

A) $\neg p \vee q$ B) $p \Rightarrow q$ C) $\neg q \Leftrightarrow p$ D) $p \wedge q$

$F \vee F \equiv F$ $T \Rightarrow F \equiv F$ $T \Leftrightarrow T \equiv T$ $T \wedge F \equiv F$ **Answer: C**

Suppose proposition $P \Rightarrow \neg q$ is false (F), which of the following true? … UEE 2004/2012.

A) $\neg q \vee (p \Rightarrow q)$ C) $(\neg q \vee p) \Leftrightarrow q$

B) $\neg q \vee (q \Rightarrow \neg p)$ D) $(p \wedge q) \Leftrightarrow \neg q$.

Solution: $p \Rightarrow \neg q \equiv F$ iff, $p \equiv T$, $\neg q \equiv F$ $q \equiv T$, $\neg p \equiv F$

A) $\neg q \wedge (p \Rightarrow q)$ C) $(\neg q \vee p) \Leftrightarrow q$

$F \wedge (T \Rightarrow T)$ $(F \vee T) \Rightarrow T$

$F \wedge T \equiv F$ $T \Leftrightarrow T \equiv T$

A) $\neg p \vee (q \Rightarrow \neg p)$ C) $(p \wedge p) \Leftrightarrow \neg q$

$F \vee (T \Rightarrow F)$ $(T \wedge T) \Rightarrow F$

$F \vee F \equiv F$ $T \Leftrightarrow F \equiv F$

Answer: C

7) If $(p \wedge \neg q) \Rightarrow \neg r$ is false, then the truth value of p,q and r respectively are ____________

A) T,F,F B) F,T,T C) T,F,T D) F,T,T

Solution: $(p \wedge \neg q) \Rightarrow \neg r$ is false if and only if

$p \wedge \neg q \equiv T$ and $\neg r \equiv F$, so that

$p \equiv T$, $\neg q \equiv T$, $q \equiv F$ and $r \equiv T$

$\therefore P \equiv T$, $q \equiv F$ and $r \equiv T$

Answer: C

8) If the truth value of $(\neg p \Leftrightarrow \neg q) \Rightarrow (P \vee \neg r)$ is false then determine the truth value of

a) $q \Rightarrow p$ b) $\neg p \Leftrightarrow r$ c) $\neg q \wedge \neg r$ d) $(\neg p \Leftrightarrow r) \wedge p$

Solution: $(\neg p \Leftrightarrow \neg q) \Rightarrow (p \vee \neg r) \equiv F$ iff $p \vee \neg r \equiv F$ and $(\neg p \Leftrightarrow \neg q) \equiv T$

- From $p \vee \neg r \equiv F$, we have, $P \equiv F$, $\neg r \equiv F$ and $r \equiv T$
- From $(\neg P \Leftrightarrow \neg q) \equiv T$, we have $\neg q \equiv T$, $\neg p \equiv T$, $q \equiv F$

a) $q \Rightarrow p$ b) $\neg p \Leftrightarrow r$ c) $\neg q \wedge \neg r$ d) $(\neg q \Leftrightarrow r) \wedge p$

$F \Rightarrow F \equiv T$ $T \Leftrightarrow T \equiv T$ $T \wedge F \equiv F$ $(T \Leftrightarrow T) \wedge F \equiv F$

9) Suppose the value of $p \Rightarrow q$ is T. What can be said about the value of $(p \wedge q) \Leftrightarrow (p \vee q)$.

Solution: $p \Rightarrow q \equiv T$, we have three possibility

Case I. $(T \Rightarrow T)$ if $P = T$, $q = T$, then $(p \wedge q) \Leftrightarrow (p \vee q) \equiv (T \wedge T)$ $\Leftrightarrow (T \vee T) \equiv T \Leftrightarrow T = T$.

Case II. $(F \Rightarrow F)$ if $P = F$, $q = F$, then

$(p \wedge q) \Leftrightarrow (p \vee q) \equiv (F \wedge F) \Leftrightarrow (F \vee F) \equiv F \Leftrightarrow F = T$.

Case III. $(F \Rightarrow T)$ if $P = F$, $q = T$, then

$(p \wedge q) \Leftrightarrow (p \vee q) \equiv (F \wedge T) \Leftrightarrow (F \vee T) \equiv F \Leftrightarrow T \equiv F$.

10) Suppose the value of $p \Leftrightarrow q$ is T what can be said about the value of: a) $p \Leftrightarrow \neg q$ b) $\neg q \Leftrightarrow q$

Solution: $p \Leftrightarrow q \equiv T$ iff both p and q have the same truth value, i.e $T \Leftrightarrow T$ or $F \Leftrightarrow F$

Case I. ($T \Leftrightarrow T$) if $P = T$, $q = T$,

a) $p \Leftrightarrow \neg q \equiv T \Leftrightarrow F \equiv F$ b) $\neg p \Leftrightarrow q \equiv F \Leftrightarrow T \equiv F$

Case II. ($F \Leftrightarrow F$) if $P = F$, $q = F$,

a) $p \Leftrightarrow \neg q \equiv F \Leftrightarrow T \equiv F$ b) $\neg p \Leftrightarrow q \equiv T \Leftrightarrow F \equiv T$

11) Let P, q, r have truth values F,F and T respectively, then which of the following have truth value "false"

A. $r \Leftrightarrow (\neg p \wedge \neg q)$ C. $\neg q \Rightarrow (p \vee \neg r)$

B. $\neg p \wedge (q \Rightarrow \neg r)$ D. $p \vee (\neg(q \wedge r))$

Solution: we have, $p \equiv F$, $q \equiv F$, and $r \equiv T$

A. $r \Leftrightarrow (\neg p \wedge \neg q)$
$T \Leftrightarrow (T \wedge T) \equiv T$

C. $\neg q \Rightarrow (p \vee \neg r)$
$T \Rightarrow (F \vee F) \equiv T \Rightarrow F \equiv F$

B. $\neg p \wedge (q \Rightarrow \neg r)$
$T \wedge (F \Rightarrow F)$
$T \wedge T \equiv T$

D. $p \vee (\neg(q \wedge r))$
$F \vee (\neg(F \wedge T)$
$F \vee T \equiv T$

Answer: C

12) If p,q and r are proposition such that $(p \Leftrightarrow r) \Rightarrow (\neg q \Leftrightarrow p)$ has truth value "false" then which of the following have truth value "false"

A. $\neg r \Rightarrow P$ B. q C. $\neg r$ D. $p \vee q$

Solution: we have $(p \Leftrightarrow r) \Rightarrow (\neg q \vee p) \equiv F$

Then $P \Leftrightarrow r \equiv T$ and $\neg q \vee p \equiv F$

- From $\neg q \vee p \equiv F$, we have $\neg q \equiv F$, $q \equiv T$, $P \equiv F$
- From $p \Leftrightarrow r \equiv T$, we have $r \equiv F$ since $P \equiv F$

A. $\neg r \Rightarrow p$ B. $q \equiv T$ C. $\neg r = T$ D. $p \vee q$

$T \Rightarrow F \equiv F$ $F \vee T \equiv T$ **Answer: A**

Equivalent proposition

Two compound statement **R** and **Q** are said to be equivalent if they have the same truth value for all possible cases (they have identical truth table)

Notation: If **R** and **Q** are equivalent, then denoted by $\mathbf{R \equiv Q}$

Each of the following pair are equivalent

- $P \Rightarrow q \equiv \neg p \vee q$
- $P \Rightarrow q \equiv \neg q \Rightarrow \neg p \leftarrow$ contra positive law
- $\neg(p \vee q) \equiv \neg p \wedge \neg q \leftarrow$ Demorgan's law

$\Rightarrow F =$

- $\neg(p\wedge q) \equiv \neg p \vee \neg q$ ← Demorgan's law
- $P \Rightarrow (q \wedge r) \equiv (p \Rightarrow q) \wedge (p \Rightarrow r)$
- $P \Rightarrow (q \vee r) \equiv (p \Rightarrow q) \vee (p \Rightarrow r)$
- **Algebra of proposition**

i) Idempotent law

a) $p \vee p \equiv p$ b) $p \wedge \neg p \equiv F$

ii) Complement law

a) $p \vee \neg p \equiv T$ b) $p \wedge \neg p \equiv F$

iii) Distributive law

a) $p\wedge (q\vee r) \equiv (p\wedge q) \vee (p\wedge r)$ b) $p\vee (q \wedge r) \equiv (p\vee q) \wedge (p\vee r)$

iv) Demorgan's law

a) $\neg(p\vee q) \equiv \neg p\wedge\neg q$ b) $\neg(p \wedge q) \equiv \neg p\vee\neg q$

v) Implication law

a) $p\Rightarrow q \equiv \neg p\vee q$ b) $p\Rightarrow (q \wedge r) \equiv \neg p\vee(q\wedge r)$

vi) Bi-implication law

a) $p\Leftrightarrow q \equiv (p\Rightarrow q) \wedge (q \Rightarrow p)$

vii) Commutative law

a) $(p \wedge q) \equiv q \wedge p$ b) $(p \vee q) \equiv (q\vee p)$

Contradiction and Tautology

- **Tautology** A compound statement is said to be tautology if it is **true** for all possible combination.
  (In a truth table of a tautology, there will be only "T" in the last column.
- **Contradiction** A compound statement is said to be contradiction if it is **false** for all possible combination.
  (In a truth table of contradiction, there will be only "F" in the last column.

Note: **If $R \equiv Q$, then $R \Leftrightarrow Q$ is tautology.**

Illustrative example

13) Which of the following compound propositions are tautology, contradiction, or neither.

a) $(p \Rightarrow q) \Leftrightarrow(\neg p\vee q)$ f) $(p\Rightarrow\neg q)\wedge(p\Rightarrow q)$

b) $(p\Rightarrow q) \Leftrightarrow(\neg q\Rightarrow\neg p)$ g) $[p\Rightarrow(p\vee r)]\Leftrightarrow(p\Rightarrow q)\vee(p\Rightarrow r)$

c) $p \Rightarrow(p\wedge q)$ h) $(\neg p\vee q) \Rightarrow (p\wedge\neg q)$

d) $p\vee(\neg(p\wedge q))$ i) $\neg(p\wedge q) \Rightarrow (p\Rightarrow q)$

e) $(p\wedge q) \wedge (\neg(p\vee q]$ j) $[p\vee(\neg q\vee\neg p)] \Rightarrow q$

Solution: a) $p \Rightarrow q \equiv \neg p\vee q$

$\therefore (\neg p\vee q) \Leftrightarrow (\neg p \vee q) \equiv T$ for all possible case

$\therefore$ tautology

b) $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$

$(p \Rightarrow q) \Leftrightarrow (q \vee \neg q)$ Explanation: $\neg q \Rightarrow \neg p \equiv p \Rightarrow q = \neg p \vee q$

$(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$

$(p \Rightarrow q) \Leftrightarrow (p \Rightarrow q) \equiv T$ for all possible case

$\therefore$ It is tautology

c) $p \Rightarrow (p \wedge q) \quad = \neg p \vee (p \wedge q)$

$\equiv (\neg p \vee p) \wedge (\neg p \vee q) \leftarrow$ Distributive law

$\equiv (T) \wedge (\neg p \vee q)$

$\equiv \quad \neg p \vee q$

But $P \Rightarrow (p \vee q)$ is tautology.

$\therefore p \Rightarrow (p \wedge q)$ is neither.

d) $p \vee (\neg(p \wedge q)$ we can simplify as

$\equiv p \vee (\neg p \vee \neg q)$

$\equiv (p \vee \neg p) \vee \neg q$

$\equiv T \vee \neg q$

$\equiv T$ for all possible case.

$\therefore p \vee (\neg(p \wedge q)$ is tautology

p	q	$\neg(p \wedge q)$	$p \vee \neg(p \wedge q)$
T	T	F	T
T	F	T	T
F	T	T	T
F	F	T	T

• the 4th column is T

Explanation

e) $(p \wedge q) \wedge (\neg(p \vee q]$ simplified as   using truth table

$\equiv (p \wedge q) \wedge (\neg p \wedge \neg q)$

$\equiv ((p \wedge q) \wedge \neg p) \wedge \neg q$

$\equiv (p \wedge \neg p) \wedge (q \wedge \neg q)$

$\equiv F \wedge F \equiv F$

p	q	$p \wedge q$	$\neg(p \vee q)$	$(p \wedge q) \wedge (\neg(p \vee q)$
T	T	T	F	F
T	F	F	F	F
F	T	F	F	F
F	F	F	T	F

$\therefore (p \wedge q) \wedge \neg(p \vee q)]$ is contradiction

a) $(p \Rightarrow \neg q) \wedge (p \Rightarrow q)$

By using truth table

P	Q	$\neg q$	$p \Rightarrow q$	$p \Rightarrow \neg q$	$(p \Rightarrow \neg q) \wedge (p \Rightarrow q)$
T	T	F	T	F	F
T	F	T	F	T	F
F	T	F	T	T	T
F	F	T	T	T	T

- From the last column (column 6), we see that $(P \Rightarrow \neg q) \wedge (p \Rightarrow q)$ is neither tautology nor contradiction.

b) $P \Rightarrow (q \vee r) \equiv (p \Rightarrow q) \vee (p \Rightarrow r)$

$\therefore p \Rightarrow (q \vee r) \Leftrightarrow (p \Rightarrow q) \vee (p \Rightarrow r)$ is tautology

i.e It is true for all possible combination.

c) $(\neg p \vee q) \Rightarrow (p \wedge \neg q)$, using truth table.

p	q	$\neg p$	$\neg q$	$\neg p \vee q$	$p \wedge \neg q$	$(\neg p \vee q) \wedge (p \wedge \neg q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	F	F

From the table we see that the last column is not all T or not all F

$\therefore$ neither tautology nor contradiction

i) $\neg(p \wedge q) \Rightarrow (p \Rightarrow q)$, using truth table

p	q	$\neg(p \wedge q)$	$(p \Rightarrow q)$	$\neg(p \wedge q) \Rightarrow (p \Rightarrow q)$
T	T	F	T	T
T	F	T	F	F
F	T	T	T	T
F	F	T	T	T

From the table, last column is not all T

$\therefore \neg(p \wedge q) \Rightarrow (p \Rightarrow q)$ is neither.

j) $[p \vee (\neg q \vee \neg p)] \Rightarrow q$ is simplified as

$[p \vee (\neg p \vee \neg q)] \Rightarrow q$

$[(p \vee \neg p) \vee \neg q)] \Rightarrow q$

$(T \vee \neg q) \Rightarrow q$

$T \Rightarrow q$

$\equiv q$

$\therefore$ It is neither

Explanation

Using table

p	q	$\neg(p \wedge q)$	$p \vee (\neg(p \wedge q)) \Rightarrow q$
T	F	T	F
T	T	F	T
F	F	T	F
F	T	T	T

14) Which of the following is **not a tautology**? …EHEECE

A. $p \vee (q \Rightarrow \neg p)$ C. $(p \Rightarrow q) \Rightarrow \neg(p \wedge \neg q)$

B. $(p \wedge q) \Rightarrow (p \vee q)$ D. $p \Rightarrow (p \wedge q)$

Solution. A. $p \vee (q \Rightarrow \neg p) \equiv p \vee (\neg q \vee \neg p) \equiv p \vee (\neg p \vee \neg q)$

$\equiv (p \vee \neg p) \vee \neg q \equiv T \vee \neg q \equiv T$

B. $(p \wedge q) \Rightarrow (p \vee q) \equiv \neg (p \wedge q) \vee (p \vee q) \equiv (\neg p \vee \neg q) \vee (p \vee q)$

$\equiv (\neg p \vee p) \vee (\neg q \vee q) \equiv T \vee T \equiv T$

C. $(p \Rightarrow q) \Rightarrow \neg(p \wedge \neg q) \equiv (\neg p \vee q) \Rightarrow (\neg p \vee q) \equiv T$

$\Rightarrow (p \wedge q)$ simplifies

Explanation

$\neg p \vee (p \wedge q) \equiv (\neg p \vee p) \wedge (\neg p \vee q)$

$\equiv T \wedge (\neg p \vee q)$

$\equiv \neg p \vee q \equiv p \Rightarrow q$

s not tautology

$\therefore$ **Answer D**

p	q	$p \Rightarrow (p \wedge q)$
T	T	T
T	F	F
F	T	T
F	F	T

erse and contrapositive

If $p \Rightarrow q$ is the implication (direct proposition) then

i) $q \Rightarrow p$ is called its converse

ii) $\neg q \Rightarrow \neg p$ is called its contrapositive

iii) $\neg p \Rightarrow \neg q$ is called its inverse

Illustrative Example

For each statement, give its converse and contrapositive

a) If $x = 3$ then $x^2 - 4 = 5$

b) If x is an integer, then 2x is an even integer

c) Either $4 + 2 \neq 8$ or $5 + 2 = 6$

Solution: a) i) Converse: If $x^2 - 4 = 5$ then $x = 3$

ii) Contrapostive: If $x^2 - 4 \neq 5$ then $x \neq 3$.

b) i) Converse: If 2x is an even integer then x is an integer

ii) Contrapostive: If 2x is not an even integer then x is not an integer

c) Recall $\neg p \vee q \equiv p \Rightarrow q$

i) implication (direct) If $4 + 2 = 8$ then $5 + 2 = 6$

ii) Converse: If $5 + 2 = 6$ then $4 + 2 = 8$

iii) Contrapostive: If $5 + 2 \neq 6$ then $4 + 2 \neq 8$

Quantifiers

- A quantifiers is a specific collection of numbers, objects, or persons.
- A quantifiers change an open statements to statements by replacing variable (s)
- There are two important ways of changing an open statement in to statement with out replacing values for the variables or pronouns. This is done by attaching one of the two symbols, known as quantifiers before an open statement. These two symbol are

i) $\exists$, called the existential quantifiers

The **notation** $\exists x$ read as:

- *There exists x*
- *For at least one x*
- *For some x*
- $(\exists x)P(x) \equiv$ There exist x in universe that satisfies property P.

ii) $\forall$, called the universal quantifiers

The notation $\forall x$ read as:

- *For all x*
- *For each x*
- *For every x*

$(\forall x)P(x) \equiv$ For all x in given universe that satisfies property P.

Example a) Let: $P(x): x^2 - 4 = 0$, $u = R$, then i) $(\exists x)P(x)$ is true

ii) $(\forall x)P(x)$ is false,

Illustrative example

16. Determine the truth value of each statements let the universe U = Real number (R)

a)	$(\exists x)(4x - 3 = -2x + 1)$	f)	$(\exists x)(x^2 = 3)$
b)	$(\exists x)(x^2 + 2x + 3 = 0)$	g)	$(\forall x)(x^2 + 4x + 3 = 0)$
c)	$(\forall x)(x^2 + 2x + 3 > 0)$	h)	$(\exists x)(x^2 + 4x + 3 = 0)$
d)	$(\forall x)(x^2 + 1 > 1)$	i)	$(\forall x)(x^2 + 2x + 3 \neq 0)$
e)	$(\exists x)(x^2 = -4)$	j)	$(\exists x)(x^2 > 0)$

Solution: U = R

a) Let $x = \frac{2}{3}$, then $\frac{8}{3} - 3 = \frac{-4}{3} + 1 \therefore (\exists x)(4x - 3 = -2x + 1)$ is True

b) $x \in R$, then for each x, $x^2 + 2x + 3 \neq 0$,
$\therefore (\exists x)(x^2 + 2x + 3 = 0)$ is F

c) For each real number x, $x^2 + 2x + 3 > 0$, $\therefore (\forall x)(x^2 + 2x + 3 > 0)$ is T

d) Let $x = 0$, $0^2 + 1 = 1$. $\therefore (\forall x)(x^2 + 1 > 1)$ is false.

e) No real number x, that satisfies, $x^2 = -4$
$\therefore (\exists x)(x^2 = -4)$ is false, But if $x \in$ complex number then $(\exists x)(x^2 = -4)$ is true.

f) Let $x = \sqrt{3}$, $\Rightarrow \sqrt{3}^2 = 3 \therefore (\exists x)(x^2 = 3)$ is True.

g) Let $x = 1 \Rightarrow 1^2 + 4(1) + 3 \neq 0 \therefore (\forall x)(x^2 + 4x + 3 = 0)$ is False

h) Let $x = -1$, so that $(-1)^2 + 4(-1) + 3 = 0$ $(\exists x)(x^2 + 4x + 3 = 0)$ is T

i) For each x, $x^2 + 2x + 3 \neq 0 \therefore (\forall x)(x^2 + 2x + 3 \neq 0)$ is True

j) Let $x = 0$ so that $0^2 = 0 \therefore (\forall x)(x^2 > 0)$ is False
But $(\exists x)(x^2 > 0)$ is true

17. Let U = The set of integers.
Let, $P(x) \equiv x$ is prime, $e(x) \equiv x$ is even and $R(x) \equiv x$ is odd.
Determine the truth value of each statement.

a) $(\exists x)[P(x) \wedge R(x)]$
b) $(\forall x)[P(x \wedge R(x)]$
c) $(\exists x)[P(x) \Rightarrow e(x)]$
d) $(\forall x)[P(x \Rightarrow e(x)]$
e) $\neg[(\exists x)(P(x) \Rightarrow e(x)]$
f) $\neg[(\forall x)(P(x) \Rightarrow e(x)]$
g) $(\forall x)[(P(x) \wedge R(x)) \Rightarrow e(x)]$
h) $(\forall x)[P(x) \vee R(x)]$

Solution:

a) Let x = 3, then $P(3) \wedge R(3)$ is true
$\therefore (\exists x)[P(x) \wedge R(x)]$ is true.

b) Let x = 9, then $P(9) \wedge R(9)$ is False
$\therefore (\forall x)[P(x \wedge R(x)]$ is False

c) Let x = 2, then $P(2) \Rightarrow e(2)$ is true.
$\therefore (\exists x)[P(x \Rightarrow e(x)]$ is true

d) Let x = 3, as counter example, then $P(3) \Rightarrow e(3)$ is false
therefore, $(\forall x)[P(x) \Rightarrow e(x)]$ is false

e) negation of true is false from (c), therefore,
$\neg[(\exists x)(P(x \Rightarrow e(x)]$ is false
Note: $\neg[(\exists x)(P(x \Rightarrow e(x)] \equiv (\forall x)(P(x) \wedge \neg e(x)]$

f) Negation of false is true from (d)
$\therefore [(\forall x)(P(x) \Rightarrow e(x)]$ is true
Note: $\neg[\forall x)(P(x) \Rightarrow e(x)] \equiv (\exists x)(P(x) \wedge \neg e(x)]$

g) Let x = 3, as counter example, then
$P(3) \wedge R(3) \Rightarrow e(3)$ is false
$\therefore (\forall x)[(P(x) \wedge R(x) \Rightarrow e(x)]$ is false

h) $(\forall x)[P(x) \vee R(x)]$ is true

Relation ship between quantifiers

i) $\neg(\exists x)P(x) \equiv (\forall x)\neg P(x)$.
ii) $\neg(\forall x)P(x) \equiv (\exists x)\neg P(x)$.
iii) $\neg[(\forall x)(P(x) \Rightarrow R(x))] \equiv (\exists x)(P(x) \wedge \neg R(x))$.
iv) $\neg[(\exists x)(P(x) \Rightarrow R(x)] \equiv (\forall x)(P(x) \wedge \neg R(x))$.

18. Give the negation of each statement and determine the truth values for each part.

a) $(\exists x)(4x - 2 = -3x + 12)$
b) $(\forall x)(x^2 + 1 = 0)$
c) $(\exists x)(x^2 + 1 \leq 0)$
d) $(\forall x)(x^2 > 0)$
e) $(\forall x)(x^2 - 1 > 0)$

Solution:

a) $\neg (\exists x)(4x-2= -3x+12) \equiv (\forall x)(4x - 2 \neq -3x + 12)$ is false

b) $\neg(\forall x)(x^2 + 1 = 0) \equiv (\exists x)(x^2 + 1 \neq 0)$ is false

c) $\neg(\exists x)(x^2 + 1 \leq 0) \equiv (\forall x)(x^2 + 1 > 0)$ is true

d) $\neg(\forall x)(x^2 > 0) \equiv (\exists x)(x^2 \leq 0)$ is true, let $x = 0$

e) $\neg(\forall x)(x^2 - 1 > 0) \equiv (\exists x)(x^2 - 1 \leq 0)$ is true, let $x = 0$

19. Suppose $e(x) \equiv x$ is even

$p(x) \equiv x$ is prime

$d(x) \equiv x$ is divisible by 2.

Which one of the following has the truth value "F" on the set of natural numbers? ….. UEE 2004/12

A) $(\exists x)[e(x) \wedge P(x)]$ C) $(\exists x)[e(x) \wedge \neg d(x)]$

B) $(\forall x)[e(x) \Rightarrow d(x)]$ D) $(\forall x)[e(x) \vee \neg d(x)]$

Solution: A) Let $x = 2$, then $e(2) \wedge p(2) \equiv$ True

B) True

C) $\neg(\forall(x)(e(x) \Rightarrow d(x))) \equiv (\exists x)(e(x) \wedge \neg d(x)) \equiv$ False

D) For each x, $e(x) \vee \neg d(x)$ is true.

Answer: C

20. Let $P(x)$: $x^2 + x$ is positive. Which of the following is equivalent to $\neg(\exists x)p(x)$? …..EHEECE 2002

A. $(\exists x)\neg(x^2 + x > 0)$ C) $(\exists x)(x^2 + x \leq 0)$

B) $(\forall x)(x^2 + x < 0)$ D) $(\forall x)(x^2 + x \leq 0)$

Solution: $\neg(\exists x)(x^2 + x > 0) \equiv (\forall x)(x^2 + x \leq 0)$

Answer: D

21. Let $p(x) \equiv x$ is positive, $Q(x) \equiv x > 5$ then the negation of $(\forall x)(Qx) \Rightarrow P(x)$ …EHEECE.

A. There is x such that $x \leq 5$ and x is positive.

B. There is x such that $x > 5$ and x is not positive.

C. There is x such that $x \leq 5$ and x is not positive.

D. There is x such that $x > 5$ and x is positive.

Solution: $\neg(\forall x)(Q(x) \Rightarrow P(x)) \equiv (\exists x)(Q(x) \wedge \neg P(x))$

$\therefore$ Answer B

22. If $(P \wedge Q) \Rightarrow (\neg S \vee R)$ is false which of the following is true ….EHEECE.

A. $S \Rightarrow R$ B. $\neg p \vee \neg S$ C. $(p \wedge \neg Q) \Leftrightarrow R$ D. $P \wedge R$

Solution: $(P \wedge Q) \Rightarrow (\neg S \vee R) \equiv F$ so that

$\neg S \vee R \equiv F$ and $(P \wedge Q) \equiv T$

$R \equiv F$, $\neg S \equiv T$, and $P \equiv T$, $Q \equiv T$

Thus,

A) $S \Rightarrow R$ B) $\neg P \vee \neg S$ C) $(P \wedge \neg Q) \Leftrightarrow R$ D) $T \wedge F \equiv F$

$T \Rightarrow F \equiv T$ $F \vee F \equiv F$ $(T \wedge F) \Leftrightarrow F \equiv T$

Answer:

23. Which of the following propositions is equivalent to $p \vee q$?...EHEECE 98/06.

A. $\neg p \Rightarrow q$ B. $\neg(\neg p \vee \neg q)$ C) $\neg(p \wedge q)$ D) $p \Rightarrow q$

Solution: A) $\neg p \Rightarrow q \equiv \neg(\neg p) \vee q) \equiv p \vee q$ $\therefore$ Answer A

If $\neg p \vee Q \Rightarrow (R \wedge \neg R)$ is true then which of the following is necessarily true? .....UEE 2003/11.

A. $R \Rightarrow Q$, B. $(Q \Rightarrow P) \wedge R$ C. $P \wedge \neg R$ D. $(Q \wedge R) \Rightarrow P$

Solution: $R \wedge \neg R \equiv F$, so that $\neg P \vee Q \equiv F$,

$\neg P \equiv F$, $P \equiv T$, $Q \equiv F$, since P is true

$\therefore (Q \wedge R) \Rightarrow P \equiv T$ $\therefore$ Answer D

Which of the following is correct statement about the proposition $(p \Rightarrow q) \Rightarrow (\neg p \vee q)$? The proposition

A. is true only when q is true.

B. is false when q is false

C. is true for all possible combination of the truth value of p and q.

D. has the same truth value as that of q.

Solution: using truth table

p	q	$p \Rightarrow q$	$\neg p \vee q$	$(p \Rightarrow q) \Rightarrow (\neg p \vee q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	T	T	T
F	F	T	T	T

Thus, $(p \Rightarrow q) \Rightarrow (\neg p \vee q) \equiv T$ for all cases; $\therefore$ Answer C

Quantifiers occurring in combination when two variables are involved.

Let $P(x, y): x + y = 6 \longleftarrow$ an open statement you can change these open statement by attaching the following four cases.

i) $(\exists x)(\exists y)p(x, y) \equiv$ There is some x and some y that satisfies the relation $P(x, y)$

ii) $(\exists x)(\forall y)p(x, y) \equiv$ There is fixed x which stands to every y in the relation $p(x, y)$

iii) $(\forall x)(\exists y)p(x, y) \equiv$ For every x there is y such that $p(x, y)$ holds.

iv) $(\forall x)(\forall y)p(x,y) \equiv$ Every x and every y satisfies the relation $p(x, y)$

Illustrative Example

Let universe, U = the set of real number

Let $P(x, y): x + y = 8$, then determine the truth value of the following.

a) $(\exists x)(\exists y)p(x, y)$ c) $(\forall x)(\exists y)p(x, y)$

b) $(\exists x)(\forall y)p(x, y)$ d) $(\forall x)(\forall y)p(x, y)$

Solution:

a) $(\exists x)(\exists y)p(x, y)$ means there is real number x and y that satisfies $x + y = 8$.
Let $x = 3$, and $y = 5$
$\therefore (\exists x)(\exists y)p(x, y)$ is true.

b) $(\exists x)(\forall y)p(x, y)$ means there is fixed x which standard for every y that satisfy $x + y = 8$ is false. Let $x = 2$, then
$(\forall y)(2 + y = 8)$ is false

c) $(\forall x)(\exists y)p(x, y)$ means, for every x, we can find corresponding y value that satisfies $x + y = 8$ is true.
$x + y = 8$ is true
i.e. Let $x = 3$, then $y = 5$
Let $x = 0$, then $y = 8$
$\therefore (\forall x)(\exists y)p(x, y)$ is true

d) $(\forall x)(\forall y)p(x, y)$ is false, let $x = 1, y = 5 \Rightarrow 1 + 5 = 6 \neq 8$

27. Let $p(x, y)$: $x < y$ and $T(x, y)$: $y + x = y$ then determine the truth value of each of the following, if the universe is the set of real numbers:

a)	$(\exists x)(\exists y)p(x, y)$	d)	$(\exists x)(\exists x)T(x, y)$
b)	$(\exists x)(\forall y)p(x, y)$	e)	$(\exists x)(\forall y)T(x, y)$
c)	$(\forall x)(\exists y)p(x, y)$	f)	$(\forall x)(\exists y)T(x, y)$

Solution:

a) $(\exists x)(\exists y)(x < y)$ True

b) $(\exists x)(\forall y)p(x, y)$ means there is fixed x which is less than every real number y is false

c) $(\forall x)(\exists y)p(x, y)$ means, for every x, there is y such that x < y is true.

d) $(\exists x)(\exists x)T(x, y)$ is true

e) $(\exists x)(\forall y)(y + x = y)$ is true
$(\exists x)(\forall y)T(x, y)$ means there is fixed x which is standard for all y, let $x = 0$.

f) $(\forall x)(\exists y)(y + x = y)$ is false
$(\forall x)(\exists y)T(x, y)$ means for every x, we can find y.
Let $x = 3 \Rightarrow y + 3. = y \Rightarrow 3 = 0$ false

28. Let U = the set of real number. Determine the truth value of each of the following.

A. $(\exists y)(\forall x)(x + y = 0)$ E) $(\exists x)(\forall y)(xy = y)$
B. $(\forall x)(\exists y)(x + y = 0)$ F) $(\exists x)(\exists y)(x+y = 5$ and $x-y=1)$
C. $(\forall x \neq 0)(\exists y)(xy = 1)$
D. $(\exists x)(\forall y)(xy = 0)$

Solution:

a) $(\exists y)(\forall x)(x + y = 0)$ is false

b) $(\forall x)(\exists y)(x + y = 0)$ is true for every x, there is y, such that $x + y = 0$

c) $(\forall x \neq 0)(\exists y)(xy = 1)$ is true

d) $(\exists x)(\forall y)(xy = 0)$ is true there is fixed x which is true for all y, let $x = 0$

e) $(\exists x)(\forall y)(xy=y)$ is true. Let $x = 1$, then it is true for all y

f) $(\exists x)(\exists y)(x+y=5$ and $x - y = 1)$ is true, let $x = 3, y = 2$

29. Which of the following is a true statement on the set of real number.

A) $(\forall x)(\exists y)(y = x + 5)$ C) $(\exists x)(\forall y)(x = y + 3)$
B) $(\exists y)(x^2 + 1 = 0)$ D) $(\forall x)(\forall y)\ (x \geq y) \wedge (y \neq x)$

Solution: $(\forall x)(\exists y)(y = x + 5)$ is true **Answer: A**

30. If x and y are non- negative integers which of the following is not rue? .... UEE 2004/12

A) $(\forall x)(\exists y)(y > x^2 - 1)$ C) $(\exists y)(\forall x)(y \leq x^2 - 1)$
B) $(\exists x)(\forall y)(y > x^2 - 1)$ D) $(\exists y)(\exists x)\ (y \leq x^2 - 1)$

Solution: A) Let $x = 2, \Rightarrow y > 4 - 1 \Rightarrow y > 3$
$\therefore (\forall x)(\exists y)(y > x^2 - 1)$ is true

B) $(\exists x)(\forall y)(y > x^2 - 1)$ means there is fixed x such that $y > x^2 - 1$ which is less than every y is false.

C) $(\exists y)(\forall x)(y \leq x^2 - 1)$ is true. There is fixed y which is less than every x such that $y \leq x^2 - 1$ is $y = -1$

D) $(\exists y)(\exists x)\ (y \leq x^2 - 1)$ is true

Arguments And Validity

- As introduction to this section, we consider the following example.
  i) If p is true and $p \Rightarrow q$ is true.
  What can be concluded about q?
  ii) If p and $p \wedge q$ is true,
  What can be concluded about q?
  i) $p \equiv T, p \Rightarrow q \equiv T, \quad \therefore q \equiv T$
  ii) $p \equiv T, p \wedge q \equiv T, \quad \therefore q \equiv T$
- In the above example, the statement, p, $p \Rightarrow q$ are called **premises or hypothesis** and q is called **conclusion.**

A logical deduction (argument form) is an assertion that a given statements, $P_1, P_2, P_3, \ldots P_n$ called **hypothesis** or **premises** yield another statement Q called the conclusion. Such logical deduction is denoted by $P_1, P_2, P_3, \ldots P_n \vdash Q$.

In the above example, the argument form is denoted by

i) $\frac{p \qquad p \Rightarrow q}{q}$ the same as $p, p \Rightarrow q \vdash q$ ii) $\frac{p \wedge q}{q}$ the same as $p \wedge q \vdash$

An argument from $P_1, P_2, P_3, \ldots P_n \vdash Q$ is said to be

i) Valid if Q is true, when ever all the premise $P_1, P_2, \ldots P_n$ are true.

ii) Invalid if Q is False, when ever all the premise $P_1, P_2, \ldots P_n$ are true.

Illustrative Example

30. Decide whether each of the following argument forms is valid or invalid.

a) $\neg p \Rightarrow \neg q$ d) $p \Rightarrow \neg q, p, r \Rightarrow q \vdash \neg r$ $\frac{q}{p}$

b) $p \Rightarrow q$ e) $\neg q \Rightarrow \neg p, r \Rightarrow p, \neg q \vdash r$ $\frac{\neg q \Rightarrow r}{p}$

c) $p, \neg p \Rightarrow q \vdash q$ f) $(\neg r \wedge \neg s), (\neg s, \Rightarrow p) \Rightarrow r \vdash$

Solution: • Start from simple statement

• Premise is always true.

a) 1) $q \equiv T \leftarrow$ premise

2) $\neg p \Rightarrow \neg q \equiv T \leftarrow$ premise

3) $\neg q \equiv F$ and $\neg p \equiv F$ from (1) and (2)

4) $p \equiv T \leftarrow$ from 3.

$\therefore$ Argument form is in valid

b) By using truth table $p \Rightarrow q, \neg q \Rightarrow r \vdash p$

Row	p	q	r	$\neg p$	$p \Rightarrow q$	$\neg q \Rightarrow r$
1	T	T	T	F	T	T
2	T	T	F	F	T	T
3	T	F	T	T	F	T
4	T	F	F	T	F	F
5	F	T	T	F	T	T
6	F	T	F	F	T	T
7	F	F	T	T	T	T
8	F	F	F	T	T	F

… the table in row 5, row 6 and row 7, the premise $p \Rightarrow q$, $\neg q \Rightarrow r$ are true. But the conclusion P is false.

$\therefore$ Argument form is in valid due to Row 5, Row 6 and Row 7.

3) using truth table $p, \neg p \Rightarrow q \vdash q$

Row	p	q	$\neg p$	$\neg p \Rightarrow q$
	T	T	F	T
	T	F	F	T
	F	T	T	T
	F	F	T	F

From the table, the premise P, and $\neg p \Rightarrow q$ are true in the row 1 and row 2. But the conclusion q is false in row 2.

$\therefore$ Argument form is invalid

d) $p \Rightarrow \neg q$, p, $r \Rightarrow q \vdash \neg r$, start from simple statement

1. $p \equiv T \longleftarrow$ premise
2. $p \Rightarrow \neg q \equiv T \longleftarrow$ premise
3. $\neg q \equiv T \longleftarrow$ from (2), $\therefore q \equiv F$
4. $r \Rightarrow q \equiv T \longleftarrow$ premise
5. $q \equiv F$, $r \equiv F \longleftarrow$ from (4), definition ($\Rightarrow$)
6. $\neg r \equiv T \longleftarrow$ from (5)

$\therefore$ Argument form is valid

e) $\neg q \Rightarrow \neg p$, $r \Rightarrow p$, $\neg q \vdash r$

start from simple statement

1. $\neg q \equiv T \longleftarrow$ premise
2. $\neg q \Rightarrow \neg p \equiv T \longleftarrow$ premise
3. $\neg p \equiv T$, and $p \equiv F \longleftarrow$ from (2)
4. $r \Rightarrow p \equiv T \longleftarrow$ premise
5. $p \equiv F$, $r \equiv F \longleftarrow$ from (4), and definition ($\Rightarrow$)

$\therefore$ Argument form is invalid

f) $(\neg r \wedge \neg s)$, $\neg(s \Rightarrow p) \Rightarrow r \vdash \neg p$

1. $\neg r \wedge \neg s \equiv T \longleftarrow$ premise
2. $\neg r \equiv T$, $r \equiv F$ and $\neg s \equiv T$ and $S \equiv F$
3. $(\neg s \Rightarrow p) \Rightarrow r) \equiv T \longleftarrow$ premise
4. $p \equiv F$, $\neg p \equiv T$

$\therefore$ Argument form is invalid

Prove the following are valid arguments by direct Method.

3) If the rain does not come, the crops are ruined and the people will starve. The crops are not ruined or the people will not starve. Therefore, the rain comes $\neg p \Rightarrow q$

Solution: $p \equiv$ The rain comes

$q \equiv$ The crops are ruined

$r \equiv$ The people will starve

The argument forms can be write in symbols

$\neg p \Rightarrow (q \wedge r), \neg q \vee \neg r \vdash p$

1. $\neg q \vee \neg r \equiv T \longleftarrow$ premise
2. $\neg(q \wedge r) \equiv T \longleftarrow$ from (1) and Demorgan's law
3. $q \wedge r \equiv F \longleftarrow$ from (2) and rule of $(\neg)$
4. $\neg p \Rightarrow (q \wedge r) \equiv T \longleftarrow$ premise
5. $\neg p \equiv F \longleftarrow$ from (4) and rule of $(\Rightarrow)$
6. $p \equiv T \longleftarrow$ from (5)

$\therefore$ Argument form is valid

32. If the team is late, them it can not play the game. If the referee is here, then the team can play the game. The team is late. Therefore, the referee is not here

Solution: Let $p \equiv$ The team is late

$q \equiv$ It can play the game

$r \equiv$ The referee is here

The argument form write as

$p \Rightarrow \neg q, r \Rightarrow q, p \vdash \neg r$

1. $p \equiv T \longleftarrow$ premise
2. $p \Rightarrow \neg q \equiv T \longleftarrow$ premise
3. $\neg q \equiv T \longleftarrow$ from (1) and (2)
4. $q \equiv F \longleftarrow$ from (3)
5. $r \Rightarrow q \equiv T \longleftarrow$ premise
6. $r \equiv F \longleftarrow$ from (5)
7. $\neg r \equiv T \longleftarrow$ from (1) and rule $(\neg)$

$\therefore$ Argument form is valid

Rule of Inference

Here are certain valid deductions called rules of inference

1. **Modes ponens:** $p, p \Rightarrow Q \vdash Q$
2. **Modes Tollens:** $\neg Q, p \Rightarrow Q \vdash \neg p$
3. **Principle of syllogism:** $p \Rightarrow Q, Q \Rightarrow R \vdash p \Rightarrow R$
4. **Principle of Detachment:** $p \wedge Q \vdash p, Q.$
5. **Modes Tollendo ponens:** $\neg p, p \vee Q \vdash Q$
6. **Modes ponendo Tollens:** $\neg(p \wedge Q), p \vdash \neg Q$
7. **Principle of equivalence:** $p \Leftrightarrow Q, p \vdash Q$

33. Investigate the validity of the following. If he works hard, he will be successful. If he is successful then he will be happy. Therefore hard work leads to happiness.

Solution: Let $p \equiv$ he works hard

$q \equiv$ he will be successful

$r \equiv$ he will be happy.

Then, the argument form write as symbol $p \Rightarrow q, q \Rightarrow r \vdash p \Rightarrow r$

By the rule of inference (syllogism). It is valid

4. Decide the validity of each of the following

a) $\neg p \vee q, r \Rightarrow p, r \vdash q$

b) $p \Rightarrow q, \neg r \Rightarrow \neg q \vdash \neg r \Rightarrow \neg p$

c) $p \Rightarrow (q \vee r), \neg r, p \vdash q$

Solution: Start from the simple statement.

a)
1. $r \equiv T \longleftarrow$ premise
2. $r \Rightarrow p \equiv T \longleftarrow$ premise
3. $p \equiv T \longleftarrow$ from (2) and rule of ($\Rightarrow$)
4. $\neg p \vee q \equiv T \longleftarrow$ premise
5. $\neg p \equiv F \longleftarrow$ from (3) and then (4)
6. $q \equiv T \longleftarrow$ from (4) and rule of ($\vee$)

$\therefore$ Argument form is valid

b)
1. $p \Rightarrow q \equiv T \longleftarrow$ premise
2. $q \Rightarrow r \equiv T \longleftarrow$ premise and contrapoistive of $\neg r \Rightarrow \neg q$
3. $p \Rightarrow q, q \Rightarrow r \vdash p \Rightarrow r \longleftarrow$ principle of syllogism

$\therefore$ Argument form is valid

c)
1. $\neg r \equiv T \longleftarrow$ premise
2. $r \equiv F \longleftarrow$ from (1)
3. $p \equiv T \longleftarrow$ premise
4. $p \Rightarrow (q \vee r) \equiv T \longleftarrow$ premise
5. $p \equiv T, q \equiv T \longleftarrow$ from (3), (4)

5. If 6 is even, then 2 does not divide 7. either 5 is not prime or 2 divide 7. But 5 is prime. Therefore, 6 is not even.

Solution: Let $p \equiv$ 6 is even

$q \equiv$ 2 divides 7

$r \equiv$ 5 is prime.

Hence, the argument form is

$p \Rightarrow \neg q, \neg r \vee q, r \vdash \neg p$

1. $r \equiv T \longleftarrow$ premise
2. $\neg r \equiv F \longleftarrow$ from (1) and rule ($\neg$)
3. $\neg r \vee q \equiv T \longleftarrow$ premise
4. $q \equiv T \longleftarrow$ from (3)
5. $p \Rightarrow \neg q \equiv T \longleftarrow$ premise
6. $p \equiv F \longleftarrow$ from (5) and rule ($\Rightarrow$)
7. $\neg p \equiv T$

$\therefore$ Argument form is Valid

36. Let $p \equiv$ I pass the examination
$q \equiv$ I study hard
Which of the following represents the argument "I pass the examination if I study hard. I don't study hard. Therefore, I pass the examination?" .... (EHEECE)

A. $p \Rightarrow q, \neg q \vdash p$
B. $p \Rightarrow q, q \vdash p$
c) $q \Rightarrow p, \neg q \vdash p$
d) $q \Rightarrow p, \neg q \vdash \neg p$

Solution: $q \Rightarrow p, \neg q \vdash p$ ∴ **Answer: C**

37. Consider the following argument form. Production is high if rain continues. Rain does not continue.
Therefore, either production is low or rain continues.
Let $p \equiv$ production is low $q \equiv$ rain continues
The following table is also given about p and q.

Row	P	q	$\neg p$	$\neg q$	$\neg q \Rightarrow q$	$q \Rightarrow \neg p$	$p \vee q$
1	T	T	F	F	T	F	T
2	T	F	F	T	T	T	T
3	F	T	T	F	T	T	T
4	F	F	T	T	F	T	F

Which of the following is necessarily true .... UEE

A. The argument form is valid due to row 2.
B. The argument form is valid due to row 2 and 3.
C. The argument form is invalid due to row 4.
D. The argument form is invalid due to row 1 and row 3.

Solution: The argument $q \Rightarrow \neg p, \neg q \vdash p \vee q$.
The premise $q \Rightarrow \neg p$, $\neg q$ are true in row 2 and row 3, but the conclusion $p \vee q$ is false in row 4
∴ Argument form is invalid due to row 4, **Answer: C**

Exercise

1. Let $p \equiv F$, $q \equiv F$, and $r \equiv T$ respectively. Determine the truth-value for each of the following.
   a) $(p \wedge \neg r) \Rightarrow \neg q$
   b) $p \Rightarrow (q \vee r)$
   c) $(p \wedge q) \Rightarrow \neg(q \vee r)$
   d) $[(p \vee q)] \Leftrightarrow [(\neg p) \wedge (\neg q)]$
   e) $[p \Rightarrow (q \vee r)] \Rightarrow [(p \wedge \neg q) \Rightarrow r]$
2. Find the negation for each of the following
   a) some students do not study their lesion
   b) $3 + 6 \leq 7$ or $4 - 1 = 5$ c) If $5 < 3$ then $-3 < -5$
   d) $3 < 1$, if and only if $3 - 1$
   e) You will study or else you will pass your examination
   f) You will study if and only if you will pass your examination
   g) All Ethiopians are honest

3. For each statement, give its **inverse, converse, contrapositive**
   a) Either a number is not prime or it is odd
   b) If $5 < 3$ then $-3 < -5$
   c) If there are no cloud, then it is not raining
   d) Items are available in necessary for price not to be high.
   e) It is sufficient that every Ethiopia works hard in order for poverty to be eradicated.

4. Determine whether the following are tautologies, contradictions or neither.

a)	$p \Rightarrow (p \wedge q)$	e) $[p \Rightarrow (q \vee r)] \Rightarrow [(p \wedge q) \Rightarrow r]$
b)	$p \Rightarrow (p \vee q)$	f) $p \vee \neg(p \wedge q)$
c)	$(p \vee q) \Rightarrow q$	g) $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$
d)	$(p \wedge q) \Rightarrow q$	h) $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (\neg r \Rightarrow \neg p)$

Determine the validity of the following

i)
   a) $p \Rightarrow q, \neg q \vdash \neg p$ d) $\neg p, p \Leftrightarrow q \vdash \neg(p \wedge q)$
   b) $\neg p \Rightarrow q, p \vdash \neg p$ e) $p \Rightarrow \neg q, p, r \Rightarrow q \vdash \neg r$
   c) $p \Rightarrow \neg q, q \vdash \neg p$ f) $p \Rightarrow q, \neg r \Rightarrow \neg q \vdash p \Rightarrow r$

ii)
   a) If the rains does not come, the crops are ruined and the people will starve. The crops are not ruined or the people will not starve. Therefore the rain comes.
   b) If trees are cut soil erosion increase, soil erosion does not increase only if trees are planted. Therefore trees are cut.
   c) If it rains, I go to the cinema. I do not go to the cinema. Therefore, it does not rain.

Let $U = \{1, 2, 3, 4\}$

Let $P(x)$: x is prime $H(x)$: x is divisible by 2.

$R(x)$: x is odd and prime $Q(x)$: $x \leq 4$

Determine the truth – value of each of the following

a) $(\exists x)\, P(x)$

b) $(\exists x)[P(x) \wedge H(x)]$

c) $(\exists x)[P(x) \Rightarrow H(x)]$

d) $(\forall x)[R(x) \Rightarrow p(x)]$

e) $\neg[(\forall x)(P(x) \Rightarrow H(x)]$

f) $(\forall x)[P(x) \vee Q(x)] \Rightarrow (\exists x)[H(x) \Rightarrow R(x)]$

Simplify a) $\neg(p \Rightarrow q) \vee (p \Rightarrow q)$ c) $(\neg p \Rightarrow q) \wedge (p \vee \neg q)$

b) $(p \Leftrightarrow q) \wedge \neg(p \Leftrightarrow q)$

Unit Fiv

Statistics and probability

Statistics: is the science dealing with

- collecting data
- organizing data
- analysis and interpretation of quantitative data

Types of data

- Qualitative data
- Quantitative data

i) **Qualitative data:**

Not measurable, we can not expressed in number

Example: Color, ear lob attachment, sex, religion nationality, inter beauty, honesty, intelligence, educational qualification....

ii) **Quantitative data:** It is measurable, we can expressed in number

Example: Age, weight, height, income, number of population, score in test.

iii) **Quantitative data:** is categorized in to two variable

These are:

a) Discrete variable: It is expressed or measured by whole number

Example: Number of student, Number of chair, Number of car

Number of ho

b) **Continuous variable:** It is expressed or measured fraction ~~or~~ decimal.

Example: height, weight, age, size, temperature. distance, income in test

Frequency distribution method of data presentation

i) Discrete frequency distribution

List of all data values obtained with their respective frequencies

ii) Grouped frequency distribution

Used to present large amount of data in class.

Ungrouped data: These are data in their original form with out arranged or summarized

Grouped data:- data organized and summarized in freq distribution are grouped data

The following are basic concept that used in frequency distribution

i) Cumulative frequency: is the sum of frequency of all the classes from the first to up to class.

Example: If f_1, f_2, f_3, f_4 are frequency of 4 classes, then the cumulative frequency of these classes are f_1, f_1+f_2, $f_1+f_2+f_3$ and $f_1 + f_2 + f_3 + f_4$ respectively

These is called "less than cumulative frequency.

If the process is done from the total frequency to back ward is called **more than frequency.**

ii) Class interval $= \dfrac{\text{(maximum value) - (minimum value)}}{\text{number of class required}}$

iii) Class limits is the lower and the upper elements in a given class

- Lower class limit is the first number in given class.
- Upper class limit is the second number in given class.

iv) Class mark (x_c) $= \dfrac{(\text{lower class limit}) + (\text{upper class limit})}{2}$

v) class boundaries

Lower class boundary = (lower class limit) – (0.5)

Upper class boundary = (upper class limit) + (0.5)

Illustrative Example

1. The following are scores of 50 student in mathematics exam.

55	44	45	48	69	30	47	50	66	38	58	44	42
46	47	53	54	35	50	62	49	82	40	46	29	47
75	25	46	28	43	50	47	62	30	56	48	30	
49	39	53	70	46	32	61	48	35	60	51	42+	

82−25

$\frac{57}{6}$ 9.5

Then

a) prepare a grouped frequency distribution using 6 classes

b) Find class interval

c) Determine the cumulative frequency distribution.

d) What is the upper class limit of 3rd class.

e) What is the lower class limit of 5th class

f) Find the cumulative frequency of 3rd class

g) How many students gets more than 54 marks

h) How many students gets less than 54 marks

i) Determine class boundaries

j) Determine class mark

k) Determine the correction factor

l) How many students gets more than 25
m) How many students gets more than 35

Solution: Here

b) $\text{class interval} = \frac{(\text{maximum value}) - (\text{minimum value})}{\text{number of class required}}$

$$= \frac{82-25}{6} = 9.5 \approx 10$$

Score (x) (class limit)	Number of student (frequency)	Cumulative frequency: Less than cumulative frequency	Cumulative frequency: More than cumulative frequency	Class boundaries	Class mark (x_i)
25-34	7	7	50	24.5-34.5	29.5
35-44	9	16	43	34.5-44.5	39.5
45-54	22	38	34	44.5-54.5	49.5
55-64	7	45	12	54.5-64.5	59.5
65-74	3	48	5	64.5-74.5	69.5
75-84	2	50	2	74.5-84.5	79.5

c) see the 3rd and 4th column
d) 54
e) 65
f) 38
g) 34 student get more than 54.
h) 38 student get less than 54.
i) see the 5th column.
j) see the 6th column.
k) correction factor $= \frac{35-34}{2} = 0.5$
l) 50 student gets more than 25.
m) 43 student gets more than 35.

Measures of Location
(Measures of central tendency)

It is typical representative of a set of data.

- The average of a given set of data is measure of central tendency (location)

There are types of averages in common use:

1. Mean
2. Median
3. Mode
4. Quartile
5. Decile
6. Percentile

i) $mean(\bar{x}) = \frac{x_1 + x_2 + x_3 + \dots + x_n}{n} = \frac{\sum x_i}{n}$ ← mean of raw data

$$= \frac{f_1x_1 + f_2x_2 + f_3x_3 + \ldots + f_nx_n}{f_1 + f_2 + f_3 + \ldots + f_n} = \frac{\sum f_ix_i}{f_i} \leftarrow \text{mean of}$$

discrete frequency distribution

$$= \frac{f_1m_1 + f_2m_2 + f_3m_3 + \ldots + f_nm_n}{f_1 + f_2 + f_3 + \ldots + f_n} = \frac{\sum f_im_i}{f_i} \leftarrow \text{mean of}$$

grouped frequency distribution

Where m is the class mark ($x_c = m$) ← class mark

ii) **Median** When observation are arranged in increasing or decreasing order of magnitude, the middle value is the median.

- **Median of Raw data is given by**

34
25
59

- $$M_d = \begin{cases} \left(\frac{n+1}{2}\right)^{th} \text{ item if } n \text{ is odd} \\ \left(\frac{\left(\frac{n}{2}\right)^{th} \text{item} + \left(\frac{n}{2}+1\right)^{th} \text{item}}{2}\right) \text{item if n is even} \end{cases}$$

Where n is number of observation

- $$M_d = B_L + \left(\frac{\frac{n}{2} - cf_b}{f_c}\right) i \leftarrow \text{ Median for grouped frequency distribution}$$

Where, B_L = Lower class boundary of the median class

n = total frequency

Cf_b = cumulative frequency just before the median class

f_c = frequency of the median class.

i = the size of class interval.

Combined mean

If $\overline{x}_1, \overline{x}_2$ are mean of n_1, n_2 observation respectively then the combined mean ($\overline{x}$) is

$$\overline{x} = \frac{n_1\overline{x}_1 + n_2\overline{x}_2}{n_1 + n_2} \longleftarrow \text{combined mean}$$

Illustrative example

2. One group of 50 student has a mean score 30 in exam. A second group of 100 student has a mean score of 40. find the mean of all the 150 student.

Solution: $n_1 = 50,\ \bar{x}_1 = 30,\qquad n_2 = 100,\ \bar{x}_2 = 40$

$$\therefore\ \bar{x} = \frac{50\times30+100\times40}{50+100} = \frac{5500}{150} = 36.7$$

2. a) Find the mean and median of the number 8, 3, 5, 10, 12

Solution: $\bar{x} = \dfrac{8+3+5+10+12}{5} = \dfrac{35}{5} = 7.6 \longleftarrow$ mean

To find the median first a write in increasing order, 3, 5, $\boxed{8, 10}$, 12
Since n = 6 is even number

$$\therefore M_d = \frac{8+10}{2} = \frac{18}{2} = 9 \longleftarrow \text{median}$$

b) The following frequency distribution tables,
Find i) the mean ($\bar{x}$) ii) the median (M_d)

Mark	3-5	6-8	9-11	12-14	15-17	18-20
Number of student frequ.	4	3	6	4	2	1

Solution: Here

- To find the mean first determine the class mark (x_c)
- Class mark (x_c) = $\dfrac{3+5}{2} = 4,\ \dfrac{6+8}{2} = 7,\ \dfrac{9+11}{2} = 10,$ 13, 16, 19
- To find the class boundaries, use correction factor, which is $\dfrac{6-5}{2} = 0.5 \longleftarrow$ correction factor
  - $3 - 0.5 = 2.5$ and $5 + 0.5 = 5.5$
  - $fx_c \longleftarrow$ product of frequency and class mark.

fx_c: $4\times4 = 16,\ 3\times7 = 21,\ 6\times10 = 60, \ldots$ Etc

Mark	Class	Class	Frequency	Cumulative

(class limit)	boundaries	mark(x_c)		frequency	
3-5	2.5-5.5	4	4	4	16
6-8	5.5-8.5	7	3	4+3 = 7	21
9-11	8.5-11.5	10	6 Md	7+6=13	60
12-14	11.5-14.5	13	4	13+4 = 17	52
15-17	14.5-17.5	16	2	17+2=19	32
18-20	17.5-20.5	19	1	19+1 = 20	19
Total			$\sum f = 20$		$\sum fx_c = 200$

- $\sum f = 20 \longleftarrow$ total frequency
- $\sum fx_c = 200 \longleftarrow$ Multiplying x_c by its corresponding frequency and adding

i) $$\therefore \ \bar{x} = \frac{\sum fx_c}{\sum f} = \frac{200}{20} = 10 \longleftarrow \text{mean}$$

ii) To find the median of grouped frequency distribution use,

$$M_d = B_L + \left(\frac{\frac{n}{2} - cf_b}{f_c} \right) i$$

- Median class is $\left(\frac{n}{2}\right) = \left(\frac{20}{2}\right)^{th}$ item = 10^{th} item. It lies on 3^{rd} class (8.5 – 11.5)
- $B_L = 8.5 \longleftarrow$ lower boundary of median class.
- $Cf_b = 7 \longleftarrow$ cumulative frequency just before the median class
- $f_c = 6 \longleftarrow$ frequency of the median class
- $i = 5.5 - 2.5 = 3 \longleftarrow$ class interval width
- $n = 20 \longleftarrow$ total frequency

$$\therefore \ M_d = 8.5 + \left(\frac{\frac{20}{2} - 7}{6} \right) 3 = 10 \longleftarrow \text{median}$$

c) The following frequency distribution tables

Find i) the mean
ii) the median iii) the mode

Daily income	6 – 10	11 – 15	16 – 20	21 – 25	26 – 30	31 – 35
Frequency	2	3	4	6	8	1

Solution: Here

First determine,

- class boundaries
- class mark
- cumulative frequency
- $fx_c \longleftarrow$ (frequency)(class mark)

Class boundaries	5.5-10.5	10.5-15.5	15.5-20.5	20.5-25.5	25.5-30.5	30.5-35.5
Frequency	2	3	4	6	0	1
Class mark (x_c)	8	13	18	23	28	33
Cumulative frequency	2	5	9	15	15	16
fx_c	16	39	72	138	0	33

i) $$\therefore \bar{x} = \frac{\sum fx_c}{\sum f} = \frac{16+39+72+138+0+33}{2+3+4+6+0+1} = \frac{298}{16} = 18.6 \leftarrow$$

mean

ii) To find the median first determine the median class by

$$\left(\frac{n}{2}\right)^{th} = \left(\frac{16}{2}\right)^{th} \text{ item} = 8^{th} \text{ item}$$

Hence the median class lies on 3^{rd} class. It is (15.5 – 20.5)

Thus,

- $B_L = 15.5 \longleftarrow$ lower boundary of median class.
- $Cf_b = 5 \longleftarrow$ cumulative frequency just before the median class
- $f_c = 4 \longleftarrow$ frequency of the median class
- $i = 10.5 - 5.5 = 5 \longleftarrow$ class interval width
- $n = 16 \longleftarrow$ total frequency

$$\therefore M_d = 15.5 + \left(\frac{\frac{16}{2} - 5}{4} \right) 5 = 19.25 \longleftarrow \text{median}$$

Mode (m_0)

Definition: The mode is a value which occurs most frequently in a set of observation:

- The mode may not exist and even if it does not exist it may not be unique.

1. Find the mode of the following data.
   a) 2, 1, 0, 4, 3, 3, 2, 5, -8, 7, -2, 3
   b) 2, 5, 3, 1, 7, 8, 9, 2, 1, 4, 5, 3, 1, 2, 5, 4, 6, 0, 1, 2, 5
   c) -1, 4, 7, 6, 9, 1, 0, 5, 8

   Solution:

   a) mode is $\hat{x} = 3$
   b) modes are $\hat{x} = 1$, $\hat{x} = 2$, $\hat{x} = 5$. Here there are 3 modal value.
   c) mode does not exist.

Mode of grouped date

Mode of grouped data in given uniform class interval can be obtained by the formula

$$Mode = M_0 = B_L + \left(\frac{d_1}{d_1 + d_2} \right) i$$

Where: B_L = lower class boundary of Modal class

d_1 = the difference between the frequency of the modal class and the frequency of the preceding class (pre-modal class)

d_2 = the difference between the frequency of the modal class and the frequency of next class

i = size of the class interval

Illustrative Example

11) a) Find the mode.

Class Interval	Frequency
4 – 6	14
7 – 9	39

10 – 12	27
13 – 15	16

Solution: The modal class is the 2^{nd} class because its frequency is the largest.

$B_L = 6.5$, $d_1 = 39 - 14 = 25$

$d_2 = 39 - 27 = 12$

$i = 9 - 6 = 3$

$$\therefore\ m_0 = B_L + \left(\frac{d_1}{d_1 + d_2}\right)i = 6.5 + \left(\frac{25}{25+12}\right)3 = 8.52$$

b) Fnd the mode.

Marks	0 – 9	10 – 19	20 – 29	30 – 39	40 – 49
Frequency	10	16	25	18	15

Solution: The modal class is 3^{rd} class because its frequency is the largest.

$\therefore B_L = 19.5$, $d_1 = 25 - 16 = 9$,

$d_2 = 25 - 18 = 7$, $i = 29 - 19 = 10$

$$\therefore\ m_0 = 19.5 + \left(\frac{9}{9+7}\right)10 = 25.125$$

Quartiles, deciles and percentiles

- The median divides, the distribution into two equal parts.
  - The quartiles divides, the distribution in to four equal parts.
  - The deciles divides, the distribution into ten equal parts.
  - Percentiles in to 100 equal parts.

Note: Quartiles, deciles and percentiles measures of location

Quartiles, deciles and percentiles for un grouped data.

Quartiles are values that divides set of data into four equal parts. There are three quartiles, Q_1, Q_2 and Q_3.

To find quartiles for ungrouped data

- Arrange the data in increasing order
- If the number of observation is:

a) **odd,** $Q_k = \left(\frac{k(n+1)}{4}\right)^{th}$ **item** **b) Even** $Q_k = \left(\frac{\left(\frac{kn}{4}\right) + \left(\frac{kn}{4}+1\right)}{2}\right)$

Illustrative Example

12. Find Q_1, Q_2 and Q_3 for the following date. 3, 6, 10, 25, 19, 14, 17, 22, 26

Solution: Arranging in increasing order, we get 3, 6, 10, 14, 17, 19, 22, 25, 26

$$Q_1 = \left(\frac{1(9+1)}{4}\right)^{th} \text{ item} = (2.5)^{th} \text{ item}$$

here Q_1 lies half way between 2^{nd} and 3^{rd} item

$\therefore Q_1 = (2^{nd})$ item + ½(3^{rd} item -2^{nd} item)

$= 6 + ½(10 - 6) = 6 + 2 = 8$

Or

$$Q_1 = \frac{6+10}{2} = 8$$

$$Q_2 = \left(\frac{2(9+1)}{4}\right)^{th} \text{ item} = 5^{th} \text{ item}$$

$\therefore Q_2 = 17 = \text{median}$

$$Q_3 = \left(\frac{3(9+1)}{4}\right)^{th} \text{ item} = 7.5^{th} \text{ item}$$

$$\therefore Q_3 = \frac{7^{th}\ item + 8^{th}\ item}{2} = \frac{22+25}{2} = 23.5$$

13. Find Q_1, Q_2 and Q_3

x	8	12	13	15	17	18	24
f	12	18	20	2	4	4	1

Solution: n = 61

$$Q_k = \left(\frac{k(n+1)}{4}\right)^{th} \text{ item} \Rightarrow Q_1 = \left(\frac{61+1}{4}\right)^{th} \text{ item} = (15.5)\text{th}$$

item

$$= \frac{15^{th}\ item + 16^{th}\ item}{2} = \frac{12+12}{2} = \frac{24}{2} = 12$$

$$Q_2 = \left(\frac{2(61+1)}{4}\right) = 31^{th} \text{ item} = 13 = \text{median}$$

$$Q_3 = \left(\frac{3(61+1)}{4}\right)^{th} \text{ item} = (46.5)^{th} \text{ item}$$

$$= \frac{46^{th}\ item + 47i^{th}\ item}{2} = \frac{13+13}{2} = 13$$

Deciles

Deciles are values that divide a set of data in to ten equal parts. There are D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8, D_9.

To find deciles for un grouped data

- Arrange the data in increasing order of magnitude
- If the number of observation is:

a) Odd, $D_i = \left(\frac{i(n+1)}{10}\right)^{th}$ item

b) even, $D_i = \left[\frac{\left(\frac{in}{10}\right) + \left(\frac{in}{10}+1\right)}{2}\right]^{th}$ item

Illustrative Example

14. Find D_2, D_3 and D_8 for each of the following data sets:

a) 18, 2, 4, 6, 10, 7, 9, and 11

Solution: Arranging in increasing order of magnitude, we get
2, 4, 6, 7, 9, 10, 11, 18

$$D_2 = \left(\frac{\frac{2(8)}{10} + \left(\frac{2(8)}{10}+1\right)}{2}\right)^{th} \text{ item} = (2.1)^{th} \text{ item}$$

Hence D_2 is between 2^{nd} and 3^{rd} item, i.e. $x_2 + 0.1(x_3 - x_2)$

$$\therefore D_2 = 4 + 0.1(6-4) = 4 + 0.2 = 4.2$$

$$D_3 = \left(\frac{\frac{3(8)}{10} + \left(\frac{3(8)}{10} + 1 \right)}{2} \right)^{th} \text{item} = \left(\frac{2.4 + 3.4}{2} \right)^{th} \text{item} = (2.9)^{th} \text{ item}$$

D_3 is between 2^{nd} item and 3^{rd} item i.e.

$x_2 + 0.9(x_3 - x_2)$ $\therefore D_3 = 4 + 0.9(6 - 4) = 5.8$

$$D_8 = \left(\frac{\frac{8(8)}{10} + \left(\frac{8(8)}{10} + 1 \right)}{2} \right)^{th} \text{item} = (6.9)^{th} \text{ item}$$

$\therefore D_8$ is between 6^{th} item and 7^{th} item

i.e. $x_6 + 0.9(x_7 - x_6)$

$\therefore D_8 = 10 + 0.9(11 - 10) = 10.9$

Percentiles

Data divided into 100 equal parts. There are $P_1, P_2, P_3 \ldots P_{99}$

Note: Percentiles are not the same as percentages.

To find percentiles

- Arrange the data in increasing order of value
- If the number of observation is

a) Odd, $P_t = \left(\frac{t(n+1)}{100} \right)^{th}$ item

b) Even, $P_t = \left(\frac{\left(\frac{tn}{100} \right) + \left(\frac{tn}{100} \right) + 1}{2} \right)^{th}$ item

Illustrative example

15. Find P_{24}, P_{42} and P_{75} for the following data.

a) 23, 36, 40, 44, 29, 27, 19, 7, 3

$$P_{24} = \left(\frac{24(9+1)}{100}\right)ith \text{ item} = (2.4)^{th} \text{ item}$$

$\therefore$ P_{24} is between 2^{nd} and 3^{rd} item

i.e. $x_2 + 0.4(x_3 - x_2)$

$\therefore$ $P_{24} = 7 + 0.4(19 - 7) = 11.8$

$$P_{42} = \left(\frac{42(9+1)}{100}\right)ith \text{ item} = 4.2 \text{ item}$$

Here, P_{42} lies between 4^{th} and 5^{th} item $= x_4 + 0.2\,(x_5 - x_4)$

$\therefore$ $P_{42} = 23 + 0.2(27 - 23) = 23.8$

$$P_{75} = \left(\frac{75(9+1)}{100}\right)ith \text{ item} = (7.5)^{th} \text{ item}$$

Here, P_{75} lies between 7^{th} and 8^{th} item

i.e. $x_7 + 0.5(x_8 - x_7)$

$\therefore P_{75} = 36 + 0.5(40 - 36) = 38$

Quartiles, Deciles and percentiles for grouped data

Each formula for grouped data is also given as follow

1. Quartiles, $Q_k = B_L + \left[\dfrac{\frac{kn}{4} - cf_b}{f_c}\right]i$

   Where, k = 1, 2, 3

   B_L = lower class boundary of k^{th} quartile class

   Cf_b = the cumulative frequency before the k^{th} quartile class

   f_k = frequency in the k^{th} quartile class

   i = the size of class

2. Deciles, $D_j = B_L + \left[\dfrac{\frac{jn}{10} - cf_b}{f_c}\right]i$

   Where: f_c = frequency of j^{th} – deciles class

3.	Percentiles. $P_i = B_L + \left[\dfrac{\dfrac{in}{100} - cf_b}{f_c}\right] i$

$\boxed{\text{Note, } Q_2 = D_5 = P_{50}}$

16. Find Q_1, Q_3, D_4, D_8, P_{12}, P_{24}, P_{68} and P_{99} of each of the following data set.

Age	3 – 12	13 – 22	23 – 32	33 – 42	43 – 52
F	4	12	10	7	2

Solution: First, find the cum. frequency

Age	3 – 12	13 – 22	23 – 32	33 – 42	43 – 52
F	4	12	10	7	2
Cf	4	16	26	33	35

To find Q_1

$$Q_1 = \left(\frac{1\times 35}{4}\right)^{th} \text{ item} = (8.75) \text{ item}$$

It is found in 2^{nd} class

$\therefore B_L = 12.5 \longleftarrow$ lower class boundary of 2^{nd} class

$cf_b = 4 \longleftarrow$ cum.freq. before 2^{nd} class of quartile

$f_c = 12 \longleftarrow$ freq. of 2^{nd} class

$i = 22 - 12 = 10 \longleftarrow$ class size

$$\therefore Q_1 = 12.5 + \left[\frac{\frac{35}{4} - 4}{12}\right](10) = 16.45$$

To find Q_3

$$Q_3 = \left(\frac{3\times 35}{4}\right) = (26.25)^{th} \text{ item, it is found in } 4^{th} \text{ class}$$

$\therefore B_L = 32.5 \longleftarrow$ lower class boundary of 4^{th} class

$cf_b = 26 \longleftarrow$ cum.freq. before 4^{th} class of quartile

$f_c = 7 \longleftarrow$ freq. of 4^{th} class

i = 10

$$\therefore Q_3 = 32.5 + \left[\frac{\frac{3 \times 35}{4} - 26}{7}\right](10) = 32.857$$

To find D_4

$$D_4 = \left(\frac{4 \times 35}{10}\right)^{th} \text{ item} = (14)^{\text{th}} \text{ item}$$

It is found in 2nd class

$\therefore B_L = 12.5 \longleftarrow$ lower class boundary of 2nd class

$cf_b = 4 \longleftarrow$ cum.freq. before 2nd class of decile

$f_c = 12 \longleftarrow$ freq. of 2nd class

$i = 10 \longleftarrow$ class size

$$\therefore D_4 = 12.5 + \left[\frac{\frac{4 \times 35}{10} - 4}{12}\right](10) = 20.83$$

To find D_8

$$D_8 = \left(\frac{8 \times 35}{10}\right)^{th} \text{ item} = (28)^{\text{th}} \text{ item}$$

It is found in 4th class

$\therefore B_L = 32.5 \longleftarrow$ lower class boundary of 4th class

$cf_b = 26 \longleftarrow$ cum.freq. before 4th class of decile

$f_c = 7 \longleftarrow$ freq. of 4th class

$i = 10 \longleftarrow$ class size

$$\therefore D_8 = 32.5 + \left[\frac{\frac{8 \times 35}{10} - 26}{7}\right](10) = 35.35$$

To find P_{12}

$$P_{12} = \left(\frac{12 \times 35}{100}\right) \text{item} = (4.2) \text{ item}$$

It is found. in 2^{nd} class

$\therefore B_L = 12.5 \leftarrow$ lower class boundary of 2^{nd} class

$cf_b = 4 \leftarrow$ cum.freq. before 2^{nd} class of percentile

$f_c = 12 \leftarrow$ freq. of 2^{nd} class

$$\therefore P_{12} = 12.5 + \left[\frac{\frac{12 \times 35}{100} - 4}{12}\right](10) = 12.66$$

To find P_{24}

$$P_{24} = \left(\frac{24 \times 35}{100}\right)^{th} \text{ item} = (8.4)^{th} \text{ item}$$

It is found in 2^{nd} class

$\therefore B_L = 12.5$

$$\therefore P_{24} = 12.5 + \left[\frac{\frac{24 \times 35}{100} - 4}{12}\right](10) = 16.16$$

To find P_{68}

$$P_{68} = \left(\frac{68 \times 35}{100}\right)^{th} \text{ item} = (23.8)^{th} \text{ item}$$

It is found in 3^{rd} class

$\therefore B_L = 22.5 \leftarrow$ lower class boundary of 3^{rd} class

$cf_b = 16 \leftarrow$ cum.freq. before 3^{rd} class of percentile

$f_c = 10 \leftarrow$ freq. of 3^{rd} class

$$\therefore P_{68} = 22.5 + \left[\frac{\frac{68 \times 35}{100} - 16}{10}\right](10) = 30.3$$

To find P_{99}

$$P_{99} = \left(\frac{99 \times 35}{100}\right)^{th} \text{ item} = (34.65) \text{ item}$$

It is found in the 5^{th} class.

$\therefore B_1 = 42.5 \leftarrow$ lower class boundary of 5^{th} class

$cf_b = 33 \leftarrow$ cum.freq. before 5^{th} class of percentile

$f_c = 2 \leftarrow$ freq. of 5^{th} class

$$\therefore P_{99} = 42.5 + \left[\frac{\frac{99 \times 35}{100} - 33}{2}\right](10) = 50.75$$

17. Find D_4, D_8, D_9, P_{12}, P_{24}, P_{70} for each of the following data set

x	12	16	17	19	21	22	28
f	12	18	20	2	4	4	1

Solution: $\Sigma f = n = 61$

$$D_k = \left(\frac{k(n+1)}{10}\right)^{th} \text{ item}$$

$$\therefore D_4 = \left(\frac{4(61+1)}{10}\right)^{th} \text{ item} = (24.8)^{th} \text{ item}$$

$$\therefore D_4 = x_{24} + 0.8(x_{25} - x_{24}) = 16 + (0.8)(16 - 16) = 16$$

$$D_8 = \left(\frac{8(61+1)}{10}\right)^{th} \text{ item} = (49.6)^{th} \text{ item}$$

$$= x_{49} + 0.6(x_{50} - x_{49}) = 17 + 0.6(17 - 17) = 17$$

$$D_9 = \left(\frac{9(61+1)}{10}\right)^{th} \text{ item} = (55.8)^{th} \text{ item}$$

$$= x_{55} + 0.8(x_{56} - x_{55}) = 21 + 0.8(21 - 21) = 21$$

$$P_{12} = \left(\frac{12(61+1)}{100}\right)^{th} \text{ item} = (7.44)^{th} \text{ item}$$

$$= x_7 + 0.44(x_8 - x_7) = 12 + 0.44(12 - 12) = 12$$

$$P_{24} = \left(\frac{24(61+1)}{100}\right)^{th} \text{ item} = (14.88)^{th} \text{ item} = 16$$

Application of Quantiles

(Quartiles, deciles and percentiles are called quantiles).

Note: Quartiles
- Q_1 (lower quartile)
- Q_2 (middle quartile)

Q_3 (upper quartile

Q_1(lower) quartile $\equiv \left(\frac{1}{4}\right)^{th}$ of the values are less than or equal to Q_1

and $\left(\frac{3}{4}\right)^{th}$ of the values greater than it.

Q_3 (upper) quartile $\equiv \left(\frac{3}{4}\right)^{th}$ of the values are less than or equal to Q_3

and $\left(\frac{1}{4}\right)^{th}$ of the value are greater than it. $Q_2 = D_5 = P_{50}$

Illustrative example

17. Find Q_1,Q_2 and Q_3 of the following data. It is marks obtained in a mathematics exam (out of 60)

Mark	30–34	35–39	40–44	45–49	50–54	55–59
Number of student	7	12	3	5	3	6
Cumlative frequency	7	19	22	27	30	36

a) If students in the top 25% are to be awarded certificate of "Best in math". What is the minimum mark for a certificate

b) If student whose scores are in the bottom 25% of the marks are considered as failures, then what is the maximum failing mark?

c) Find the percentile of student who get more than 50

d) If student get more than 75 percentile find the minimum mark he/she get.

e) Find the pass mark if 70% of the student to pass.

Solution: To find Q_1

$$Q_1 = \left(\frac{1\times 36}{4}\right)^{th} \text{ item} = 9^{th} \text{ item, which falls in } 2^{nd} \text{ class } (35-39)$$

$B_L = 34.5 \longleftarrow$ Lower class boundary of 2^{nd} class.

$Cf_b = 7 \longleftarrow$ the cum. Frequence before 2^{nd} class.

$f_k = 12 \longleftarrow$ the number of frequence in the 2^{nd} class.

$i = 5 \longleftarrow$ the size of class interval.

$$\therefore Q_1 = 34.5 + \left(\frac{\frac{1\times 36}{4} - 7}{12}\right)(5) = 35.3$$

- Interpretation

  $Q_1 = 35.3$ means $\left(\frac{1}{4}\right)^{th}$ or 25% of the student gets less than or equal to 35.3 and $\left(\frac{3}{4}\right)^{th}$ or 75% of the student scores greater than 35.3

To find Q_3

$Q_3 = \left(\frac{3\times 36}{4}\right)^{th}$ item = 27th item which falls in 4th class (45 – 49)

$\therefore B_L = 44.5 \longleftarrow$ lower class bourdry of 3rd class.

$Cf_b = 22 \longleftarrow$ the cumfreq. Before 3rd class.

$f_k = 5 \longleftarrow$ the number of frequency in the 3rd class.

$i = 5$

$$\therefore Q_3 = 44.5 + \left[\frac{27-22}{5}\right]5 = 49.5$$

- Interpretation

  $Q_3 = 49.5$ means $\left(\frac{3}{4}\right)^{th}$ or 75% student gets less than or equal to 49.5 and $\left(\frac{1}{4}\right)^{th}$ or 25% of the student score greater than 49.5

a) $Q_3 = 49.5 \longleftarrow$ minimum mark to award certificate "Best in math"

b) $Q_1 = 35.3 \longleftarrow$ maximum failing mark

c) Let the percentile be P_k, Hence $P_k > 50$

$$P_k = B_L + \left(\frac{\frac{k\times 36}{100} - cf_b}{f_c}\right)(i) \Rightarrow 50 + \left(\frac{\frac{k\times 36}{100} - 27}{3}\right)5 > 50$$

$\Rightarrow (0.12k - 9)5 > 0 \Rightarrow 0.6k - 45 > 0 \Rightarrow 0.6k > 45$

$\therefore k > \frac{45}{0.6} = 75 \qquad \therefore P_{75}$

d) More than 75 percentile means

$\Rightarrow P_{25} = 34.5 + \left[\frac{9-7}{12}\right)5 = 35.3$

$35.3 \longleftarrow$ the minimum mark he/she get.

e) If 70% is to pass then 30% is to fail

Therefore, answer is the 30th percentile which is P_{30}

To find P_{30}

$P_{30} = \left(\frac{30 \times 36}{100}\right)^{th}$ item = $(10.8)^{th}$ item

It is found in 2nd class (35 – 39)

- $B_L = 34.5 \longleftarrow$ lower boundary of 2nd class.
- $Cf_b = 7 \longleftarrow$ cumulative frequency just before 2nd class if P_{30}
- $f_c = 12 \longleftarrow$ frequency of the 2nd class of P_{30}
- $i = 5 \longleftarrow$ class width

$\therefore \quad P_{30} = 34.5 + \left(\frac{10.8-7}{12}\right)5 = 36.08 \leftarrow$ minimum pass mark

Measures of dispersion

The scatter or spread of the items of a distribution is known as a dispersion or variation. It is the degree to which numerical data tend to spread about an average value.

Measure of centeral tendency is not enough to describe the nature of the data. It is therefore better to use both central tendency and dispersion.

Consider the following data set

Group	Data value	Total	Mean	Variance
A	5,5,5,5,5	25	5	0
B	3,6,4,6,6,	25	5	2
C	2,5,8,0,10	25	5	17

From the table the means are identical but variation in the given set of data

$\therefore$ Measure of dispersion, helps us to know the degree of uniformity and consistency of the series.

- If the difference between the item is large the dispression also large and vice versa.
- If the difference between is very small, measure of central tendency (Average) represents and describes the data adequately.

The common measure of dispersions are

* Range * Variance * Standard deviation

Range

The difference between the two extreme values Range = $x_{max} - x_{min}$

- It depends on extreme value

For group data

Range is the diffenee between the upper class bounds of the highest class and the lower class boundary of the lowest class.

Illustrative example

18) Find the range for the following table

x	6 – 11	12 – 17	18 – 23
f	5	6	9

Range = 23.5 – 5.5 = 18

Variance and standard deviation

They are the most commonly used measure of dispersion.

Definition: Variance is the mean of the squared deviation of each item from the mean.

- Standard deviation = $\sqrt{variance}$
- Variance $(S^2) = \frac{(x_1 - \bar{x})^2 + (x^2 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n}$

$$\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}$$

- Standard deviation $S = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}}$

For frequency distribution

- Variance $= \dfrac{\sum_{i=1}^{n} f_i\left(x_i - \bar{x}\right)^2}{\sum_{i=1}^{n} f_i}$

Illustrative example

19) Find the variance and standard deviation of the population function

a) 1,2,6,3

b)

x	3	4	5	6	7
f	2	1	3	3	1

Solution: $\bar{x} = \dfrac{1+2+6+3}{4} = 3$

a)

x	$x-\bar{x}$	$\left(x-\bar{x}\right)^2$
1	$1-3=-2$	4
2	$2-3=-1$	1
3	$3-3=0$	0
6	$6-3=3$	9

$\sum\left(x-\bar{x}\right)^2 = 4+1+0+9 = 14$

$\therefore$ Variance $\left(\delta^2\right) = \dfrac{\sum\left(x-\bar{x}\right)^2}{n} = \dfrac{14}{4} = 3.5$

$\therefore$ Standard deviation $\left(\delta^2\right) = \sqrt{3.5}$

b) Mean $\left(\bar{x}\right) = \dfrac{2\times3+1\times4+3\times5+3\times6+1\times7}{2+1+3+3+1} = \dfrac{6+4+15+18+7}{10}$

$= \dfrac{50}{10} = 5$

X	f	$x-\bar{x}$	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
3	2	$3-5=-2$	4	8
4	1	$4-5=-1$	1	1
5	3	$5-5=0$	0	0

6	3	6–5= 1	1	3
7	1	7 – 5 = 2	4	4

$$\therefore \text{Variance} = \frac{\sum f(x-\bar{x})^2}{\sum f} = \frac{8+1+0+3+4}{2+1+3+3+1} = \frac{16}{10}$$

$\therefore$ Standard deviation = $\sqrt{1.6}$

20) Find the range, variance and standard deviation for each of the following data.

Class	0 – 4	5 – 9	10 – 14	15 – 19	20 – 24
Frequency	3	4	1	2	1

Solution:

- Range 24.5 – (-0.5) = 25

To find variance

Class	Freq	Class mark (x_e	fx_c	$x_c - \bar{x}$	$(x_c - \bar{x})^2$	$f(x_c - \bar{x})^2$
0 – 4	3	2	6	– 7	49	147
5 – 9	4	7	28	–2	4	16
10 – 14	1	12	12	0	0	0
15 – 19	2	17	34	8	64	128
20 – 24	1	22	22	13	169	169

- Mean $(\bar{x}) = \frac{\sum fx_c}{\sum f} = \frac{102}{11} \approx 9$

- Variance $(\delta^2) = \frac{\sum f(x_c - \bar{x})^2}{\sum f} = \frac{460}{11} = 41.7$

Standard deviation = $\sqrt{41.7} \approx 6.4$

21) The sum of eight numbers is 48 and the sum of their squares is 296

Find a) mean b) variance c) standard deviation

Solution: mean $(\bar{x}) = \frac{48}{8} = 6$

$$\text{Variance} = \overline{x^2} - (\bar{x})^2$$

Variance = $\frac{296}{8} - 6^2 = 37 - 36 = 1$ S.D = $\sqrt{1} = 1$

If the standard deviation of $x_1, x_2, x_3, \ldots x_n$ is 5, then find standard deviation of $3x_1 + 2$, $3x_2 + 2$, $3x_3 + 2$ . . .

Solution: Recall grade lesson

Note. Adding constant to a data new variance = old variance

- Multiplying constant k to data
- New variance = k^2 times old variance

$\therefore$ New variance = $(3)^2(25)$

$\therefore$ New S.D = $\sqrt{(3^2)(25)} = (3)(5) = 15$

3... If marks obtained by 9 students in two test (out of 10)

Test I	10	9	8	7	6	5	4	3	2
Test II	7	5	9	4	7	5	7	4	6

a) determine the range, variance and standard deviation of each test.

b) Which test has highest variation.

c) Which test has more consistence score

d) Which test has lesser consistence score

Solution:

- Range of test I, $10 - 2 = 8$
- Range of test II, $9 - 4 = 5$
- Mean of test I =

$$\frac{10+9+8+7+6+5+4+3+2}{9} = \frac{54}{9} = 6$$

- Mean of test II = $\frac{7+5+9+4+7+5+7+4+6}{9} = 6$
- Variance of test I =

$$\frac{(10-6)^2+(9-6)^2+(8-6)^2+\ldots+(2-6)^2}{9} = 6.7$$

- Standard dev. of test I = $\sqrt{6.7} = 2.58$
- Variance of test II = $\frac{(7-6)^2+(5-6)^2+\ldots+(6-6)^2}{9}$

$$= \frac{1+1+9+4+1+1+1+4+0}{9} \approx 2.44$$

$\therefore$ standard dev. of test II $= \sqrt{2.44} \approx 1.56$

Hence:

a)

	Range	Variance	S.D
Test I	8	6.7	2.58
Test II	5	2.44	1.56

b) Test I has highest variation.

c) Test II has more consistence score

- Lesser S.D is more consistence
- Greater S.D is less consistence.

d) Test I has lesser consistence score.

Permutations and combination

i) Fundamental principles of counting

Multiplication principle

If one event can occur in **m** ways and second event can occur n ways then both events can occur in **mn** ways.

Illustrative Example

24. There are 5 gates to go from A to B and 4 different gates to go from B to C. How many different ways to go from A to C

Solution:

There are $(5 \times 4) = 20$ ways

25. In how many different way can five books be arranged on shelf

Solution:

1^{st} book has 5 choices
2^{nd} book has 4 choices
3^{rd} book has 3 choices
4^{th} book has 2 choices
5^{th} book has 1 choices

$\therefore$ The total number of arrangement

$5 \times 4 \times 3 \times 2 \times 1 = 120$ ways

26. In how many different ways can 6 people be seated in a row

Solution: $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ ways

27. Abebe has 4 pair of trousers, 5 shirts and 2 pair of shoes. In how many different ways can he dress.

Solution: $4 \times 5 \times 2 = 40$ ways

28. In how many ways can a president, vice-president and secretary be elected from a committee of 12 people

Solution: There are 12 people to place in 3 positions. Thus

$$12 \times 11 \times 10 = 1320 \text{ ways}$$

29. There are 6 girls and 7 boys for group work. In how many different ways can a girl-boy couple be formed for group work.

Solution: $6 \times 5 = 30$ ways

30. How many four-digit numbers can be formed from the digit 1,3,4,5,6 and 7 where the digit is used at most once (Repetitions are not allowed)

a) If the number must be odd
b) If the number must be even
c) If the number greater than 5000
d) If the number less than 4000.

Solution: Here

a) * selecting thousand places we can fill either of 6 digit
* selecting hundreds places = 5 way
* selecting tens places = 4 way
* selecting the unit places = 2 way
$\therefore 6 \times 5 \times 4 \times 2 = 240$ way

b) $6 \times 5 \times 4 \times 1 = 120$ way
c) $3 \times 5 \times 4 \times 3 = 180$ way
d) $2 \times 5 \times 4 \times 3 = 120$ way

31. From question #30, if repetition is allowed i.e. the digit used more than once then

a) $6 \times 6 \times 6 \times 4 = 864$ way
b) $6 \times 6 \times 6 \times 2 = 432$ way
c) $3 \times 6 \times 6 \times 6 = 648$ way
d) $2 \times 6 \times 6 \times 6 = 432$ way

Definition: Factorial Notation
For each positive integer n,
$n! = n(n-1)(n-2) \ldots\ldots 3.2.1.$ also $0! = 1$

Example: Simplify: a) $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

b) $$\frac{8!}{5!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = 336$$

c) $5! + 3! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 + 3 \cdot 2 \cdot 1 = 126$

Permutation

A Permutation is an arrangement of objects.

The number of permutation of **n** different object taken r at a time is $P(n,r)=\dfrac{n!}{(n-r)!}$

- The permutation of n different objects taken r at a time means an arrangement of r out of the n-object with attention given to the order of arrangement.

Illustrative Example

32. Find a) All the permutation of A B C

b) All the permutation of A B C D

c) All the permutation of 1, 2, 3, 7

***Solution*:** 1^{st} place we can fill either of the 3 letter A, B or C

2^{nd} place we can fill either of the two remaining

3^{rd} place we can fill the remaining one letters.

$\therefore$ P(3, 3)$=\dfrac{3!}{(3-3)!}=\dfrac{3!}{0!}=\dfrac{3!}{1!}=\dfrac{3!}{1}=3\times2\times1=6$ different arrangement

b) The 4 letters, A, B, C, D can be arranged as P(4, 4) =

$$P(4, 4)=\frac{4!}{(4-4)!}=\frac{4\times3\times2\times1}{1}=24$$ different arrangement

c) The digit 1, 2, 3, 7 can be arranged

- Thousand place we can fill either of the four digit.
- Hundred place we can fill either of the remaining three digit.
- Tens place we can fill either of the two digit
- Unit place we fill the remaining digit.

$\therefore 4\times3\times2\times1=24$ different 4 digit number we can make

$\therefore \dfrac{4!}{(4-4)!}=\dfrac{4!}{0!}=4\times3\times2\times1=24$ different 4 digit # make

33. a) How many permutation are there of 3 people selected from a group of 10 people.

b) In how many ways can a first, a second, and third prize be awarded in class of 9 student.

c) A row of six seats in class room is to be filled by selecting individual from a group of ten students.
In how many ways can the seats

d) In how many ways can 7 books be arranged in shelf

e) There are 8 different colours how many ways of 3 different colour are possible

Solution: Here

a) $P(10,3)=\dfrac{10!}{(10-3)!}=\dfrac{10!}{7!}=10\times9\times8=720$

b) $P(9,3)=\dfrac{9!}{(9-3)!}=\dfrac{9\times8\times7\times6!}{6!}=504$

c) $P(10,6)=\dfrac{10!}{(10-6)!}=\dfrac{10!}{4!}=10\times9\times8\times7\times6\times5=151{,}200$

d) $P(7,7)=\dfrac{7!}{(7-7)!}=7!=7\times6\times5\times4\times3\times2\times1=5040$ way

e) $P(8,3)=\dfrac{8!}{(8-3)!}=\dfrac{8!}{5!}=8\times7\times6=336$ way

34. a) How many three digit numbers can be formed from 3, 4, 5, 6 and 7

i) If repetition allowed

ii) If each digit used at most once i.e. (repetition not allowed)

Solution: i) $5\times5\times5=125$

ii) $P(5,3)=\dfrac{5!}{(5-3)!}=5\times4\times3=60$

Permutation of some identical object

The number of permutation of n objects of each

n_1 are alike object of 1st kind,

n_2 are alike object of 2nd kind.

n_3 are alike object of kth kind, is

$\dfrac{n!}{n_1!n_2!\dots n_k}$ where $n=n_1+n_2+\dots+n_k$

Illustrative example

35. Find the number of arrangement (permutation) can be made from.

a) 4 black, 3 red, 2 white

b) 2 black, 2 red, 2 white

Solution: Here

a) $n = 4 + 3 + 2 = 9$ 9 ~~4~~ 4 ~~7~~ 7 ~~3~~ ...

$$\therefore \frac{9!}{4!3!2!} = \frac{9\times8\times7\times\cancel{6}\times5}{3\times\cancel{2}\times2} = 630$$

b) $n = 2 + 2 + 2 = 6$

$$\therefore \frac{6!}{2!2!2!} = \frac{6\times5\times4\times3\times2\times1}{2\times2\times2} = 90$$

36. Find the number of arrangement (permutation can be made ... of the letters of the word

a) LAMP b) FOOD c) CALCULUS

Solution:

a) There are 4 different letters in word "LAMP"

$$\therefore 4P_4 = \frac{4!}{(4-4)!} = 4! = 4\times3\times2\times1 = 24$$

b) Again, there are 4 letters in "FOOD". 1F, 2'0',1D

$$\therefore \frac{4!}{1!2!1!} = \frac{4\times3\times2\times1}{2\times1} = 12 \text{ differ ways}$$

c) Although there are 8 letters in word "CALCULUS" 2C,1A,2L,2U,1S.

$\therefore$ The permutation (arrangement) will be

$$\frac{8!}{2!1!2!2!1!} = \frac{8\times7\times6\times5\times4\times3\times2\times1}{2\times1\times2\times1\times2\times1} = 5,040.$$

37. In how many ways can the letters of word :ARRANGE" be arranged? How many of these arrangements are there in which

a) the two R'S come together

b) the two R'S and the two A'S come together

Solution:

a) The number of arrangements in which the two RS come together can be obtained by treating the two R'S as one letter. Thus, total number of arrangement.

$$\therefore \frac{6!}{2!1!1!1!} = 360$$

b) $$\frac{5!}{1!1!1!1!1!} = 5! = 120$$

38. a) Find the number of arrangement (permutation) that can be made out of the letters of the word "MATHEMATICS". In how many of these permutations

b) Do all the vowels occur together

c) Do the word start with I

d) Begins with the two A

e) Begins with E and ends with C

Solution:

a) The word "MATHEMATICS" consists of 11 letters of which 2 are A_s,

2 M_s

2 T_s and the rest all different

$\therefore$ The total number of arrangements (permutations) are

$$\frac{11!}{2!2!2!}$$

b) The word "MATHEMATICS" consists of 4 vowels A, A, E and I (two are similar)

To find the number of arrangements in which the four vowels occur together, forming one letter. Thus we are left with 7 letter of which

$$\left.\begin{array}{l} 2 \text{ are } M_s \\ 2 \text{ are } T_s \\ 1 \text{ is } H \\ 1 \text{ is } C \\ 1 \text{ is } S \end{array}\right\} \text{arranged as} = \frac{7!}{2!2!}$$

The 4 vowels arranged as $= \frac{4!}{2!}$

$\therefore$ Total number of arrangements are $\left(\frac{7!}{2!2!}\right)\left(\frac{4!}{2!}\right)$

c) $\frac{10!}{2!2!2!}$ d) $\frac{9!}{2!2!}$ e) $\frac{9!}{2!2!2!}$

39. a) In 3 different physics books, 4 different chemistry books and 2 different maths books are to be arranged on shelf and each type of a book are identical. In how many can these books be arranged

Solution: $\frac{9!}{3!4!2!}$ .... Explanation $3 + 4 + 2 = 9$

40. How many different number of five digits can be made using the digit 4, 4, 2, 2, 2

Solution: There are two 4S

Three, 2S

Hence the number of different numbers can be formed

$$\frac{5!}{2!3!} = \frac{1\times2\times3\times4\times5}{1\times2\times1\times2\times3} = 10$$

Circular permutation

A circular permutation is an arrangement of element in a circular pattern.

- In circular arrangement, there is no "first place" so that fix the position of one object
- A circular arrangement of **n** object is $(n-1)!$ different way

41. **Example:** In how many ways can 6 people be seated around circular table?

Solution: Use $(n-1)!$

$\therefore (6-1)! = 5! = 5 \times 4\times 3 \times 2 \times 1 = 120$ way

Combination

A combination is a selection of objects with out considering the order

- **AB** and **BA** represent the same combination

The number of combination of **n** object taken **r** at a time is denoted by $C(n, r) = \binom{n}{r} = \frac{n!}{(n-r)!r!}$

Illustrative Example

42. Compute each of the following

a) C(8, 5) b) C(7, 3) c) C(n, n)

Solution: Here

a) $C(8,5) = \frac{8!}{(8-5)!5!} = \frac{8!}{3!5!} = \frac{8\times7\times6}{3\times2} = 56$

b) $C(7,3) = \frac{7!}{(7-3)!3!} = \frac{7\times6\times5}{3\times2} = 35$

c) $C(n,n) = \frac{n!}{(n-n)!n!} = 1$

43. If C(n, 3) = C(n, 5), find n

Solution: since C(n, r) = C(n, n-r)

Thus, C(n, 3) = C(n, n-3) $\Rightarrow$ C(n, n-3) = C(n, 5)

$\therefore$ n – 3 = 5 $\therefore$ n = 8

44. If C(n, 15) = C(n, 11), find, n

Solution: C(n, 15) = C(n, n – 15)

$\Rightarrow$ C(n, n-15) = C(n,11)

$\therefore$ n – 15 = 11

$\therefore$ n = 26

45. If C(18, r) = C(18, r + 2), then find r

Solution: C(18, r) = C(18, 18 – r)

$\Rightarrow$ C(18, 18-r) = C(18, r +2)

$\Rightarrow$ 18 – r = r + 2 $\Rightarrow$ 16 = 2r $\therefore$ r = 8

46. a) In how many way can 2 books be selected out of 10 different books?

b) In how many ways can a committee of 3 be selected from 8 people

c) In how many ways can three or more people be selected out of 5.

Solution: Here

a) $C(10,2)=\frac{10!}{8!2!}=\frac{10\times 9}{2}=45$

b) $C(8,3)=\frac{8!}{(8-3)!3!}=\frac{8\times 7\times 6}{3\times 2}=56$

c) $C(5, 3) + C(5, 4) + C(5, 5) = 10 + 5 + 1 = 16$

47. In how many ways can 3 men and 2 women be selected fro... 12 men and 8 women?

Solution: Out of 12 men select 3 men

Out of 8 women select 2 women.

$$\therefore\ C(12,3)\cdot C(8,2)=\left(\frac{12!}{(12-3)!3!}\right)\cdot\left(\frac{8!}{(8-2)!2!}\right)=6160$$

48. A number of ways of selecting a committee of 3 boys and ... girls from a group of 5 boys and 8 girls from a group of 5 bo... and 8 girls is ------------ (Entrance exam)

A) 560 B) 128 C) 40 D) 256

Solution: $C(5, 3)\cdot C(8, 5) = (10)(56) = 560$ Answer...

49. A committee of 3 members has to be formed from 5 men a... 6 women. In how many ways can this be done when th... committee consists of

a) 2 women and 1 men

b) all men

c) at least 2 men

d) all women

e) at most 2 men

Solution:

a) 2 women from 6 women can be selected $\binom{6}{2}$.

1 men from 5 men can be selected $\binom{5}{1}$

$$\therefore\ \binom{6}{2}\binom{5}{1}=\left(\frac{6!}{(6-2)!2!}\right)\left(\frac{5!}{(5-1)!1!}\right)=(15)(5)=75\text{ way...}$$

b) 3 men from 5 men selected is $C\binom{5}{3}=10$

c) at least 2 men means:

2 men and 1 women or 3 men and no women

$$\therefore C\binom{5}{2}\cdot\binom{6}{1}+\binom{5}{3}\cdot\binom{6}{0}=(10)(6)+(10)(1)=70$$

d) All women means 3 of them are women.

$$\therefore \binom{6}{3}\cdot\binom{5}{0}=20 \text{ ways}$$

e) at most 2 men: 2 men or 1 men

2 men and 1 women or 1 men and 2 women

$$\therefore \binom{5}{2}\cdot\binom{6}{1}+\binom{5}{1}\cdot\binom{6}{2}=(10)(6)+(5)(15)=135$$

50. A set contains 5 elements. How many of its subset has:

a) exactly 3 element

b) at least 3 element

c) at most 3 element

Solution:

a) $C\binom{5}{3}=\frac{5!}{2!3!}=\frac{5\times 4}{2}=10$

b) at least 3 element mean that (3, 4 or 5 element)

$$\therefore \binom{5}{3}+\binom{5}{2}+\binom{5}{1}+\binom{5}{0}=10+10+5+1=26$$

51. A box contains 5 white, 2 Red and 3 Black balls. If three balls are selected at random. Find the number of ways such that

a) One ball is black c) exactly 2 white

b) 3 of them white d) 1 white, 1 red and 3 black

Solution: Total ball: 5 + 2 + 3 = 10ball

a) $\binom{3}{1}\binom{7}{2}=(3)(21)=63$ c) $\binom{5}{2}\binom{5}{1}=(10)(5)=50$

b) $\binom{5}{3}\binom{5}{0}=10$ d) $\binom{5}{1}\binom{2}{1}\binom{3}{3}=5\cdot(2)\cdot 1=10$

Binomial theorem

Binomial theorem for positive integers if n is a positive integer, then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n}b^n$$

Illustrative example

52. Expand: $(2x^2 + 3)^4$

Solution:

$$(2x^2+3)^4 = \binom{4}{0}(2x^2)^4 + \binom{4}{1}(2x^2)^3(3) + \binom{4}{2}(2x^2)^2(3)^2 + \binom{4}{3}(2x^2)\ldots$$

$$= 16x^8 + 96x^6 + 216x^4 + 216x^2 + 81$$

Example: $\left(\sqrt{x} - 2y\right)^5$

$$\left(\sqrt{x} - 2y\right)^5 = \binom{5}{0}\sqrt{x}^5 + \binom{5}{1}\sqrt{x}^4(-2y)^1 + \binom{5}{2}\sqrt{x}^3(-2y)^2$$

$$+\binom{5}{3}\sqrt{x}^2(-2y)^3 + \binom{5}{4}\sqrt{x}(-2y)^4 + \binom{5}{5}(-2y)^5$$

$$= x^{\frac{5}{2}} - 10x^2y + 40x^{\frac{3}{2}}y^2 - 80xy^3 + 80\sqrt{x}y^4 - 32y^5$$

53. Find the 4th term in the expansion of $(2x^3 - 3y^2)^5$

Solution: with $a = 2x^3$, and $b = -3y^2$, and using the binomial theorem

with k = 3 and n = 5 we have $\binom{5}{3}(2x^3)^2(-3y^2)^3 = -1080x^6y^6$

$\therefore$ The fourth term is $-1080x^6y^6$

54. Find the indicated term without expanding

a) $(3x - y)^{10}$, eighth term

b) $(x + 2y)^{12}$, the fourth term

c) $\left(\frac{3}{x} - \frac{x}{3}\right)^{13}$, the seventh term

Solution: a) a = 3x, b = -y, k = 7, and n = 10, we have

$$\binom{10}{7}(3x)^3(-y)^7 = -3240x^3y^7$$

b) a = x, b = 2y, k = 3, and n = 12 $\therefore$

$$\binom{12}{3}x^9(2y)^3 = -960x^7y^3$$

c) a = $3x^{-1}$, b = $\frac{1}{3}x$, k = 6 and n = 1 $\therefore$

$$\binom{13}{6}(3x^{-1})^7\left(\frac{1}{3}x\right)^6 = \binom{13}{6}(3)x$$

55. Find the coefficient of x^5 y^3 in the expansion of $\left(\frac{5}{2}x - \frac{2}{5}y\right)^8$

Solution: $\binom{8}{k}\left(\frac{5}{2}x\right)^{8-k}\left(\frac{-2}{5}y\right)^k = x^5y^3$

$\therefore$ k = 3

$$\therefore \binom{8}{k}\left(\frac{5}{2}x\right)^5\left(\frac{-2}{5}y\right)^3 \therefore \binom{8}{3}\left(\frac{5}{2}x\right)^5\left(\frac{-2}{5}y\right)^3$$

$$\therefore (56)\left(\frac{5^5}{2^5}\right)\left(\frac{(-2)^3}{5^3}\right)x^5y^3 = -350$$

56. Find whether there is any term-containing x^9 in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{20}$

Solution: $\binom{20}{k}(2x^2)^{20-k} \cdot (-x^{-1})^k = (\ )x^9$

$$\Leftrightarrow x^{40-2k-k} = x^9$$

$$40 - 3k = 9 \Leftrightarrow k = \frac{31}{3}$$

Which is not possible because r must be a positive integer. Therefore there is no term containing x^9 in the given expansion.

57. Find the term independent (constant) of x in the expansion of each of the following.

a) $\left(x^2-\frac{1}{x}\right)^9$ b) $\left[2x^3-\frac{1}{x}\right]^{12}$ c) $\left(x-\frac{1}{2x}\right)^{10}$

Here, we make the exponent of x zero

Solution: $\binom{9}{k}\left(x^2\right)^{9-k}\cdot\left(-x^{-1}\right)^k$

$\Rightarrow 18-2k-k=0 \Rightarrow 18-3k=0$

$k=6$

$\therefore \binom{9}{6}\left(x^2\right)^3\left(-x^{-1}\right)^6=84$

b) By making the exponent of x zero

$\binom{12}{k}\left(2x^3\right)^{12-k}\cdot\left(-x^{-1}\right)^k$

$\Rightarrow x^{36-3k-k}\cdot x=x^0 \Rightarrow 36-4k=0 \Rightarrow 4k=36$

$\therefore k=9$

$\therefore \binom{12}{9}\left(2x^3\right)^3\cdot(-1)^9x^{-9}=-1760$

c) $\left(x-\frac{1}{2x}\right)^{10}$

$\binom{10}{k}x^{10-k}\cdot\left(-\frac{1}{2}x^{-1}\right)^k$

$\binom{10}{k}x^{10-k}\cdot x^{-k}\left(-\frac{1}{2}\right)^k$

By setting exponent of x zero we get

$10-2k=0$

$k=5$

ally exclusive events:

E_1 and E_2 are mutually exclusive if and only if they have out comes, i.e. $E_1 \cap E_2 = \varnothing$

ı Rolling of die

= getting even number

= getting prime number

= {2, 4, 6}

= {2, 3, 5}

$\cap E_2 = \{2\}$

nd E_2 are not mutually exclusive event.

tossing of coin once.

and {T} are mutually exclusive event.

$\cap E_2 = \varnothing$ then E_1 and E_2 are mutually exclusive
$\cap E_2 \neq \varnothing$ then E_1 and E_2 are not mutually exclusive

endent Events: Two event E_1 and E_2 are said to be dent events if the occurrence of one does not influence irrence of the other.

le: In tossing of two coin, the event getting head {H} irst coin and the event getting head {H} on the second independent.

are independent event then $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$

ent Event: Two events said to be dependent if the ce of one affect the occurrence of the other.

: A box contain 4 Red and 6 black ball. If two balls are what is the probability that

th are black (with out replacement)

th are red: (with out replacement)

d then black

a) The two event (drawing single black ball) are it

the first draw, $P(\text{black}) = \dfrac{6}{10}$

the 2^{nd} draw, given that the first draw was black

lack) $= \dfrac{5}{9}$

Probability

Probability is measure of uncertainty involved in the happening of event so that definite value may be assigned to it.

Definition:

i) **Random experiment:** any happening whose result is uncertain.
Example: Tossing of coin, Rolling of die.

ii) **Out comes:** The possible result of the experiment.

iii) **Sample space** is the set of all possible out comes of an experiment.
Example: S = {1,2,3,4,5,6} ⟵ sample space in rolling of die.

iv) **Events:** An event is any subset of sample space.
Example: In tossing of coin once, {H,T} is event

Types of Events

a. **Simple event** containing exactly one sample point.

b. **Compound event** is an event that has more than one sample point
Example: tossing of two coin {HH, HT, TH, TT}

c. **Complement** of an event, E is the collection of all out comes in the sample space that are not in E, denoted by E'
Example: In Rolling of die
Find a) the event E getting 6 b) E'
Solution: Sample space S = {1,2,3,4,5,6}
a) E = {6} ⟵ occurrence of event
b) E' = {1,2,3,4,5} ⟵ complement of event (Non occurrence of event)

d) **Exhaustive Event:** Events are said to be exhaustive if at least one of them must necessarily occur.
Example: In rolling of die:
{1}, {2}, ..., {6}, {1, 2}, {3, 6}, {1,2,3,4,5,6}
are exhaustive event.
Example: In tossing of coin:
{H}, {T}, {HT} are exhaustive event

Let $E_1, E_2, \ldots E_n$ forms set of exhaustive event then
$E_1 \cup E_2 \cup E_3 \cup \ldots E_n$ = subset.

e) **Mutually exclusive events:**

Two events E_1 and E_2 are mutually exclusive if and only if they have no common out comes, i.e. $E_1 \cap E_2 = \varnothing$

Example: In Rolling of die

Let E_1 = getting even number

Let E_2 = getting prime number

$\Rightarrow E_1 = \{2, 4, 6\}$

$\Rightarrow E_2 = \{2, 3, 5\}$

Thus $E_1 \cap E_2 = \{2\}$

$\therefore E_1$ and E_2 are not mutually exclusive event.

Example: In tossing of coin once.

: {H} and {T} are mutually exclusive event.

Note: i) If $E_1 \cap E_2 = \varnothing$ then E_1 and E_2 are mutually exclusive

ii) If $E_1 \cap E_2 \neq \varnothing$ then E_1 and E_2 are not mutually exclusive

f) **In dependent Events:** Two event E_1 and E_2 are said to be independent events if the occurrence of one does not influence the occurrence of the other.

Example: In tossing of two coin, the event getting head {H} on the first coin and the event getting head {H} on the second coin are independent.

* If E_1 and E_2 are independent event then $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$

g) **Dependent Event:** Two events said to be dependent if the occurrence of one affect the occurrence of the other.

Example: A box contain 4 Red and 6 black ball. If two balls are drawing what is the probability that

a) both are black (with out replacement)

b) both are red: (with out replacement)

c) Red then black

Solution: a) The two event (drawing single black ball) are dependent

- on the first draw, $P(\text{black}) = \frac{6}{10}$
- on the 2^{nd} draw, given that the first draw was black

$P(\text{black}) = \frac{5}{9}$

$\therefore$ P(black then black) = $\frac{6}{10} \cdot \frac{5}{9} = \frac{30}{90} = \frac{1}{3}$

b) Probability of getting Red, in the first draw P(Red) = $\frac{4}{10}$

- Probability of getting Red, in the 2^{nd} draw, given that the first draw was Red, P(Red) = $\frac{3}{9}$

$\therefore$ P(Red then Red) = $\frac{4}{10} \cdot \frac{3}{9} = \frac{12}{90} = \frac{2}{15}$

c) P(Red then Black) = $\frac{4}{10} \cdot \frac{6}{9} = \frac{24}{90} = \frac{4}{15}$

d) P(Black then Red) = $\frac{6}{10} \cdot \frac{4}{9} = \frac{24}{90} = \frac{4}{15}$

Equally likely event: Each element (event) has equal chance to select.

Probability of an Event

Probability can be measured by three different approaches

i) The classical (mathematical) approach
ii) The empirical (relative frequency) approach
iii) The axiomatic approach

i) The classical approach

Definition: If all the out comes of random experiment are equally likely out comes, then the probability of an event E is

$$P(E) = \frac{n(E)}{n(S)}$$

Example: A fair die is tossed once. Find the probability that

a) E_1: An odd number appears.
b) E_2: An even number appears

Solution:

a) $S = \{1, 2, 3, 4, 5, 6\}$, $n(S) = 6$
$E_1 = \{1, 3, 5\}$, $n(E_1) = 3$

$$\therefore P(E_1) = \frac{n(E_1)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

b) $n(S) = 6$, $E_2 = \{2, 4, 6\}$, $n(E_2) = 3$

$$\therefore P(E_2) = \frac{n(E_2)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

ii) The empirical approach

$$P(E) = \frac{\text{frequency of } E}{\text{total number of observation}}$$

Example 1: The following table gives a distribution of daily income.

Income	18-24	25-31	32-37	38-44	45-51
No. of work	5	40	15	23	17

Find the probability that:

a) their income less than or equal to 31

b) their income 32 or above

c) their income either between 32 -27 or 18 -24

Solution: Total number of observation = 5+40+15+23+17= 100

a) $p(income \leq 31) = \frac{5+40}{100} = \frac{45}{100} = \frac{9}{20}$

b) $p(Income \geq 32) = \frac{15+23+17}{100} = \frac{55}{100} = \frac{11}{20}$

c) $p(Income\, 32-37 \text{ or } 18-24) = \frac{15+5}{100} = \frac{20}{100} = \frac{1}{5}$

2. 100 bulb were found to be defective in lot of 800 bulb. Find the probability that a bulb selected from the lot is defective is

$$p(Def) = \frac{f_D}{800} = \frac{100}{800} = \frac{1}{8}$$

ii. The axiomatic approach

Let S be the sample space of an experiment and E an event. Then **probability** P (E) of **E** is given by

$$P(E) = \frac{n(E)}{n(S)}$$

Since E is a subset of S, we see that

$$0 \leq n(E) \leq n(S)$$

$$0 \leq \frac{n(E)}{n(S)} \leq \frac{n(S)}{n(S)} \longleftarrow \text{Dividing both side by n(S)}$$

$$0 \leq \frac{n(E)}{n(S)} \leq 1$$

$$0 \le \frac{n(E)}{n(S)} \le 1$$

$$0 \le p(E) \le 1 \longleftarrow \text{axiom property}$$

Note: i) if p(E) = 0 then E = ∅. (Impossible event)
ii) if p(E) = 1, then E = S (Certain event)
iii) p(not E) = 1 – P(E) = p(E') where E' = S\E

Odds in favour of and odds against an event

If p(E) = m and p(E') = n then the ratio:

i) $\frac{p(E)}{p(E')} = \frac{m}{n} \rightarrow$ odd in favor of Event E (ratio of success to failures)

ii) $\frac{p(E')}{p(E)} = \frac{n}{m} \rightarrow$ odd against an event E

Note: p (E') = 1 – p(E) and (E') = (not E)
n(S) = n(E) + n(E') = m + n

Illustrative Example

52. The odd infavour of an event are 4:7. Find probability of occurrence (P(E))

Solution: $\frac{p(E)}{p(E')} = \frac{4}{7} \Rightarrow \frac{p(E)}{1-p(E)} = \frac{4}{7}$

$\Rightarrow$ 7P(E) = 4 – 4p(E) $\Rightarrow$ 11p(E) = 4

$\therefore$ p(E) = $\frac{4}{11}$ and p(E') = $1 - \frac{4}{11} = \frac{7}{11}$

54. The odd against certain event are 2:3. Find
a) the probability of its occurrence
b) the probability of non-occurrence.

Solution: a) $\frac{p(E')}{p(E)} = \frac{2}{3} \Rightarrow \frac{1-p(E)}{p(E)} = \frac{2}{3} \Rightarrow 3 - 3p(E) = 2p(E)$

$\therefore 5\ P(E) = 3$

$$\therefore p(E) = \frac{3}{5}$$

b) $p(E') = 1 - p(E) = 1 - \frac{3}{5} = \frac{2}{5}$

55. A box containing 4 green, 5 black and 6 white marbles. One marble is drawn at random.

i) Find the probability:

a) green marble.

b) not green marble

ii) Find c) the odds in favor of drawing green marble

d) the odds against drawing green marble

e) the odds against drawing white marble

Solution: Here

a) $p(\text{green}) = \frac{4}{4+5+6} = \frac{4}{15}$

b) $p(\text{not green}) = 1 - p(\text{green}) = 1 - \frac{4}{15} = \frac{11}{15}$

ii) c) $$\frac{p(E)}{p(E')} = \frac{p(green)}{p(not\ green)} = \frac{\frac{4}{15}}{\frac{11}{15}} = 4:11$$

d) $$\frac{p(E')}{p(E)} = \frac{p(not\ green)}{p(green)} = \frac{\frac{11}{15}}{\frac{4}{15}} = 11:4$$

e) $$\frac{p(E')}{p(E)} = \frac{p(not\ white)}{p(white)} = \frac{1-p(white)}{p(white)} = \frac{1-\frac{6}{15}}{\frac{6}{15}} = \frac{\frac{9}{15}}{\frac{6}{15}} = 9:6$$

56. If $p(E) = 0.6$, Find

a) The odd in favor of event E.

b) The odd against the event E.

Solution: Let = O(E) odd infavor of E

a) $O(E) = \frac{P(E)}{P(E')} = \frac{0.6}{1-0.6} = \frac{0.6}{0.4} = \frac{3}{2} \leftarrow$ odd infavor of E.

b) $O(E') = \frac{P(E')}{P(E')} = \frac{1-0.6}{0.6} = \frac{0.4}{0.6} = \frac{4}{6} = \frac{2}{3} \leftarrow$ odd against E.

Rules of probability

I. Addition rule

If E_1 and E_2 are any two events then

i) $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

ii) $P(E_1 \cup E_2) = P(E_1) + P(E_2)$, if E_1 and E_2 are mutually exclusive i.e. $E_1 \cap E_2 = \varnothing$

Note: $P(E_1 \text{ or } E_2) = P(E_1 \cup E_2)$

II. Multiplication Rule

If E_1 and E_2 are any two events, the probability that both events occur denoted by $P(E_1 \text{ and } E_2) = P(E_1 \cap E_2) = P(E_1E_2)$ is given by

i) $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$ If E_1 and E_2 are independent event

ii) $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2|E_1)$ if E_1 and E_2 are dependent event (conditional probability)

$= P(E_2) \cdot P(E_1|E_2)$

Condional Probability: When occurrence of one event depends on the occurrence of another event, we say the second event is conditioned by the first event.

This is called **conditional probability.**

i) The conditional probability of event E_1 given that E_2 has already occurred is defined by $P(E_1|E_2) = \frac{P(E_1 n E_2)}{P(E_2)}$

ii) If event E_2, given that E_1 has already occurred is defined by $P(E_2|E_1) = \frac{P(E_1 n E_2)}{P(E_1)}$

Note: The vertical bar is read "given"

Illustrative Example

57. A box containing 6 red ball, 4 white ball and 5 blue balls. One ball is drawn at random. Find the probability that is:

a) red or white
b) white or blue
c) neither red nor blue
d) not blue
e) odd in favor or red or white
f) odd against white or blue

Solution: Let R = Red, B = Blue, W = White by using addition rule

a) $P(R \text{ or } W) = P(R) + P(W) = \frac{6}{15} + \frac{4}{15} = \frac{10}{15} = \frac{2}{3}$

b) $P(W \text{ or } B) = P(W) + P(B) = \frac{4}{15} + \frac{5}{15} = \frac{9}{15} = \frac{3}{5}$

c) P(neither red nor blue) = 1 − P(R or W)

$$= 1 - \left(\frac{6}{15} + \frac{4}{15}\right) = 1 - \frac{2}{3} = \frac{1}{3}$$

d) $P(\text{not } B) = 1 - P(B) = 1 - \frac{5}{15} = \frac{10}{15} = \frac{2}{3}$

e) $O(R \text{ or } W) = \dfrac{P(R)+P(W)}{1-(P(R)+P(W))} = \dfrac{\frac{2}{3}}{1-\frac{2}{3}} = 2:1$

f) Exercise left for the student

58. A number is selected at random from the set of integers from 1 to 20 inclusive. Find the probability that the number selected is:

a) Multiple of 2 or 3 b) multiple of 2 and 3

Solution: Let S = {1, 2, 3, ….20} $\therefore n(S) = 20$

Let E_1 = selecting multiple of 2,

E_2 = event selecting multiple of 3.

Then, $E_1 = \{2,4,6,8,10,12,14,16,18,20\}$, $n(E_1) = 10$

$E_2 = \{3, 6,9,12,15,18\}$, $n(E_2) = 6$

Since, E_1 n $E_2 = \{6, 12, 18\}$

$\therefore$ E_1 and E_2 are not mutually exclusive

a) $P(E_1 \text{ or } E_2) = P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$$= \frac{10}{20} + \frac{6}{20} - \frac{3}{20} = \frac{13}{20}$$

b) $P(E_1 \text{ and } E_2) = P(E_1 \cap E_2) = P(E_1) \cdot P(E_2|E_1)$

$= P(E_1) \cdot P(E_2) \longleftarrow$ Since E_1 and E_2 are independent event $(P(E_2|E_1) = P(E_2))$

$$= \left(\frac{10}{20}\right) \cdot \frac{6}{20} = \frac{3}{20}$$

59. A die and coin is tossed together. Find thee probability of g...
a) six and head
b) odd number and head
c) 3 and tail
d) even number and t...

Solution: Tossing die = {1, 2, 3, 4, 5 , 6}

Tossing coin = {H, T}

Tossing coin and die together is $6 \times 2 = 12$ out come

$\therefore$ S = {1H, 2H, 3H, 4H, 5H, 6H, 1T, 2T, 3T, 4T, 5T, 6...

Thus,

a) $p(\text{six and head}) = p(\text{six}) \cdot p(\text{head}) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$. Event...
{6H}

b) $p(\text{odd and head}) = p(\text{odd}) \cdot p(\text{head}) = \frac{3}{6} \cdot \frac{1}{2} = \frac{1}{4}$. Even...
{1H,3H,5H}

c) $p(\text{3 and tail}) = p(\text{getting 3}) \cdot p(\text{tail}) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$. Ev...
{3T}

d) $p(\text{even and tail}) = p(\text{even}) \cdot p(\text{tail}) = \frac{3}{6} \cdot \frac{1}{2} = \frac{1}{4}$. Event =
{2T,4T,6T}

60) A box contains 4 white and 3 red balls. If two balls are dra... after the other with out replacement. Find the probability th...
a) both are red.
b) both are white
c) the first ball is red and the second is white.
d) the first ball is white and the second is red
e) both are different color

Solution: Let R = Red and W = White, then

a) $P(\text{R and R}) = \frac{3}{7} \cdot \frac{3-1}{7-1} = \frac{3}{7} \cdot \frac{2}{6} = \frac{1}{7} \longrightarrow$ (Explanation: $\frac{3-1}{7-1}$ is one red ball is removed from 1^{st} draw)

b) $P(\text{W and W}) = \frac{4}{7} \cdot \frac{4-1}{7-1} = \frac{4}{7} \cdot \frac{3}{6} = \frac{2}{7} \longrightarrow \frac{4-1}{7-1}$ is one white ball is removed on 1^{st} draw

c) $P(R \text{ and then } W) = \frac{3}{7} \cdot \frac{4}{7-1} = \frac{3}{7} \cdot \frac{4}{6} = \frac{2}{7} \longrightarrow$ 7-1 is Red is removed on the 1st draw

d) $P(W \text{ and then } R) = P(W) \cdot P(R|W)$

$= \frac{4}{7} \cdot \frac{3}{7-1} = \frac{4}{7} \cdot \frac{3}{6} = \frac{2}{7} \longrightarrow$ 7-1 is White is removed on the 1st draw

e) $P(RW \text{ or } WR) = P(RW) + P(WR) =$

$= P(R) \cdot P(W|R) + P(W) \cdot P(R|W) = \frac{3}{7} \cdot \frac{4}{6} + \frac{4}{7} \cdot \frac{3}{6} = \frac{4}{7}$

Method II, a) $P(RR) = \frac{C(3,2)}{C(7,2)} = \frac{1}{7}$, b) $P(WW) = \frac{C(4,2)}{C(7,2)} = \frac{2}{7}$,

1) A box contains 4 red, 3 white and 5 blue balls. Three balls are drawn one after the other. Find the probability of getting:

a) all 3 are red. Drawing with i) replacement
ii) out replacement

b) all 3 are white.
i) With replacement ii) with out replacement

c) all the balls are of different colour.

d) the balls are drawn in the order red, white, blue. Each ball is
i) replaced ii) not replaced

Solution: Let Red = R_1, R_2, R_3 (1st, draw 2nd draw, 3rd draw of red ball respectively W = W_1, W_2, W_3 (1st, 2nd, 3rd draw of white ball respectively.

Then

a) i) If each ball is replaced, then R_1, R_2 and R_3 are in dependent event

$\therefore P(R_1 R_2 R_3) = P(R_1).P(R_2).P(R_3) = \frac{4}{12} \cdot \frac{4}{12} \cdot \frac{4}{12} = \frac{1}{27}$

ii) If ball is not replaced, then R_1, R_2 and R_3 are dependent event

$\therefore P(R_1 R_2 R_3) = P(R_1).P(R_2|R_1).P(R_3|R_1 R_2) = \frac{4}{12} \cdot \frac{3}{11} \cdot \frac{2}{10} = \frac{1}{55}$

b) i) $P(W_1W_2W_3) = P(W_1)\cdot P(W_2)\cdot$

$P(W_3) = \frac{3}{12}\cdot\frac{3}{12}\cdot\frac{3}{12} = \frac{1}{64} \leftarrow$ (each ball is replaced i.e independent)

ii) $P(W_1W_2W_3) = P(W_1)\cdot P(W_2|W_1)\cdot P(W_3|W_1W_2) \leftarrow$ (No replaced dependent)

$= \frac{3}{12}\cdot\frac{2}{11}\cdot\frac{1}{10} = \frac{1}{220} \rightarrow P(W_1) = \frac{3}{12}$

$P(W_2|W_1) = \frac{3-1}{12-1} = \frac{2}{11}$

> ***Note***: We can use: $P(W_1W_2W_3) = \frac{C(3,3)}{C(12,3)}$

c) P(1 of each colour is draw)

$= \frac{C(4,1)\cdot C(3,1)\cdot C(5,1)}{C(12,3)} = \frac{4\times3\times5}{210} = \frac{3}{11}$

d) i) If each ball is replaced then R,W,B are independent

$\therefore P(RWB) = P(R)\cdot P(W)\cdot P(B) = \frac{4}{12}\cdot\frac{3}{12}\cdot\frac{5}{12} = \frac{5}{144}$

ii) If each ball is not replaced then events are dependent.

$\therefore P(RWB) = P(R)\cdot P(W|R)\cdot P(B|RW) = \frac{4}{12}\cdot\frac{3}{11}\cdot\frac{5}{10} = \frac{1}{22}$

62) From a deck of 52 playing cards, one card is drawn. Find probability that it is

a)	either king or Queen	d)	either king or black
b)	either Queen or Jack.	e)	either heart or king
c)	ether Queen or Red.	f)	either an Ace or Hea

***Solution*: Here:**

A standard deck of 52 playing cards made up as shown.

Black (B)	Spade	A ♠	K ♠	Q ♠	J ♠	10 ♠	9 ♠	8 ♠	7 ♠	6 ♠	5 ♠	4 ♠	3 ♠	2 ♠
	Clube	A ♣	K ♣	Q ♣	J ♣	10 ♣	9 ♣	8 ♣	7 ♣	6 ♣	5 ♣	4 ♣	3 ♣	2 ♣
Red (R)	Heart (H)	A ♥	K ♥	Q ♥	J ♥	10 ♥	9 ♥	8 ♥	7 ♥	6 ♥	5 ♥	4 ♥	3 ♥	2 ♥
	Diamond	A ♦	K ♦	Q ♦	J ♦	10 ♦	9 ♦	8 ♦	7 ♦	6 ♦	5 ♦	4 ♦	3 ♦	2 ♦

a) $P(K \text{ or } Q) = P(K) + P(Q) = \frac{4}{52} + \frac{4}{52} = \frac{2}{13}$

b) $P(Q \text{ or } J) = P(Q) + P(J) = \frac{4}{52} + \frac{4}{52} = \frac{2}{13}$

c) $P(Q \text{ or } R)=P(Q)+P(R) - P(QnR)= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{7}{13}$

d) $P(K \text{ or } B) = P(K)+P(B) - P(KnB)= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{7}{13}$

e) $P(H \text{ or } K) = P(H) + P(K) - P(HnK) =$

$\frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

f) $P(A \text{ or } H) = P(A)+P(H) - P(AnH) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13}$

63) Two cards are drawn from pack of 52 cards. Find the probability that

a) all are Jack if the first card was replaced
b) All are king if the first card was not replaced
c) The first is an Ace and the second is king if the first cad was not replaced

***Solution:* Here**

a) $P(J \text{ and } J) = P(J) \cdot P(J) = \frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169} \leftarrow$ the 1^{st} king was replaced

b) $P(K \text{ and } K) = \frac{4}{52} \cdot \frac{4-1}{52-1} = \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221} \leftarrow$ the 1st king was not replaced

c) $P(AK) = P(A) \cdot P(K|A) = \frac{4}{52} \cdot \frac{4}{52-1} = \frac{4}{52} \cdot \frac{4}{51} = \frac{4}{663}$

64) Two balls are selected from a box of containing 7 red and 5 bl one after the other. Find the probability that

a) the second ball selected is blue given that the first was re ball.

b) The second ball selected is red ball given that the first wa blue ball.

c) The second ball selected is blue given that the first was blue ball

d) The second ball is red knowing that the first ball is red

Solution: $P(\text{Red then blue}) = P(R) \cdot P(B|R) = \frac{7}{12} \cdot \frac{5}{11}$

a) $\therefore P(B|R) = \frac{5}{11} \leftarrow$ given that Red has already occurred

b) $P(\text{Band then Red}) = P(B) \cdot P(R|B) = \frac{5}{12} \cdot \frac{7}{11}$

$\therefore P(R|B) = \frac{7}{11} \leftarrow$ given that the blue ball has already occurred.

c) $P(\text{blue and then blue}) = P(B_1) \cdot P(B_2|B_1) = \frac{5}{12} \cdot \frac{4}{11}$

$\therefore P(B_2|B_1) = \frac{4}{11}$

d) $P(\text{Red then Red})\ P(R_1) \cdot P(R_2|R_1) = \frac{7}{12} \cdot \frac{6}{11} \quad \therefore P(R_2|R_1) =$

65) From a pack of 52 playing cards, three cards are drawn one af the other with out replacement. Find the probability that ki Queen and Jack will be obtained respectively?

Solution: $P(KQJ) = P(K) \cdot P(Q|K) \cdot P(J|QK)$

$$= \frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4}{50} = \frac{8}{16575}$$

If A and B are independent event and $P(A) = \frac{1}{3}$, $P(B) = \frac{3}{4}$ find $P(A \cup B)$

Solution:

$P(A\cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - (P(A) \cdot P(B))$

$$= \frac{1}{3} + \frac{3}{4} - \left(\frac{1}{3} \cdot \frac{3}{4}\right) = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

If a pair of dice is thrown. Find the probability that the sum of the top number is

a) 7 b) greater than 10, c) 11 d) either 6 or 10

e) 5 f) neither 6 nor 10 g) total of 7 or 11

Solution: $S = \{(1,1),(1,2),(1,3),(1,4),(1,5)(1,6),(2,1)...(5,6),(6,6)$

$\therefore n(S) = 36$

a) let E = Event that the sum is 7

Thus, $E = \{(2,5),(5,2,(1,6)(6,1),(3,4),(4,3)\}$, Then $n(E) = 6$

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

b) Let E = Event sum greater than 10 i.e, 11 or 12

$\therefore E = \{(5,6),(6,5),(6,6)\}$, $n(E) = 3$

$$\therefore P(E) = \frac{3}{36} = \frac{1}{12}$$

c) Let E = Event sum is 11

$\therefore E = \{(5,6),(6,5)\}$, $n(E) = 2$

$$\therefore P(E) = \frac{2}{36} = \frac{1}{18}$$

d) Let E = Event sum 6 or 10.

Sum 6 = $\{(1,5),(5,1),(2,4),(4,2),(3,3),\}$

Sum 10 = $\{(4,6),(6,4),(5,5)\}$

$\therefore E = \{(1,5),(5,1),(2,4),(4,2),(3,3),(4,6),(6,4),(5,5)\}$

$$\therefore P(E) = \frac{8}{36} = \frac{2}{9}$$

e) Let E = Event sum 5, $\therefore E = \{(1,4),(4,1),(2,3),(3,2)\}$

$$\therefore P(E) = \frac{5}{36}$$

f) P(neither 6 nor 10) = 1 − p (sum either 6 or 10) = $1-\frac{2}{9}$ ...

g) Event E = {(1,6),(6,1),(2,5),(5,2),(3,4),(4,3),(5,6),(6,5)}

$\therefore P(E) = \frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9}$

68) If $P(A) = \frac{2}{5}, P(B) = \frac{1}{3}, P(A \cup B) = \frac{1}{2}$

Find a) P(A|B) b) P(B|A)

Solution:

a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\frac{1}{2} = \frac{2}{5} + \frac{1}{3} - P(A \cap B) \Rightarrow P(A \cap B) = \frac{7}{30}$, but

$P(AnB) = P(B) \cdot P(A|B)$

$\therefore P(A|B) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{7}{30}}{\frac{2}{5}} = \frac{7}{12}$

69) If $P(A) = \frac{1}{2}, P(B) = \frac{1}{3}, P(A \cap B) = \frac{1}{4}$ Find

a) P(AUB) b) P(A|B) c) P(B|A)

Solution: Here

a) $P(AUB) = P(A) + P(B) - P(AnB) = \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$

b) $P(A|B) = \frac{P(AnB)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}$ c)

Probability and Combination analysis

70) A box contains 5 red, 3 white and 4 blue balls. If 3 balls ... selected at random. Find the probability that

a) all are red d) all the balls are different col...

b) all are white e) two are red and one is blue

c) at least one ball is white

Solution: Here

a) all are red means that 3 of them is red from 5 red

$$\therefore P(\text{all red}) = \frac{C(5,3)}{C(12,3)} = \frac{10}{210} = \frac{1}{21}$$

b) $P(3 \text{ white}) = \dfrac{C(3,3)}{C(12,3)} = \dfrac{1}{210}$

c) P(at least 1 white) = P(1w or 2 white or 3 white)

$$= \frac{C(3,1)C(9,2) + C(3,2)C(9,1) + C(3,3)C(9,0)}{C(12,3)}$$

$$= \frac{(3)(36) + (3)(9) + 1}{210} = \frac{136}{210}$$

d) $P(\text{all are different color}) = \dfrac{C(5,1) \cdot C(3,1) \cdot C(4,1)}{C(12,3)} = \dfrac{1}{7}$

e) $P(2 \text{ red and } 1 \text{ blue}) = \dfrac{C(5,1) \cdot C(4,1)}{C(12,3)} = \dfrac{(10) \times 4}{210} = \dfrac{4}{21}$

71) Out of group of 5 girls and 4 boys, if three of them are selected for group work leader. What is the probability that

a) all are girls b) all are boys c) at least one girl

Solution:

a) $P(\text{all girls}) = \dfrac{C(5,3) \cdot (4,0)}{C(9,3)} = \dfrac{10}{84} = \dfrac{5}{42}$

b) $P(\text{all boys}) = \dfrac{C(4,3) \cdot (5,0)}{C(9,3)} = \dfrac{4}{84} = \dfrac{1}{21}$

c) $P(\text{at least one girl}) = 1 - P(\text{all boys}) = 1 - \dfrac{1}{21} = \dfrac{20}{21}$

72. In pack of ten books, three are geometry books while the rest are book on algebra. If three books are drawn at random, what is the probability that one of them is geometry book and the others are book on algebra

A. $\dfrac{21}{40}$ B. $\dfrac{21}{100}$ C. $\dfrac{2}{21}$ D. $\dfrac{7}{10}$

Solution:

$$\frac{C(3,1)C(7,2)}{C(10,3)}=\frac{(3)(21)}{120}=\frac{21}{40}$$

Answer:

73. A jar contains 3 red and 5 green balls identical in all aspects except colour. If two balls are drawn at random, then the probability that the two balls drawn are not green is

A. $\frac{9}{14}$ B. $\frac{15}{28}$ C. $\frac{3}{28}$ D. $\frac{5}{14}$

Solution: P(not green) = 1 – p(green)

$$=1-\frac{C(5,2)C(3,0)}{C(8,2)}=1-\frac{10}{28}=\frac{18}{28}=\frac{9}{14}$$

Answer: A

74. A box contains 8 items of which 3 are defective. If a man selects two items are drawn from the box together, what is the probability that

a) both items are defective b) both are non-defective
c) one item is defective d) at least one is defective
e) at most one is defective

Solution: from 8 item $\begin{cases}\text{3 defective item}\\ \text{5 non – defective item}\end{cases}$

a) 2 defective items out of 3 defective items = C(3, 2)

$$\therefore p(both\ defective)=\frac{C(3,2)}{C(8,2)}=\frac{3}{28}$$

b) 2 non – defective out of 5 non-defective

$$\therefore p(both\ non\ defective)=\frac{C(5,2)}{C(8,2)}=\frac{10}{28}$$

c) One item defective means
The other one is non-defective

$$\therefore p(1=defective)=\frac{\binom{3}{1}\binom{5}{1}}{\binom{8}{2}}=\frac{(3)(5)}{28}=\frac{15}{28}$$

d) $p(x\geq 1)=p(x=1)+p(x=2)$

$$=\frac{C(3,1)C(5,1)+C(3,2)}{C(8,2)}=\frac{15}{28}+\frac{3}{28}=\frac{18}{28}=\frac{9}{14}$$

e) p(at most one defective) = p(no defective) + p(1-defective)

$$=\frac{C(5,2)C(3,0)+C(5,1)C(3,1)}{C(8,2)}=\frac{10+15}{28}=\frac{25}{28}$$

75. A bag contains 3 fresh and 5 spoiled oranges of equal size. If three oranges are to be drawn at random from the bag, what is the probability that at least one of the oranges is fresh?

A. $\frac{15}{28}$ B. $\frac{15}{56}$ C. $\frac{45}{56}$ D. $\frac{23}{28}$

Solution: $p\begin{pmatrix}\text{at least one of} \\ \text{the orange fresh}\end{pmatrix} = p(1\text{fresh}) + p(2\text{fresh}) + p(3\text{fresh})$

$$=\frac{C(3,1)C(5,2)+C(3,2)C(5,1)+C(3,3)}{C(8,3)}$$

$$=\frac{30+15+1}{56}=\frac{46}{56}=\frac{23}{28}$$

Answer: C

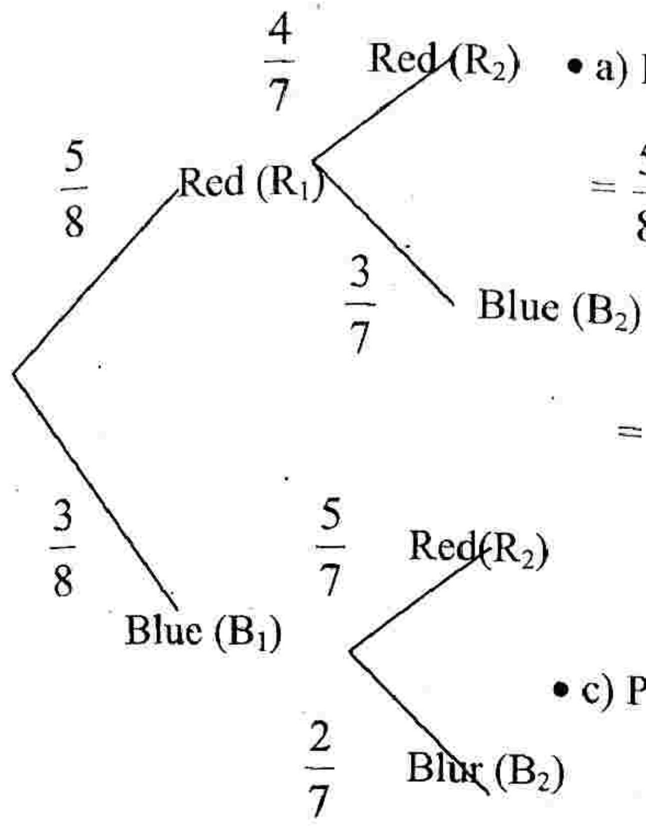
Calculating probability using tree diagram

When we have two or more events the probability of any out come we can obtain in simple ways from tree diagram.

76. A box contains 5 red balls, 3 blue balls. Two balls are drawn one after the other with out replacement. Find the probability that:

a) both are red
b). both are blue
c) not the same colour
d) the 2^{nd} ball is blue given that the first ball is ed.
e) the 2^{nd} ball is red given that the first ball is blue
f) 2^{nd} ball is red given that 1^{st} ball is red
g) 2^{nd} ball is blue given that 1^{st} ball is blue.

Solution: Let R_1 = red on 1^{st} draw and B_1 = blue on 1^{st} draw.
R_2 = red on 2^{nd} draw B_2 = blue on 2^{nd} draw.



• a) $P(R_1R_2) = P(R_1)\cdot P(R_2|R_1)$

$$= \frac{5}{8}\cdot\frac{4}{7} = \frac{5}{14}$$

• b) $P(B_1B_2) = P(B_1).P(B_2|B_1)$

$$= \frac{3}{8}\cdot\frac{2}{7} = \frac{3}{28}$$

• c) P(not the same colour)

= P(Red then blue or blue then red)

$$= P(RB) + P(BR) = P(R)\cdot P(B|R) + P(B)P(R|B)$$

$$= \frac{5}{8}\cdot\frac{3}{7} + \frac{3}{8}\cdot\frac{5}{7} = \frac{15}{28}$$

Or P(not the same colour)

= 1 – P (same colour) = 1 – P(both red) – P(both blue)

$$= 1 - \frac{5}{14} - \frac{3}{28} = \frac{15}{28}$$

d) $P(B|R) = \frac{3}{7}$ ← probability that 2nd ball is blue

Given that 1st ball is red.

e) $P(R|B) = \frac{5}{7}$ ← probability that 2nd ball is red.

f) $P(R_2|R_1) = \frac{4}{7}$ ← probability that 2nd ball is red give that 1

ball is red

g) $P(B_2|B_1) = \frac{2}{7}$

Repeated Trials

Let p be the probability of success of an event
q be the probability of its failure.
n be repeated the trail n times
Now out of n trail r successes, n – r failure.
$\therefore$ P(r successes) = $c(n, r)\, p^r \cdot q^{n-r}$

The probability of getting two head in 5 to sses of coin.

Solution: $P(2) = C(5, 2)\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = \frac{5}{16}$

78. The probability of getting 6 correct answer in 10 question each question has 4 multiple choice.

Solution: $p = \frac{1}{4},\ q = \frac{3}{4}\ \therefore\ p(6) = c(10,6)\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^4$.

Solved Problem

79) If P(x) = 0.4, P(y) = 0.6 and P(x and y) = 0.15 then
a) Are the events x and y dependent or independent.
b) find i) $P(x|y)$ ii) $P(y|x)$

Solution: Here

a) $P(x \text{ and } y) = P(x) \cdot P(y|x) \Rightarrow 0.15 = (0.4)\, P(y|x)$

$\therefore P(y|x) = \frac{0.15}{0.4} = 0.375$

This shows it is dependent because
$P(y|x) \neq P(y) \Rightarrow (0.375 \neq 0.6)$

b) $P(x \text{ and } y) = p(y) \cdot P(x|y)$
$0.15 = (0.6)\, P(x|y)$
$\therefore P(x|y) = 0.25$

78) There are 3 red and 9 green ball in a box. What is the probability of drawing at random with out replacement green then another green.

A) $\frac{9}{16}$ B) $\frac{6}{11}$ C) $\frac{9}{12}$ D) $\frac{1}{4}$

Solution: P(green then green) = $\frac{9}{12} \cdot \frac{8}{11} = \frac{6}{11}$

81) How many possible 3 member committees can be selec... from an organization that contains 10 members

A. 120 B) 360 C) 240 D) 720

Solution: $C(10, 3) = \frac{10!}{(10-3)!3!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120$, **Answer: A**

82) One box contains 2 red ball and 4 blue marbles. A second bo... contains 3 red ball and 2 blue ball. What is the probability th... drawing one ball from each box will pick 2 red ball.

A) $\frac{3}{10}$ B) $\frac{1}{5}$ C) $\frac{2}{5}$ D) $\frac{1}{2}$

Solution: Let R_1 = event "Red" ball from 1[st] box
R_2 = event "Red" ball from 2[nd] box

Then $P(R_1) = \frac{2}{2+4} = \frac{2}{6}$, *and* $P(R_2) = \frac{3}{3+2} = \frac{3}{5}$

Since R_1 and R_2 are in different box, there fore they a... independent

$$\therefore P(R_1 \cdot R_2) = P(R_1) \cdot P(R_2) = \frac{2}{6} \cdot \frac{3}{5} = \frac{1}{5}$$ **Answer: B**

83) A black ball and 4 red balls are in a box. If two balls are drawn with out replacement, what is the probability of getting.

a) black ball on the first draw and red ball on the second.

b) two red ball

c) second ball is black given that the first ball was red.

Solution: Let B = event black and R = Red

Then a) $P(BR) = P(B) \cdot P(R|B) = \frac{1}{5} \cdot \frac{4}{4} = \frac{1}{5}$

b) $P(R_1 \text{ and } R_2) = P(R_1) \cdot P(R_2|R_1) = \frac{4}{5} \cdot \frac{3}{4} = \frac{3}{5}$

c) we know that $P(RB) = P(R) \cdot (B|R) = \frac{4}{5} \cdot \frac{1}{4}$

$$\therefore P(B|R) = \frac{1}{4} \leftarrow \text{required probability.}$$

A probability that a student to pass mathematics exam is $\frac{2}{3}$ and probability that he passes both mathematics and an English exam is $\frac{14}{45}$.

The probability that he pass at least one exam is $\frac{4}{5}$. What is the probability that he pass the English exam is ________

A) $\frac{7}{15}$ B) $\frac{44}{45}$ C) $\frac{4}{9}$ D) $\frac{8}{15}$

Solution: P (math) = $\frac{2}{3}$; P(math $\cap$ Eng) = $\frac{14}{45}$.

P (math $\cup$ Eng) = $\frac{4}{5}$, Now we find P(Eng)

$\therefore$ P (math $\cup$ Eng) = P(math) + P(Eng) − P(math $\cap$ Eng)

$$\frac{4}{5} = \frac{2}{3} + P(Eng) - \frac{14}{45}$$

$$\therefore P(Eng) = \frac{4}{5} - \frac{2}{3} + \frac{14}{45} = \frac{36 - 30 + 14}{45} = \frac{4}{9}$$

85) A school has three class room for grade 11, namely, 11A, 11B and 11C.
The number of students in these class rooms is 28, 20 and 22 respectively. All student took an exam and the average score of student of 11A, 11B and 11C is 60,70 and 70 respectively. What is the average score in this examination for all grade 11 students …UEE 2004/12

A) 66 B) 66.67 C. 65 D) 65.67

Solution: $\overline{X} = \frac{60 \times 28 + 20 \times 70 + 22 \times 70}{28 + 20 + 22} = \frac{4620}{70} = 66$ **Ans.: A**

86) A three – digit library identification card is to be printed from the numbers 0,1,2,3,4,5 in such a way that the first is non – zero and no number is to be repeated. How many such cards can be printed? … UEE 2004/12

A) 100 B) 120 C) 150 D) 180

Answer: A

Solution: 5×5×4 = 100,

87) A student needs to select 3 books from 3 different mathematics, 3 different physics and 1 history book. What is the probability that one of them is mathematics and the other two are either physics or history books . . . UEE 2004/12

A) $\frac{3}{35}$ B) $\frac{9}{35}$ C) $\frac{15}{35}$ D) $\frac{18}{35}$

Solution: $\frac{C(3,1)C(4,2)}{C(7,3)} = \frac{3\times 6}{35} = \frac{18}{35}$

Answer: D

88) Items produced by certain company are subjected to two kinds of defects D_1 and D_2. out of total production, if 5% have defect D_1, have Defect D_2 and 2% have both defect S, then what is probability for an item to have defect D_2, given that it has defect D_1

A) 0.2 B) 0.05 C) 0.1 D) 0.4

Solution: $P(D_1 \text{ and } D_2) = 0.02$, $P(D_1) = 0.05$

$\Rightarrow P(D_1 \text{ and } D_2) = P(D_1).P(D_2|D_1) \Rightarrow 0.02 = 0.05\, P(D_2|D_1)$

$\Rightarrow P(D_2|D_1) = \frac{0.02}{0.05} = \frac{2}{5} = 0.4$,

Answer: D

Supplementary exercise

1. Find the mean, mode(s) and median of the following data 1, 9, 4, 3, 10, 2, 8, 5.
2. Find the mean, mode(s) and median of the following distribution

v	-1	0	2	3	5
f	5	7	4	1	3

3. Given 1, 2, x, 5, y, 8. Find the values of x and y if the mode of the resulting numbers is 5 and the mean is 4.
4. What number should be included in the following data so that the median is 5.2? 4, 7, 11, 5, 3
5. If the mean of a, b, c, d is k then what is
   a) a + d, b + d, c + d, 2d b) ac, bc, c^2, dc
6. Find the median, lower and upper quartiles from the following table

Marks below	10	20	30	40	50	60	70	80
Number of student	15	35	60	84	94	127	198	24[illegible]

7. Find the median from the following table

V	5	7	9	11	13	15	17	19
F	1	2	7	9	11	8	5	4

8. Calculate the mean and median from the following table

Class Interval	Frequency
6.5 – 7.5	5
7.5 – 8.5	12
8.5 – 9.5	25
9.5 – 10.5	48
10.5 – 11.5	32
11.5 – 12.5	6
12.5 – 13.5	1

9. In how many different ways can 5 student be seated in a row of 5 chair?
10. How many different ways are there to select 4 different players from 10 players on a team to play four table tennis matches, where the matches are ordered.
11. How many three digit numerals can be written by using only the digits 4, 5, 6, 7, 8, and 9 if no digit is repeated in the same numeral.
12. How many integers can be represented using the digit 3, 5, 7, 9 with each digit used at most once in a numeral?
13. A map of four countries is to be colored with a different color for each country. If six color are available, in how many different ways can the map be colored.
14. Two lottery tickets are drawn from 30 for first and second prizes. Find the number of ways of doing this.
15. How many ways can a football team schedule 3 holiday game with 3 teams if they are all available on any of 5 possible dates?
16. In how many different ways can 30 aks, 4 pins and 2 apples be arranged along a property line if one does not distinguish between trees of the kind?
17. How many ways can 8 people be assigned to 2 triple 2 double rooms?
18. If $P(n, 4) = 12.P(n, 2)$ find n
19. From a group of 4 men and 5 women, how many committees of size 3 are possible
    a) with no restriction b) with 2 men and a women
    c) with 2 men and a women if a certain man must be on the committee
20. Find the value of r if $C(18, r) = C(18, r + 2)$
21. If $C(n, 3) = C(n, 5)$, find the value of $C(2n, 2)$

22. How many signals can be made with 5 different flags ... them any number at a time
23. There are 3 copies each of 4 different books. In how many ... can they be arranged on a shelf?
24. In how many ways can 2 or more ties be selected out of 8 ties
25. If k is a real number and if the middle term in the expansion ...

$\left(\frac{k}{2}+2\right)^8$ is 1120, find k

26. Find the term independent of x in

i) $\left(2x+\frac{1}{3x^2}\right)^9$ ii) $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$ iii) $\left(\frac{4}{3}x^2-\frac{3}{2x}\right)$

27. Find the coefficient of x^{18} in the expansion of $(ax^4-bx)^9$
28. If the coefficient of x^2 and x^3 in the expansion of $(3+kx)^9$ ...
29. In the expansion of $(1+x)^{2n+1}$, the coefficient of x^r and x^{r+1} ... equal. Find r
30. Find the value of r if the coefficient of $(2r+4)$th term and ... $(r-2)$th term in the expansion of $(1+x)^{18}$
31. Determine the probability that in a family of 5 children there will be at least 2 boys and 1 girl
32. find the probability of throwing a total of 7 in single throw with two dice.
33. what is the probability of throwing a total of 8 or 11 in single throw with two dice
34. A ball is drawn at random from a box containing 10 red, 30 white, 20 blue and 15 orange marbles. Find the probability that it is

a) orange or red c) not blue

b) not red or blue d) white e) red, white or blue

35. Determine the probability of three 6's in 5 tosses of fair die (Hint, P($666\bar{6}\,\bar{6}$) or P($66\bar{6}6\bar{6}$) or ($6\bar{6}66\bar{6}$) ...
36. A probability that a man will be alive in 25 years is $\frac{3}{5}$ and the probability that his wife will be alive in 25 years is $\frac{2}{3}$. Find the probability that

a) both will be alive b) only the man will be alive

c) only the wife will be alive d) at least one will be alive

Unit Six

Matrices and Determinants

A matrix is rectangular array (or arrangement) of numbers in rows and column.

Here are the general 3 × 4 matrix and

matrix $A_{3\times4} \longrightarrow$ 3 the number of Row
4 the number of column

Row by Column
→ by ↓

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

- Each entry in a matrix is called element of matrix
- Elements in the **horizontal line** are called rows.
- Elements in the **vertical line** are called column.
- Element is identified by its row and column number. We write in subscript notation like
  $\mathbf{a_{ij}}$ called $\mathbf{ij^{th}}$ element or entry of given matrix.
- $\mathbf{a_{ij}}$ indicates element in $\mathbf{i^{th}}$ row and $\mathbf{j^{th}}$ column.
- The number of rows and the number of columns are called **the size or, order dimension of matrix.**
- The dimension of the above matrix are 3 × 4 (read three by four)
- Write the number of rows it first and the number of column second.

Illustrative example

1. If $A = \begin{bmatrix} 3 & 6 & -2 & 4 & 1 \\ 7 & 0 & 5 & -6 & 8 \\ 11 & 9 & -4 & -3 & 2 \\ -8 & -1 & -7 & \frac{1}{2} & -5 \end{bmatrix}$, then

   a) Find the order of matrix A
   b) Find he value of a_{13}, a_{31}, a_{24}, a_{42}, a_{21}, a_{25} a_{52}, a_{36}
   c) Lable these element of A with variable a and subscripts.
   i) -7 ii) -3 iii) 2 iv) -6

4x3

Solution: a) $4 \times 5 \longrightarrow$ It has 4 rows and 5 column

b) $a_{13} = -2$, $a_{31} = 11 \longleftarrow 3^{rd}$ row and 1^{st} column

$a_{24} = -6$, $a_{42} = -1$, $a_{21} = 7 \longleftarrow 2^{nd}$ row 1^{st} column

$a_{25} = 8$, a_{52} = does not exist

a_{36} = does not exist

c) i) $a_{43} = -7$ ii) $a_{34} = -3$, iii) $a_{35} = 2$ iv) $a_{24} = -6$

- **Type of matrices**

These are row matrix, column matrix, square matrix, diagonal matrix, zero matrix, scalar matrix, identity matrix, and triangular matrix.

I. **Row matrix.** A matrix having exactly single row $(3\ 5\ 7)$

II. **Column Matrix.** A matrix having exactly single column $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

III. **Square matrix.** A matrix with the same number of rows and column is called a square matrix

$$\begin{pmatrix} 3 & 4 \\ 7 & 1 \end{pmatrix}_{2\times 2}$$

IV. **Diagonal matrix.** A square matrix in which all non diagonal elements are 0 is called a diagonal matrix.

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

V. **Zero matrix.** A matrix in which each element is zero is called zero matrix or null matrix for example.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{2\times 2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}_{3\times 2}$$

VI. **Scalar Matrix.** A diagonal matrix in which all the diagonal elements are equal

Example $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are scalar matrix

VII. **Identity (unit) matrix.** A scalar matrix in which each diagonal elements are all one. For example

Example $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are identity matrix.

VIII. **Upper triangular matrix.** A square matrix in which all the elements below the diagonal elements are zero

Example, $\begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 5 & 7 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$ are upper triangular.

- **Lower triangular matrix.** A square matrix in which all the elements above the diagonal elements are zero.

Example: $\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 3 & 4 & 6 \end{pmatrix}$ are lower triangular matrix.

- **Triangular matrix.** Either upper triangular or lower triangular.

Equality of matrices

Two matrices $A_{m\times n}$ and $B_{m\times n}$ are said to be equal, written

i) A = B if and only if they have the same order and their corresponding elements are equal ($a_{ij} = b_{ij}$ for $1 \le i \le m$ and $1 \le j \le n$

Illustrative Example

2. Find x, y, and z if the matrices

$$A=\begin{pmatrix} -5 & 5x-8 & 4 \\ 3 & 8 & y-3 \\ 10 & 3z+5 & 12 \end{pmatrix} \text{ and } B=\begin{pmatrix} -5 & 2x+3 & 4 \\ 3 & 8 & 5y-6 \\ 10 & -3z+5 & 12 \end{pmatrix}$$

are equal

Solution: If A = B, then corresponding element must be equal

$5x - 8 = 2x + 3$

$5y - 6 = y - 3$ and $3z + 5 = -3z + 5$

Solving, $5x - 8 = 2x + 3 \Rightarrow 3x = 11 \Rightarrow x = \dfrac{11}{3}$

$$5y - 6 = y - 3 \Rightarrow 3y = 3 \Rightarrow y = 1$$
$$3z + 5 = -3z + 5 \Rightarrow 6z = 0 \Rightarrow z = 0$$

Addition and subtraction of matrices

If A and B are two matrices of the same order then the sum of A and B denoted by **A + B** obtained by adding the corresponding element of A and B

Let $A = a_{ij}$, $B = b_{ij}$, and $C = c_{ij}$ be $m \times n$ matrices.
Then $C = A + B$ if and only if $c_{ij} = a_{ij} + b_{ij}$
And $C = A - B$ if and only if $c_{ij} = a_{ij} - b_{ij}$

Properties of matrix addition

i. $A + B = B + A$ (commutative property)
ii. $(A + B) + C = A + (B + C)$ (Associative)
iii. $A + 0 = 0 + A = A$ (Existence of additive identity.)
iv. $A + (-A) = 0$ (Existence of additive inverse)

Multiplication of a matrix by a scalar

The product of a matrix $\mathbf{A} = \mathbf{a_{ij}}$ by a scalar k is denoted by KA and is obtained by multiplying every element of A by K

Thus $KA = (Ka_{ij})$

If $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$, then $KA = \begin{bmatrix} ka_1 & ka_2 & ka_3 \\ kb_1 & kb_2 & kb_3 \end{bmatrix}$

Illustrative Example

3. Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}$

Find a) 2A + 3B b) 2A − B

Solution: a) $2\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} + 3\begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 2 & 6 \end{pmatrix} + \begin{pmatrix} -3 & 9 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 2 & 12 \end{pmatrix}$

b) $2\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 2 & 6 \end{pmatrix} + \begin{pmatrix} 1 & -3 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 5 & -5 \\ 2 & 4 \end{pmatrix}$

4. Let $A = \begin{pmatrix} 9 & 1 \\ 4 & 3 \end{pmatrix}$,

$B = \begin{pmatrix} 1 & 5 \\ 7 & 12 \end{pmatrix}$, find the matrix x such that $3A + 2B + \frac{1}{2}x = 0$

Solution: $3A + 2B + \frac{1}{2}x = 0 \Rightarrow x = -2[3A + 2B]$

$$\Rightarrow x = -2\left[3\begin{pmatrix} 9 & 1 \\ 4 & 3 \end{pmatrix} + 2\begin{pmatrix} 1 & 5 \\ 7 & 12 \end{pmatrix}\right] = -2\left[\begin{pmatrix} 27 & 3 \\ 12 & 9 \end{pmatrix} + \begin{pmatrix} 2 & 10 \\ 14 & 24 \end{pmatrix}\right]$$

$$\Rightarrow x = -2\begin{bmatrix} 29 & 13 \\ 26 & 33 \end{bmatrix} = \begin{bmatrix} -58 & -26 \\ -52 & -66 \end{bmatrix}$$

5. Find x, y, z and w if $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$

Solution: $\begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+4 & 6+x+y \\ -1+z+w & 2w+3 \end{bmatrix}$

Equating of the corresponding element

$\Rightarrow 3x = x + 4 \Rightarrow 2x = 4 \Rightarrow x = 2$

$\Rightarrow 3w = 2w+3 \Rightarrow w = 3$

$\Rightarrow 3z = -1 + z + w \Rightarrow 2z = -1 + 3 = 2 \Rightarrow z = 1$

$\Rightarrow 3y = 6 + x + y \Rightarrow 2y = 6 + 2 \Rightarrow y = 4$

6. Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 7 & 8 & 9 \\ -3 & 0 & 4 \end{bmatrix}$ find matrix B such that

a) A + B is a scalar matrix

b) A + B is an identity matrix

Solution: Let $B = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

Now, $A + B = \begin{bmatrix} 1 & -2 & 3 \\ 7 & 8 & 9 \\ -3 & 0 & 4 \end{bmatrix} + \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ is scalar

So that $\begin{bmatrix} 1+a_1 & -2+a_2 & 3+a_3 \\ 7+b_1 & 8+b_2 & 9+b_3 \\ -3+c_1 & 0+c_2 & 4+c_3 \end{bmatrix} = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$

$\Leftrightarrow 1 + a_1 = k \Rightarrow a_1 = k - 1$ and $-2 + a_2 = 0 \Rightarrow a_2 = 2$

$3 + a_3 = 0 \Rightarrow a_3 = -3$ and $7 + b_1 = 0 \Rightarrow b_1 = -7$, $8 + b_2 = k$

$\Rightarrow b_2 = k - 8$, $9 + b_3 = 0 \Rightarrow b_3 = 3$ again $-3 + c_1 = 0 \Rightarrow c_1 = 3, 0 +$

$c_2 = 0 \Rightarrow c_2 = 0$, and $4 + c_3 = k \Rightarrow c_3 = k - 4$

$$\therefore B = \begin{pmatrix} k-1 & 2 & 3 \\ -7 & k-8 & -9 \\ 3 & 0 & k-4 \end{pmatrix}$$

b) $\begin{pmatrix} 1 & -2 & 3 \\ 7 & 8 & 9 \\ -3 & 0 & 4 \end{pmatrix} + \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ identity matrix

$\Rightarrow a_1 + 1 = 1 \Rightarrow a_1 = 0$, $-2 + a_2 = 0 \Rightarrow a_2 = 2$, $a_3 = -3$

$\Rightarrow 7 + b_1 = 0 \Rightarrow b_1 = -7$, $8 + b_2 = 1 \Rightarrow b_2 = -7$, $b_3 = -9$

$-3 + c_1 = 0 \Rightarrow c_1 = 3$, $0 + c_2 = 0 \Rightarrow c_2 = 0$, $c_3 + 4 = 1$, $c_3 = -3$

$$\therefore B = \begin{pmatrix} 0 & 2 & -3 \\ -7 & -7 & -9 \\ 3 & 0 & -3 \end{pmatrix}$$

7. For matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$

$D = \begin{bmatrix} 3 & 3 \\ 0 & -2 \end{bmatrix}$ for what values of a, b and c the matrix D is linear combination (aA + bB + cC) of A, B and C.

A. 1, -2, 1 B. 2, 2, -2 C. 1, 1, -1 D. 2, 1, -1

Solution:

$$=\begin{bmatrix} a & a \\ a & -a \end{bmatrix}+\begin{bmatrix} 0 & b \\ b & b \end{bmatrix}+\begin{bmatrix} -c & b \\ c & c \end{bmatrix}=\begin{bmatrix} 3 & 3 \\ 0 & -2 \end{bmatrix}$$

$\Rightarrow a + 0 + (-c) = 3$ and $a + b + 0 + 3$

$\Rightarrow 0 + b + c = 0 \Rightarrow b = -c$

$\therefore a = 2,\ b = 1,\ c = 1$ **Answer: D**

8. Construct a 3×2 matrix if $A = (a_{ij})$ where $a_{ij} = 3i - 4j$

Solution: $a_{11} = 3 - 4 = -1$

$a_{12} = 3 - 4(2) = -5$

$a_{21} = 3(2) - 4 = 2$

$a_{22} = 3(2) - 4(2) = 6 - 8 = -2$

$a_{31} = 3(3) - 4(1) = 9 - 4 = 5$

$a_{32} = 3(3) - 4(2) = 9 - 8 = 1$

$$\therefore\ A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}=\begin{pmatrix} -1 & -5 \\ 2 & -2 \\ 5 & 1 \end{pmatrix}$$

Matrix Multiplication

Two matrices A and B are said to be conformable for the product AB (in this every order of A and B) if the number of columns in A is equal to the number of rows in B.

Thus, if the orders of A and B are $m \times n$ and $p \times q$ respectively then

i) AB is defined if number of columns in A = number of rows in B i.e. if $n = p$

ii) BA is defined if number of columns in B equal number of rows in A i.e. if $q = m$

The rule for multiplication of two conformable matrices is called row – by – column method

Consider $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$ and $B=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times 2}$

Orders of A and B are 3×3 respectively. AB is defined and is of order 3×2.

$$= \begin{pmatrix} 3 & 14 \\ -3 & -7 \end{pmatrix}$$

$$BA = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 0 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 0 & 1 & -3 \\ 3 & -2 & -1 & -8 \\ 8 & 4 & 9 & -5 \end{bmatrix}$$

$AB \neq BA$

Properties of Multiplication of Matrices

- $A(BC) = (AB)C$ (Associative)
- $A(B+C) = AB + AC$ (Distributive)
- $AB \neq BA$ (Not commutative)
- If A is square matrix, then $A^2 = A.A$, $A^3 = A.A.A$.....

Note: The multiplication of diagonal matrices is commutative.

Example, Let $A = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ then

Find a) AB b) BA

Solution: a) $AB = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 3 \end{pmatrix}$

b) $BA = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 3 \end{pmatrix}$

$AB = BA \longleftarrow$ diagonal matrix are commutative

If $A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} a & 1 \\ b & -1 \end{pmatrix}$ and $(A+B)^2 = A^2 + B^2$

find a and b.

Solution: $A + B = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} + \begin{pmatrix} a & 1 \\ b & -1 \end{pmatrix} = \begin{pmatrix} a+1 & 0 \\ b+2 & -2 \end{pmatrix}$

- $(A+B)^2 = \begin{pmatrix} a+1 & 0 \\ b+2 & -2 \end{pmatrix}\begin{pmatrix} a+1 & 0 \\ b+2 & -2 \end{pmatrix} = \begin{pmatrix} a^2+2a+1 \\ ab-b+2a-2 \end{pmatrix}$
- $A^2 = A \cdot A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

- $B^2 = B\cdot B = \begin{pmatrix} a & 1 \\ b & -1 \end{pmatrix}\begin{pmatrix} a & 1 \\ b & -1 \end{pmatrix} = \begin{pmatrix} a^2+b & a-1 \\ ab-b & b+1 \end{pmatrix}$
- $A^2 + B^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} a^2+b & a-1 \\ ab-b & b+1 \end{pmatrix} = \begin{pmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{pmatrix}$

Now, $(A + B)^2 = A^2 + B^2$

$$\Rightarrow \begin{pmatrix} a^2+2a+1 & 0 \\ ab-b+2a-2 & 4 \end{pmatrix} = \begin{pmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{pmatrix}$$

Thus, $a - 1 = 0 \Rightarrow a = 1$ and $b = 4$, $\therefore a = 1, b = 4$

10. If $f(x) = x^2 - 5x$ and $A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$ then find f(A).

Solution: Here, Note, $A^2 = A\cdot A$

$$f(A) = A^2 - 5A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} - 5\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$\Rightarrow A^2 = A\cdot A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1-4 & -2+2 \\ 2-2 & -4+1 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$$

$$\therefore f(A) = A^2 - 5A = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} - 5\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -8 & 10 \\ -10 & 2 \end{pmatrix}$$

11. If $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then find a) A^2 b) A^3 C) A^4 D) A^n

Solution: Here

a) $A^2 = A\cdot A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1+1 & 1+1 \\ 1+1 & 1+1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$$\therefore A^2 = 2A = 2\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

b) $A^3 = A^2\cdot A = 2A\cdot A = 2A^2 = 2\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 4\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 4A$

c) $A^4 = A^3\cdot A = 4A\cdot A = 4A^2 = 4(2A) = 8A$

19. Given, $A=\begin{bmatrix}4 & -3 & 1\\ -5 & 2 & -1\end{bmatrix}$ *and* $B=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}$

Let C_{ij} denote the element in the i^{th} row and j^{th} column of the product $C = AB$ then find a) C_{23} b) C_{21} c) C_{13}

Solution: a) $C_{23} = A_2B^3 \longleftarrow$ Product of 2nd row of A and 3rd column of B

$$\therefore C_{23} = (-5 \quad 2 \ -2)\begin{pmatrix}3\\ 6\\ 9\end{pmatrix} = -15+12-18=-21$$

b) $$C_{21} = A_2B^1 = (-5\ 2\ -2)\begin{pmatrix}1\\ 4\\ 7\end{pmatrix} = -5+8-14=-11$$

c) $$C_{13} = A_1B^3 = (4\ -3 \quad 1)\begin{pmatrix}3\\ 6\\ 9\end{pmatrix} = 12-18+9=3$$

Transpose of Matrix

If we interchange row in to column and column in to row of **mxn** matrix A we get **n ×m** matrix, denoted by A^t which is called the transpose of A

Example: a) If $A = \begin{bmatrix}1 & 2 & 3\\ 5 & 6 & 7\end{bmatrix}_{2\times3}$ *then* $A^t = \begin{pmatrix}1 & 5\\ 2 & 6\\ 3 & 7\end{pmatrix}_{3\times2}$

b) If $A = \begin{pmatrix}3 & 4\\ 5 & 6\end{pmatrix}_{2\times2}$ *then* $A^t = \begin{pmatrix}3 & 5\\ 4 & 6\end{pmatrix}_{2\times2}$

Properties of transpose matrix

i) $(A^t)^t = A$ ii) $(A + B)^t = A^t + B^t$ iii) $(AB)^t = B^tA^t$

Symmetric and skew symmetric matrices

Definition: i) A square matrix A is called **symmetric** matrix if $A^t = A$

Definition: ii) A square matrix A is called **skew symmetric** matrix if $A^t = -A$ or $A^t + A = 0$

In skew symmetric matrices A

- $A = [a_{ij}]$, $a_{ij} = -a_{ji}$
- $a_{11} = a_{22} = a_{33} = \ldots = 0 \longleftarrow$ all diagonal element 0
- $a_{12} = -a_{21}$, $a_{13} = -a_{31}$ .....
- all main diagonal element of skew symmetry matrix is zero.

To show, put i = j

$\Rightarrow \mathbf{a_{ii} = -a_{ii}} \Rightarrow \mathbf{2a_{ii}} = 0 \Rightarrow \mathbf{a_{ii}} = 0$ for $\mathbf{a_{11}}$, $i = j$

20. Let $A = \begin{pmatrix} 3 & 4 \\ 4 & 6 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 3 & 5 \\ 3 & -1 & 4 \\ 5 & 4 & 8 \end{pmatrix}$ then show that

a) $A^t = A$ b) $B^t = B$

Solution: a) $A^t = \begin{pmatrix} 3 & 4 \\ 4 & 6 \end{pmatrix} \Rightarrow A^t = A \Rightarrow A$ is symmetric

b) $B^t = \begin{pmatrix} -2 & 3 & 5 \\ 3 & -1 & 4 \\ 5 & 4 & 8 \end{pmatrix} \Rightarrow B^t = B \Rightarrow B$ is symmetric

21. Let $A = \begin{bmatrix} 0 & 4 & -6 \\ -4 & 0 & 2 \\ 6 & -2 & 0 \end{bmatrix}$ then show that A is skew symmetric

Solution: • $a_{11} = a_{22} = a_{33} = 0$

- $a_{12} = 4$ and $a_{21} = -4 \Rightarrow a_{12} = -a_{21}$
- $a_{13} = -6$ and $a_{31} = 6 \Rightarrow a_{13} = -a_{31}$

$\therefore$ A is skew symmetric.

Note: If A is a square matrix then

i) $A + A^t$ is symmetric matrix

ii) $A - A^t$ is skew symmetric

iii) $\left(\frac{A + A^t}{2}\right) + \left(\frac{A - A^t}{2}\right) = A$

22. Let $A = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$ then

a) find symmetric matrix from A

c) $\begin{vmatrix} x & x+2 \\ x+1 & 2x \end{vmatrix} = -2$ d) $\begin{vmatrix} 2x-1 & x+1 \\ x+2 & x-2 \end{vmatrix} = 9$

Solution:

a) $\begin{vmatrix} x-3 & 8 \\ 2 & x+3 \end{vmatrix} = 0 \Leftrightarrow x^2 - 9 - 16 = 0 \Rightarrow x^2 - 25 = 0 \Rightarrow x$

b) $\begin{vmatrix} 4 & 2x \\ 5x & 3 \end{vmatrix} = -8 \Leftrightarrow 12 - 10x^2 = -8 \Rightarrow -10x^2 = -20 \Rightarrow x^2 = 2$

$\therefore x = \pm\sqrt{2}$

c) $\begin{vmatrix} x & x+2 \\ x+1 & 2x \end{vmatrix} = -2 \Leftrightarrow 2x^2 - (x+2)(x+1) = -2$

$\Leftrightarrow 2x^2 - (x^2 + 3x+2) = -2$

$\Rightarrow x^2 - 3x - 2 = -2 \Rightarrow x^2 - 3x = 0 \therefore x = 0 \text{ or } x = 3$

d) $\begin{vmatrix} 2x-1 & x+1 \\ x+2 & x-2 \end{vmatrix} = 9 \longleftarrow$ exercise left for you (Tip!)

Determinant of order 3 × 3 matrices

In 3×3 matrix, the determinants are defined as follows

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Illustrative Example

26. $\begin{vmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \\ 5 & -2 & 6 \end{vmatrix} = 2\begin{vmatrix} 4 & 2 \\ -2 & 6 \end{vmatrix} - 1\begin{vmatrix} 1 & 5 \\ 5 & 6 \end{vmatrix} + 3\begin{vmatrix} 1 & 4 \\ 5 & -2 \end{vmatrix} = 5 + 4 - 66 = -6$

Consider the following illustration of **minors and cofactor.**

- Let the minors of the element: a_{11}, a_{12} and a_{23} be denoted by M_{11}, M_{12} and M_{23} respectively

Let the cofactor of the element a_{11}, a_{12} and a_{23} be denoted by A_{11}, A_{12} and A_{23} respectively. Then:

Matrix	Minor	Cofactor
$\begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{bmatrix}$	$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ $= a_{22}a_{33} - a_{32}a_{23}$	$A_{11} = (-1)^{1+1}M_{11} = M_{11}$
$\begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{bmatrix}$	$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$	$A_{12} = (-1)^{1+2}M_{12} = -M_{12}$
$\begin{bmatrix} a_{11} & a_{12} & \cancel{a_{13}} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} \\ a_{31} & a_{32} & \cancel{a_{33}} \end{bmatrix}$	$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$	$A_{23} = (-1)^{2+3}M_{23} = -M_{23}$

Minors and Cofactor

Definition: Let $A = a_{ij}$ be a square matrix of order $n > 1$

i) The **minor M_{ij}** of the element **a_{ij}** is the determinant of the matrix of order $n - 1$ obtained by deleting the **i^{th} row** and **j^{th} column** of A.

ii) The **cofactor A_{ij}** of the element a_{ij} is $A_{ij} = (-1)^{i+j}M_{ij}$.

Illustrative Example

27. Find all the minors and the cofactors of the elements in the matrix (a_{11}, a_{12}, a_{21}, a_{22})

a) $\begin{bmatrix} 6 & -4 \\ 3 & 2 \end{bmatrix}$ b) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Solution: a) $M_{11} = 2 \longleftarrow$ deleting 1^{st} row and 1^{st} column

$M_{12} = 3 \longleftarrow$ minor of -4

$M_{21} = -4$ and $M_{22} = 6$

$$\therefore \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -4 & 6 \end{bmatrix}$$ 2 -3

- Minor of 6 is 2
- Minor of -4 is 3
- Minor of 3 is -4

Cofactor $(A_{ij} = (-1)^{i+j}(M_{ij})$

Thus, $A_{11} = (-1)^{1+1}(M_{11}) = (1)(2) = 2 \longleftarrow$ cofactor of 6

$A_{12} = (-1)^{1+2}(M_{12}) = (-1)(3) = -3 \longleftarrow$ cofactor of 4

$A_{21} = (-1)^{2+1}M_{21} = (-1)(-4) = 4 \longleftarrow$ cofactor of 3

$A_{22} = (-1)^{2+2}M_{22} = (1)(6) = 6 \longleftarrow$ cofactor of 2

$$\therefore \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & 6 \end{bmatrix}$$

b) $\therefore \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} d & c \\ b & a \end{bmatrix} \leftarrow$ minor of entry $a_{11}, a_{12}, a_{21}, a_{22}$

Cofactor, of $a_{11}, a_{12}, a_{21}, a_{22}$ $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \leftarrow$ use $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$

28. If $A = \begin{pmatrix} 1 & -3 & 3 \\ 4 & 2 & 0 \\ -2 & -7 & 5 \end{pmatrix}$, then find

a) the minor of a_{11}, a_{21}, a_{23}

b) the cofactor of a_{11}, a_{21}, a_{23}

Solution: Deleting the right row and column of A.

a) Minor of a_{11} is $M_{11} = \begin{vmatrix} 2 & 0 \\ -7 & 5 \end{vmatrix} = 10 \leftarrow$ deleting 1st row and 1st column.

Minor of a_{21} is $M_{21} = \begin{vmatrix} -3 & 3 \\ -7 & 5 \end{vmatrix} = -15 + 21 = 6 \leftarrow$ deleting 2nd row and 1st column

Minor of a_{23} is $M_{23} = \begin{vmatrix} 1 & -3 \\ -2 & -7 \end{vmatrix} = -7 - 6 = -13 \leftarrow$ deleting 2nd row and 3rd column

b) Cofactor of a_{11} is $A_{11} = (-1)^{1+1}M_{11} = (1)(10) = 10$

Cofactor of a_{21} is $A_{21} = (-1)^{2+1}M_{21} = (-1)(6) = -6$

Cofactor of a_{23} is $A_{23} = (-1)^{2+3}M_{23} = (-1)(-13) = 13$

Finding determinant of 3 × 3 matrix by expanding by cofactors.

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$ is given by expanding

- Along Row I: $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$.
- Along Row III: $|A| = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$.
- Along Column II: $|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$.

29. Let $A = \begin{pmatrix} 3 & -2 & 2 \\ 6 & 1 & -1 \\ -2 & -3 & 2 \end{pmatrix}$ then determinant $|A|$ by expanding along:

a) first row b) first column c) 2^{nd} row

Solution: a) along first row, $|A| = 3A_{11} - 2A_{12} + 2A_{13}$

$$= 3(-1)^2\begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} + (-2)(-1)^3\begin{vmatrix} 6 & -1 \\ -2 & 2 \end{vmatrix} + 2(-1)^2\begin{vmatrix} 6 & 1 \\ -2 & -3 \end{vmatrix}$$

$= 3(2 - 3) + 2(12 - 2) + 2(-18 + 2) = -15$

$\therefore |A| = -15$

b) Along first column: $|A| = 3A_{11} + 6A_{21} + (-2)A_{31}$

$$= 3(-1)^2\begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} + 6(-1)^3\begin{vmatrix} -2 & 2 \\ -3 & 2 \end{vmatrix} + (-2)(-1)^4\begin{vmatrix} -2 & 2 \\ 1 & -1 \end{vmatrix}$$

$= 3(2 - 3) + -6(-4 + 6) + -2(2 - 2) = -15 \quad \therefore |A| = -15$

c) Exercise left for you

30. Solve each of the following equation:

a) $\begin{vmatrix} -1 & 3 & -6 \\ x & 3 & 0 \\ 1 & -2 & 1 \end{vmatrix} = 1$ b) $\begin{vmatrix} x+1 & 2 & 3 \\ x & 1 & x \\ 1 & 1 & 1 \end{vmatrix} = 0$

Solution:

Since the second row contains a zero, we shall expand the 2^{nd} row.

$\Rightarrow xA_{21} + 3A_{22} + 0A_{23} = 1$

$$\Rightarrow (-1)^3(x)\begin{vmatrix}3 & -6\\ -2 & 1\end{vmatrix} + 3(-1)^4\begin{vmatrix}-1 & -6\\ 1 & 1\end{vmatrix} + 0(-1)^5\begin{vmatrix}-1 & 3\\ 1 & -2\end{vmatrix} = 1$$

$-x(3-12) + 3(-1+6) + 0 = 1$

$9x + 15 = 1$

$9x = -14$

$$\therefore x = \frac{-14}{9}$$

b) $(x+1)A_{11} + 2A_{12} + 3A_{13} = 0$ (by expanding 1st row)

$$\Rightarrow (x+1)(-1)^2\begin{vmatrix}1 & x\\ 1 & 1\end{vmatrix} + 2(-1)^3\begin{vmatrix}x & x\\ 1 & 1\end{vmatrix} + 3(-1)^4\begin{vmatrix}x & 1\\ 1 & 1\end{vmatrix} = 0$$

$\Rightarrow (x+1)(1-x) - 2(x-x) + 3(x-1) = 0$

$\Rightarrow 1 - x^2 + 3x - 3 = 0 \Rightarrow x^2 - 3x + 2 = 0 \Leftrightarrow (x-2)(x-1) = 0$

$\therefore x = 2$ or $x = 1$

Determinant of triangular matrix form

- Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13}\\ 0 & a_{22} & a_{23}\\ 0 & 0 & a_{33}\end{pmatrix}$ then $|A| = a_{11}\, a_{22}\, a_{33}$

- Let $A = \begin{pmatrix} a_{11} & 0 & 0 & 0\\ a_{21} & a_{22} & 0 & 0\\ a_{31} & a_{32} & a_{33} & 0\\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix}$ = then $|A| = a_{11} \cdot a_{22}\, a_{33}\, a_{44}$

- *If A be squar matrix of order **n** in triangular form then determinant of A is the product of the elements on the main diagonal.*

  $|A| = a_{11}\, a_{22}\, A_{33} \ldots A_{nn}$.

31. Compute each of the following determinant

a) $\begin{vmatrix}3 & 4\\ 0 & 6\end{vmatrix}$ b) $\begin{vmatrix}2 & 0 & 0\\ 6 & 7 & 0\\ 5 & 4 & 1\end{vmatrix}$

c) $\begin{vmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{vmatrix}$ d) $\begin{vmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{vmatrix}$

Solution: a) $3 \times 6 = 18$ c) $(2)(-2)(1)(3) = -12$

b) $2 \times 7 \times 1 = 14$ d) $(5)(2)(-3) = -30$

Properties of Determinant

1. *If A is square matrix then* $|A| = |A^t|$, *Example*, $\begin{vmatrix} 3 & 5 \\ 4 & 6 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}$

2. Inter **change** of any two rows or column **change** the sign of the determinant

Example:

a) $\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = -\begin{vmatrix} 5 & 6 \\ 3 & 4 \end{vmatrix}$ b) $\begin{vmatrix} 2 & 0 & 1 \\ 6 & 4 & 3 \\ 0 & 3 & 5 \end{vmatrix} = -\begin{vmatrix} 6 & 4 & 3 \\ 2 & 0 & 1 \\ 0 & 3 & 5 \end{vmatrix}$

3. If the elements one row (or column) of a determinant are multiplied by scalar k, then the new determinant multiplied by $|B| = k|A|$

Example: • $\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = \begin{vmatrix} ka & b \\ kc & d \end{vmatrix} = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

• $\begin{vmatrix} ka & kb \\ mc & md \end{vmatrix} = (km)\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

• $\begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = (k)^2\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

• $\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{vmatrix} = k^3\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Example: • $\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = -2$, $\begin{vmatrix} 6 & 4 \\ 10 & 6 \end{vmatrix} = 2\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = 2(-2) = -4$

35. If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 4$ then which of the following is not true

A. $\begin{vmatrix} 2a_1 & c_1 & 3b_1 \\ 2a_2 & c_2 & 3b_2 \\ 2a_3 & c_3 & 3b_3 \end{vmatrix} = 24$ C. $\begin{vmatrix} 3a_3 & 3c_3 & 6b_3 \\ a_2 & c_2 & 2b_2 \\ a_1 & c_1 & 2b_1 \end{vmatrix} = -24$

B. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 + 4a_2 & b_3 + 4b_2 & c_3 + 4c_2 \end{vmatrix} = 4$ D. $\begin{vmatrix} \frac{1}{2}a_1 & \frac{1}{2}b_1 & \frac{1}{2}c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Solution:

A. $-(2)(3)(4) = -24$ (2^{nd} column and 3^{rd} column are inter-changed and factoring)

B. $4 \longleftarrow$ Adding multiple of row to another row, the determinant value is the same.

C. $(-(3)(2)(4) = -24 \longleftarrow$ Inter change 1^{st} row and 3^{rd} row and factoring 3 and 2

D. $\frac{1}{2} \cdot 0 = 0 \longleftarrow$ factoring $\frac{1}{2}$ and two identical row.

36. If $\begin{vmatrix} x & t+w \\ y & S+U \end{vmatrix} = m$, and $\begin{vmatrix} x & t \\ y & s \end{vmatrix} = n$, then $\begin{vmatrix} x & w \\ y & u \end{vmatrix}$ is equal

A. m + n B. m − n C. mn D. m

Solution: $\begin{vmatrix} x & t+w \\ y & S+U \end{vmatrix} = \begin{vmatrix} x & t \\ y & s \end{vmatrix} + \begin{vmatrix} x & w \\ y & u \end{vmatrix}$

$\Rightarrow m = n + \begin{vmatrix} x & w \\ y & u \end{vmatrix} \Rightarrow \begin{vmatrix} x & w \\ y & u \end{vmatrix} = m - n$

Answ

37. a) Use the properties of determinant to show that the equation of straight line through the points (x_1, y_1) and (x_2, y_2) is given by

$$\begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix} = 0$$

b) Find the equation of the line passing through the points (2, 3) and (-1, 4) (using #37a)

Solution: a) Putting x_1 for x and y_1 for y.

$$\Rightarrow \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_1 & y_1 \\ 1 & x_2 & x_2 \end{vmatrix} = 0 \leftarrow \text{we get two identical row}$$

$\therefore$ (x_1, y_1) and (x_2, y_2) are on the same line

b) putting $x_1 = 2$, $y_1 = 3$ and $x_2 = -1$, $y_2 = 4$

$$\Rightarrow \begin{vmatrix} 1 & x & y \\ 1 & 2 & 3 \\ 1 & -1 & 4 \end{vmatrix} = 1\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} + (-1)^3 x\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + y\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = 0$$

$\Rightarrow 8 + 3 - x(4 - 3) + y(-1 - 2) = 0$

$\Rightarrow -x - 3y = -11 \Rightarrow x + 3y = 11$

$\therefore$ The equation of line is $x + 3y = 11$

Removing common factors from Rows and Column.

38. Find the factor and determinant value of each of the following

a) $\begin{vmatrix} a & b \\ a^2 & b^2 \end{vmatrix}$ b) $\begin{vmatrix} 14 & -6 & 4 \\ 4 & -5 & 12 \\ -21 & 9 & -6 \end{vmatrix}$ c) $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$

Solution: Here

a) $\begin{vmatrix} a & b \\ a^2 & b^2 \end{vmatrix} = ab\begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix} = ab(b-a) \longrightarrow$ Removing factor a, b from 1^{st} and 2^{nd} column.

b) $\begin{vmatrix} 14 & -6 & 4 \\ 4 & -5 & 12 \\ -21 & 9 & -6 \end{vmatrix} = (2)(-3)\begin{vmatrix} 7 & -3 & 2 \\ 4 & -5 & 12 \\ 7 & -3 & 2 \end{vmatrix} = (-6)(0) = 0$

- 2 is common factor of row 1
- -3 is common factor of row 3
- Row 1 and Row 3 are identical

c) $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \leftarrow$ Removing factor a, b,

$= abc\begin{vmatrix} 1-1 & 1-1 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix} \leftarrow$ from 1st, 2nd and 3rd column.

- $C_1 = C_1 - C_2$
- $C_2 = C_2 - C_3$ (by property four

$= abc\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix} = abc\begin{vmatrix} a-b & b-c \\ a^2-b^2 & b^2-c^2 \end{vmatrix} \leftarrow$

Determinant by expanding 1st row

- Factor 1st and 2nd column of a – b and b – c gives:

$= abc(a-b)(b-c)\begin{vmatrix} 1 & 1 \\ a+b & b+c \end{vmatrix}$

$\therefore \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (abc)(a-b)(b-c)(c-a)$

Adjoint of a square matrix

Definition: The adjoint of a square matrix A is the transpose of the matrix obtained by replacing each element of A by its co-factor in A denoted by **adj A= $(A_{ij})^T$**

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ then *cofactor of* $A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

$$\therefore \text{adj } A = (\text{cofactor of } A)^t = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{32} & A_{33} \end{pmatrix}$$

Example a) Find adj.A if $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$

Solution: First, find the cofactor, A_{11}, A_{12}, A_{21}, A_{22}

$A_{11} = (-1)^{1+1}|w| = w$ and $A_{12} = (-1)^{1+2}|z| = -z$

$A_{21} = (-1)^{2+1}|y| = -y$ and $A_{22} = (-1)^{2+2}|x| = x$

Then, cofactor $= \begin{pmatrix} w & -z \\ -y & x \end{pmatrix}$, adjoint is transpose of cofactor

$$\therefore adjA = \begin{pmatrix} w & -y \\ -z & x \end{pmatrix}$$

Thus, $\text{adj}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ $\text{adj}\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 6 & -4 \\ -5 & 3 \end{pmatrix}$

Singular and non-singular matrices

Definition: A square matrix A is called

i) **Singular** if $|A| = 0$

ii) **Non-singular** if $|A| \neq 0$

Example: Find $|A|$ if (i) $A = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$

Solution: i) $|A| = (3)(8) - (6)(4) = 24 - 24 = 0 \leftarrow$ A is singular

ii) $|A| = 3.6 - 4.5 = 18 - 20 = -2 \leftarrow$ non singular (Invertible)

39. For what value of x the given matrix A is singular

a) $A = \begin{pmatrix} x+1 & x \\ 2x & -x \end{pmatrix}$ b) $A = \begin{pmatrix} x+2 & 6 \\ 2 & x-2 \end{pmatrix}$

Solution: a) $|A| = (x+1)(-x) - 2x^2 = -x^2 - x - 2x^2 = -3x^2 - x$

$\Rightarrow -3x^2 - x = 0 \Rightarrow x(3x+1) = 0$

$\therefore x = 0$ or $x = -\frac{1}{3} \longleftarrow$ singular

b) $|A| = 0 \Rightarrow (x+2)(x-2) - 12 = 0 \Rightarrow x^2 - 4 - 12 = 0$
$\Rightarrow x^2 - 16 = 0 \qquad \therefore x = 4$ or $x = -4$

40. For what values set of x the matrix A is non singular (Invertible)

a) $A = \begin{pmatrix} x & 3 \\ 4 & x-4 \end{pmatrix}$ b) $A = \begin{pmatrix} x-3 & -2 \\ 4 & x \end{pmatrix}$

Solution:

a) $|A| \neq 0 \Rightarrow x^2 - 4x - 12 \neq 0 \Rightarrow (x-6)(x+2) \neq 0$
$\Rightarrow x \neq 6$ and $x \neq -2$.

$\therefore$ S.S= $\mathbb{R}\setminus\{6,-2\} \rightarrow$ Invertible for all real number except 6,-2

b) $|A| \neq 0 \Rightarrow x^2 - 3x + 8 \neq 0$, for all x, $|A| \neq 0$

$\therefore$ S.S= R $\leftarrow$ Invertible for every x

Inverse of a square matrix

Definition: A square matrix A said to be **Invertible (non-singular)** if and only if there is square matrix B such that AB = BA = I where I is the identity matrix.

I. The inverse of A is denoted by A^{-1} i.e $B = A^{-1}$ so that $AA^{-1} = I_n$

II. $|AA^{-1}| = |A||A^{-1}| = |I_n| = 1 \quad \therefore |A^{-1}| = \frac{1}{|A|}$

III. $A^{-1} = \frac{1}{|A|} adj.A$ IV. $(AB)^{-1} = B^{-1}A^{-1}$ V) $(KA)^{-1} = \frac{1}{k}A^{-1}$

41. Which of the following is an invertible matrix?

A. $\begin{bmatrix} 2 & -5 \\ 4 & -10 \end{bmatrix}$ C. $\begin{bmatrix} -2 & 3 \\ 4 & -6 \end{bmatrix}$

B. $\begin{bmatrix} 2 & -8 \\ -3 & 12 \end{bmatrix}$ D. $\begin{bmatrix} 4 & -3 \\ 6 & 9 \end{bmatrix}$

Solution:

A. $\begin{vmatrix} 2 & -5 \\ 4 & -10 \end{vmatrix} = -20 + 20 = 0 \longleftarrow$ Not invertible

B. $\begin{vmatrix} 2 & -8 \\ -3 & 12 \end{vmatrix} = 24 - 24 = 0 \longleftarrow$ Not invertible

C. $\begin{vmatrix} -2 & 3 \\ 4 & -6 \end{vmatrix} = 12 - 12 = 0 \longleftarrow$ Not invertible

D. $\begin{vmatrix} 4 & -3 \\ 6 & 9 \end{vmatrix} = 36 + 18 = 54 \neq 0 \longleftarrow$ Invertible **Answer: D**

42. Using adjoint-cofactor method find the inverse of each of the following matrices

A. $A = \begin{bmatrix} 1 & 3 \\ 4 & 13 \end{bmatrix}$ B. $A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$. C. $A = \begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix}$

Solution: To find the inverse:

i) determine, $|A|$

ii) determine cofactor of A

iii) determine adj.A

iv) write $A^{-1} = \frac{1}{|A|} adj.A$

Explanation

adj $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

a) $|A| = \begin{vmatrix} 1 & 3 \\ 4 & 13 \end{vmatrix} = 13 - 12 = 1$

cofactor of A: $A_{11} = 13$, $A_{12} = (-1)^3(4) = -4$, $A_{21} = (-1)^3(3) = -3$, $A_{22} = 1$

$$\therefore Adj.A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^t = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} = \begin{pmatrix} 13 & -3 \\ -4 & 1 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adj.A = \frac{1}{1}\begin{pmatrix} 13 & -3 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 13 & -3 \\ -4 & 1 \end{pmatrix}$$

b) $|A| = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 10 - 3 = 7$, adj.A $= \begin{pmatrix} 5 & -3 \\ -1 & 2 \end{pmatrix}$

$$\therefore A^{-1} = \frac{1}{|A|} adj.A = \frac{1}{7}\begin{pmatrix} 5 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{7} & \frac{-3}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{pmatrix}$$

c) $|A|=\begin{vmatrix}2 & 5\\-1 & 3\end{vmatrix}=6+5=11$, adj.A $=\begin{pmatrix}3 & -5\\1 & 2\end{pmatrix}$

$$\therefore A^{-1}=\frac{1}{|A|}adj.A=\frac{1}{11}\begin{pmatrix}3 & -5\\1 & 2\end{pmatrix}=\begin{pmatrix}\frac{3}{11} & \frac{-5}{11}\\ \frac{1}{11} & \frac{2}{11}\end{pmatrix}$$

43. Find the inverse of each matrix:

a) $A=\begin{bmatrix}1 & 0\\0 & k\end{bmatrix}$ b) $A=\begin{bmatrix}k & 1\\1 & 1\end{bmatrix}, k\neq 1$

c) $A=\begin{bmatrix}k & 1\\1 & k\end{bmatrix}, k\neq \pm 1$

Solution:

a) $A=\begin{vmatrix}1 & 0\\0 & k\end{vmatrix}=$ k, $adj.A=\begin{pmatrix}k & 0\\0 & 1\end{pmatrix}$

$$\therefore A^{-1}=\frac{1}{k}\begin{pmatrix}k & 0\\0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\0 & \frac{1}{k}\end{pmatrix}$$

b) $A=\begin{vmatrix}k & 1\\1 & 1\end{vmatrix}=$ k, $adj.A=\begin{pmatrix}1 & -1\\-1 & k\end{pmatrix}$

$$\therefore A^{-1}=\frac{1}{|A|}adj.A=\frac{1}{k-1}\begin{pmatrix}1 & -1\\-1 & k\end{pmatrix}=\begin{pmatrix}\frac{1}{k-1} & \frac{-1}{k-1}\\ \frac{-1}{k-1} & \frac{k}{k-1}\end{pmatrix}$$

c) $A=\begin{vmatrix}k & 1\\1 & k\end{vmatrix}=k^2-1$, $adj.A=\begin{pmatrix}k & -1\\-1 & k\end{pmatrix}$

$$\therefore A^{-1}=\frac{1}{|A|}adj.A=\frac{1}{k^2-1}\begin{pmatrix}k & -1\\-1 & k\end{pmatrix}=\begin{pmatrix}\frac{k}{k^2-1} & \frac{-1}{k^2-1}\\ \frac{-1}{k^2-1} & \frac{k}{k^2-1}\end{pmatrix}$$

44. Find the inverse of each of the following

a) $A=\begin{pmatrix}1 & 2 & -1\\ 3 & 1 & 1\\ 1 & -1 & 2\end{pmatrix}$ b) $A=\begin{pmatrix}1 & 0 & 2\\ 2 & -1 & 3\\ 4 & 1 & 8\end{pmatrix}$

Solution: First find the cofactor, then adjA

a) $A_{11}=(-1)^{1+1}\begin{vmatrix}1 & 1\\ -1 & 2\end{vmatrix}=3$ $A_{12}=(-1)^{1+2}\begin{vmatrix}3 & 1\\ 1 & 2\end{vmatrix}=-5$

$$A_{13}=(-1)^{1+3}\begin{vmatrix}3 & 1\\ 1 & -1\end{vmatrix}=-4$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}2 & -1\\ -1 & 2\end{vmatrix}=-3 \qquad A_{22}=(-1)^{4}\begin{vmatrix}1 & -1\\ 1 & 2\end{vmatrix}=3$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}1 & 2\\ 1 & -1\end{vmatrix}=3$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}2 & -1\\ 1 & 1\end{vmatrix}=3 \quad A_{32}=(-1)^{3+2}\begin{vmatrix}1 & -1\\ 1 & 2\end{vmatrix}=-3$$

$$A_{33}=(-1)^{6}\begin{vmatrix}1 & 2\\ 3 & 1\end{vmatrix}=-5$$

$$\therefore \text{cof.A}=\begin{pmatrix}3 & -5 & -4\\ -3 & 3 & 3\\ 3 & -3 & -5\end{pmatrix},\ \therefore \text{adjA}=\begin{pmatrix}3 & -3 & 3\\ -5 & 3 & -3\\ -4 & 3 & -5\end{pmatrix}$$

And, $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$ = 1.3 +2.)-5) + (-1)(-4) = -3

$$\therefore\ A^{-1}=\frac{1}{|A|}adj.A=\frac{1}{-3}\begin{pmatrix}3 & -3 & 3\\ -5 & 3 & -3\\ -4 & 3 & -5\end{pmatrix}=\begin{pmatrix}-1 & 1 & -1\\ \frac{5}{3} & -1 & 1\\ \frac{4}{3} & -1 & \frac{5}{3}\end{pmatrix}$$

b) $A_{11} = (-1)^2 \begin{vmatrix} -1 & 3 \\ 1 & 8 \end{vmatrix} = -11 \qquad A_{12} = (-1)^3 \begin{vmatrix} 2 & 3 \\ 4 & 8 \end{vmatrix} = -4,$

$$A_{13} = (-1)^4 \begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} = 6$$

Similarly, $A_{21} = 2$, $A_{22} = 0$, $A_{23} = -1$

$A_{31} = 2$, $A_{32} = 1$, $A_{33} = -1$

And $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 1(-11)+0.(-4)+2.6 = 1$

$$\therefore\ A^{-1} = \frac{1}{|A|} adj.A = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}$$

45. Which of the following pair of matrices are not inverses of each other?

A. $\begin{bmatrix} 1 & 3 \\ 4 & 13 \end{bmatrix}$ *and* $\begin{bmatrix} 13 & -3 \\ -4 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ *and* $\begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ *and* $\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$ D. $\begin{bmatrix} 2 & 6 \\ 6 & 5 \end{bmatrix}$ *and* $\begin{bmatrix} \frac{5}{8} & \frac{-3}{4} \\ \frac{-3}{8} & \frac{1}{4} \end{bmatrix}$

Solution: Multiply the pair of matrices, if you get the identity matrices then, the pair of matrices are inverse of each other,

that, $\begin{bmatrix} 1 & 3 \\ 4 & 13 \end{bmatrix} . \begin{bmatrix} 13 & -3 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} . \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} . \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

But D. $\begin{bmatrix} 2 & 6 \\ 6 & 5 \end{bmatrix} . \begin{bmatrix} \frac{5}{8} & \frac{-3}{4} \\ \frac{-3}{8} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ \frac{15}{8} & \frac{-13}{4} \end{bmatrix} \longleftarrow$ Not identity

that not inverse of each other.

Answer

If the matrices $\begin{bmatrix} 3 & x+2 \\ x & 3 \end{bmatrix}$ *and* $\begin{bmatrix} x+1 & -4 \\ -x & x+1 \end{bmatrix}$ are inverses of each other. then the possible value of x is

A. 0 B. -5 C. -1 D. 2

Solution: $\begin{pmatrix} 3 & x+2 \\ x & 3 \end{pmatrix} \cdot \begin{pmatrix} x+1 & -4 \\ -x & x+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} 3x+3-x^2-2x & -12+x^2+3x+2 \\ x^2+x-3x & -4x+3x+3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow C_{11} = -x^2 + x + 3 = 1 \Leftrightarrow x^2 - x - 2 = 0$ ......(*)

$\Rightarrow (x-2)(x+1) = 0 \Leftrightarrow x = 2, x = -1$

$\Rightarrow C_{12}=0 \Rightarrow x^2+3x - 10 = 0 \Leftrightarrow (x-2)(x+5) = 0 \Leftrightarrow x = 2, x = -5$..(**)

$\Rightarrow C_{21} = 0 \Rightarrow x^2 - 2x = 0 \Leftrightarrow x(x-2) = 0 \Leftrightarrow x = 0, x = 2$.... (***)

$\Rightarrow C_{22} = 1 \Rightarrow -4x + 3x + 3 = 1 \Leftrightarrow -x + 3 = 1 \Leftrightarrow x = 2$ .....(****)

Thus, the common value from (*), (**), (***) and (****) is 2.

Answer: D

47. Let $A = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ then

Find a) A^t b) A^{-1} c) $A^t.A^{-1}$

Solution: a) $A^t = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b) to find A^{-1}

$$\text{adj. A} = \begin{bmatrix} \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} \sin\theta & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} \sin\theta & \cos\theta \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} -\sin\theta & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} \cos\theta & -\sin\theta \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \end{vmatrix} & \begin{vmatrix} \sin\theta & 0 \\ \cos\theta & 0 \end{vmatrix} & \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} \end{bmatrix} \ldots \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\therefore \text{adj.} A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } |A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= \cos\theta\cos\theta + \sin\theta\sin\theta + 0$$

$$\therefore |A| = \cos^2\theta + \sin^2\theta = 1$$

$$A^{-1} = \frac{1}{|A|} adj.\ A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \longleftarrow A^t = A^{-1}$$

c. $A^{-1}A^t \longrightarrow$ Exercise left for students

A system of equation can be written as matrix equation

System	**matrix equation**
$a_1x + b_1y = C_1$ $a_2x + b_2y = C_2$	$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$

$A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ can be solved using the inverse of A

$$A^{-1}A\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

Find the value of x, y, for each of the following matrix equation

a) $\begin{bmatrix} 3 & 4 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ b) $\begin{pmatrix} -5 & 3 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

Solution: Multiplying each side of matrix equation by the inverse

a) Let $A = \begin{bmatrix} 3 & 4 \\ 3 & 5 \end{bmatrix}, adj.A = \begin{pmatrix} 5 & -4 \\ -3 & 3 \end{pmatrix}$, $|A| = 15 - 12 = 3$

Thus,

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 5 & -4 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} & \frac{-4}{3} \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$\therefore$ x = -3 and y = 2

b) Let $A = \begin{pmatrix} -5 & 3 \\ 4 & -1 \end{pmatrix}, adj.A = \begin{pmatrix} -1 & -3 \\ -4 & -5 \end{pmatrix}, |A| = 5 - 12 = -7$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \frac{1}{-7}\begin{pmatrix} -1 & -3 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{4}{7} & \frac{5}{7} \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \Rightarrow x=2,\ y=5$$

Systems of equations with two or three variables

In this section, describes, how to solve the system of equations by matrix method.

Consider, a matrix created from a system of equation.

As an example, we start with the system $\begin{cases} x - 2y + 3z = 4 \\ 2x + y - 4z = 3 \\ -3x + 4y - z = -2 \end{cases}$

Now, using only the coefficients of the variable and constants of this system, we can write the 3 × 4 matrix

$$\begin{matrix} x - 2y + 3z = 4 \rightarrow \\ 2x + y - 4z = 3 \rightarrow \\ -3x + 4y - z = -2 \rightarrow \end{matrix}\begin{pmatrix} 1 & -2 & 3 & 4 \\ 2 & 1 & -4 & 3 \\ -3 & 4 & -1 & -2 \end{pmatrix} \leftarrow \text{this called the augmented matrix.}$$

The matrix formed by the coefficient of the variable is called **coefficient matrix**

$$\begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & -4 \\ -3 & 4 & -1 \end{pmatrix} \longleftarrow \text{coefficient matrix}$$

$$\begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} \longleftarrow \text{constant matrix (formed from the constant)}$$

Elementary Row operations

1. Interchange two rows.

   Symbolic representation: $R_i \longleftrightarrow R_j$

2. Multiply all the elements in a row by the same non zero number.

   Symbolically: $kR_j \to R_i$

3. Replace a row by the sum of that row and a multiple of any other row. **($kR_i + R_j \to R_j$)**

The goal of row operations is transforming the matrix to upper **triangular matrix** with main diagonal element 1 and 0's to the left of the 1's in all rows except the first. This called the **echelon form** of the matrix. Examples of echelon form are:

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & -1 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & -3 & -1 & -6 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 1 & -2 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 1 & -4 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The order in which the elements in the 2×3 matrix were changed is important.

1. Change a_{11} to 1
2. Change a_{21} to zero
3. Change a_{22} to 1

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \left[\begin{array}{cc|c} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \end{array}\right]$$

Illustrative Example

49. Solve by using a matrix, row operations.

a) $\begin{cases} 2x + 3y = 6 \\ 4x + 5y = 8 \end{cases}$ b) $\begin{cases} 3x - 5y = -7 \\ 5x - 3y = -1 \end{cases}$

Solution: Write the system in matrix.

$$\begin{cases} 2x + 3y = 6 \longrightarrow \\ 4x + 5y = 8 \longrightarrow \end{cases} \left[\begin{array}{cc|c} 2 & 3 & 6 \\ 4 & 5 & 8 \end{array}\right]$$

i. Change element a_{11} to 1.

$\Rightarrow$ Multiply row 1 by $\frac{1}{2}$

$$\left[\begin{array}{cc|c} 1 & \frac{3}{2} & 3 \\ 4 & 5 & 8 \end{array}\right]$$

$$R_1 \rightarrow \frac{1}{2}R_1$$

ii. Change element a_{21} to zero.
$\Rightarrow$Multiply row 1 by -4, and add it to row 2. Replace row 2 by the sum.
($-4R_1 + R_2 \rightarrow R_2$)

$$\left[\begin{array}{cc|c} 1 & \frac{3}{2} & 3 \\ 0 & -1 & -4 \end{array}\right]$$

iii. Change element a_{22} to 1.
$\Rightarrow$Multiply row 2 by -1.
The matrix is now in **echelon form.**

$$\left[\begin{array}{cc|c} 1 & \frac{3}{2} & 3 \\ 0 & 1 & 4 \end{array}\right]$$

Write the system of equations represented by the matrix.

$$\begin{cases} x + \frac{3}{2}y = 3 \Rightarrow x + \frac{3}{2}(4) = 3 \Rightarrow x = 3 - 6 = -3 \\ y = 4 \end{cases}$$

$\Rightarrow x = -3, y = 4 \therefore$ The solution is (-3, 4).

b) $$\begin{cases} 3x - 5y = -7 \longrightarrow \\ 5x - 3y = -1 \longrightarrow \end{cases} \begin{bmatrix} 3 & -5 & -7 \\ 5 & -3 & -1 \end{bmatrix}$$

i. Change a_{11} to 1.

$$\left(R_1 \longrightarrow \frac{1}{3}R_1\right)$$

$\Rightarrow$Multiply row 1 by $\frac{1}{3}$ $\begin{bmatrix} 1 & \frac{-5}{3} & -\frac{7}{3} \\ 5 & -3 & -1 \end{bmatrix}$

ii. Change a_{21} to zero

$\Rightarrow$Multiply row 1 by (-5), and add it to row 2. Put the sum in row 2. $(-5R_1 + R_2 \rightarrow R_2)$

$$\left[\begin{array}{cc:c} 1 & -\frac{5}{3} & -\frac{7}{3} \\ 0 & \frac{16}{3} & \frac{32}{3} \end{array}\right]$$

iii. Change a_{22} to 1.

$$\frac{3}{16}R_2 \longrightarrow R_2$$

$$\Rightarrow \text{Multiply row 2 by } \frac{3}{16} \left(\begin{array}{cc:c} 1 & -\frac{5}{3} & -\frac{7}{3} \\ 0 & 1 & 2 \end{array}\right)$$

Write the system of equation represented in matrix form

$$\Rightarrow \begin{cases} x - \frac{5}{3}y = -\frac{7}{3} \\ y = 2 \end{cases} \Rightarrow \begin{cases} x - \frac{5}{3}(2) = -\frac{7}{3} \\ \Rightarrow x = -\frac{7}{3} + \frac{10}{3} = \frac{3}{3} = 1 \end{cases}$$

$\Rightarrow x = 1, y = 2$

The order in which the elements in a 3×4 matrix are changed is:

1. Change a_{11} to 1
2. Change a_{21} and a_{31} to zero.
3. Change a_{22} and a_{33} to 1.
4. Change a_{32} to zero.

$$\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]$$

Illustrative Example

Solve by using a matrix row operation.

$$\begin{cases} 3x + 2y + 3z = 2 \\ 2x - 3y + 4z = 5 \\ x + 4y + 2z = 8 \end{cases} \qquad \text{b)} \quad \begin{cases} 3x - 2y - 3z = 5 \\ x + 3y - 2z = -4 \\ 2x + 6y + 3z = b \end{cases}$$

Solution:

i) Write the system in matrix form.

ii. Change a_{11} to 1

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 2 \\ 2 & -3 & 4 & 5 \\ 1 & 4 & 2 & 8 \end{array}\right]$$

⇒Inter change row 1 and row 3. $(R_1 \longleftrightarrow R_3)$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 8 \\ 2 & -3 & 4 & 5 \\ 3 & 2 & 3 & 2 \end{array}\right]$$

iii. Change a_{21} and a_{31} to zero.

Multiply row 1 by (-2) and add to row 1.

Multiply row 1 by (-3) and add to row 1.

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 8 \\ 0 & -11 & 0 & -11 \\ 0 & -10 & -3 & -22 \end{array}\right]$$

iv. Change a_{22} to 1.

⇒ Multiply row 2 by

v. Change a_{32} to zero.

⇒Multiply row 2 by 10 and add to row 3.

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & -10 & -3 & -22 \end{array}\right]$$

vi. Change a_{33} to 1.

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & -12 \end{array}\right].$$

Multiply row 3 by $\left(-\frac{1}{3}\right)$.

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Write the system of equations represented by the matrix.

$\Rightarrow x + 4y + 2z = 8$

$y = 1$

$z = 4$

Substitute y = 1 and z = 4 into the equation: x + 4y + 2z
and solve for x.

$\Rightarrow x + 4(1) + 2(4) = 8 \Rightarrow x = 8 - 4 - 8 = -4$

$\therefore\ x = -4,\ y = 1,\ z = 4$

b) **Exercise left for you. (Ans. (3, -1, 2)**

Cramer's Rule

Cramer's rule is a method of solving a system of linear equations of

A) Two variable

Let $\begin{cases} a_1x + b_1y = k_1 \\ a_2x + b_2y = k_2 \end{cases}$ the Solution of the system is given by

$$\text{i) } x = \frac{D_x}{D} = \frac{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_1 & b_2 \end{vmatrix}}, \qquad \text{ii) } y = \frac{D_y}{D} = \frac{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_1 & b_2 \end{vmatrix}}$$

$$\therefore\ \text{S.S} = \left\{\left(\frac{D_x}{D}, \frac{D_y}{D}\right)\right\} \text{ where } D \neq 0$$

B) Three variable

Let $\begin{cases} a_1x + b_1y + c_1z = k_1 \\ a_2x + b_2y + c_2z = k_2 \\ a_3x + b_3y + c_3z = k_3 \end{cases}$ the Solution of the system is given by:

$$\text{i) } x = \frac{D_x}{D} = \frac{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \qquad \text{ii) } y = \frac{D_y}{D} = \frac{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

iii) $$z=\frac{D_z}{D}=\frac{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

$$\therefore S.S=\{(x,y,z)\}=\left\{\left(\frac{D_x}{D},\frac{D_y}{D},\frac{D_z}{D}\right)\right\} \text{ where } D \neq 0$$

The three possibility of Solution

1. Case **I.** If $D \neq 0$, then the system has **exactly** one Solution (consistent)
2. Case **II.** If $D = 0$, $D_x = D_y = D_z = 0$ then the system has an **infinite number** of Solution (dependent)
3. Case **III.** If $D = 0$, $D_x \neq 0$, $D_y \neq 0$ $D_z \neq 0$, then the system has **no Solution** (inconsistent)

Illustrative Example

51. Solve by cramer's rule

a) $\begin{cases} 3x-y=-8 \\ 2x+6y=3 \end{cases}$ b) $\begin{cases} 3x+2y=4 \\ 5x+5y=9 \end{cases}$ c) $\begin{cases} x-2y=-1 \\ 2x+5y=16 \end{cases}$

Solution:

a) $$\begin{cases} 3x-y=-8 \\ 2x+6y=3 \end{cases}, \; x=\frac{D_x}{D}=\frac{\begin{vmatrix} -8 & -1 \\ 3 & 6 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ 2 & 6 \end{vmatrix}}=\frac{-48+3}{18+2}=\frac{-9}{4},$$

$$y=\frac{D_y}{D}=\frac{\begin{vmatrix} 3 & 4 \\ 5 & 9 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ 2 & 6 \end{vmatrix}}=\frac{25}{20}=\frac{5}{4}$$

b) $\begin{cases} 3x+2y=4 \\ 5x+5y=9 \end{cases}$, $x=\dfrac{D_x}{D}=\dfrac{\begin{vmatrix} 4 & 2 \\ 9 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 5 & 5 \end{vmatrix}}=\dfrac{2}{5}$, $y=\dfrac{D_y}{D}=\dfrac{\begin{vmatrix} 3 & 4 \\ 5 & 9 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 5 & 5 \end{vmatrix}}$

$\therefore$ S.S $=\left\{\left(\dfrac{2}{5},\dfrac{7}{5}\right)\right\}$

c) $\begin{cases} x-2y=-1 \\ 2x+5y=16 \end{cases}$, $x=\dfrac{D_x}{D}=\dfrac{\begin{vmatrix} -1 & -2 \\ 16 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 2 & 5 \end{vmatrix}}=\dfrac{27}{9}=3,$

$$y=\frac{D_y}{D}=\frac{\begin{vmatrix} 1 & -1 \\ 2 & 16 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 2 & 5 \end{vmatrix}}=\frac{18}{9}=2 \ \therefore \text{S.S} = \{(3,2)\}$$

52. Solve the following systems of equations

a) $\begin{cases} 3x+2y+z=-5 \\ x-y+3z=-5 \\ 2x+3y+z=0 \end{cases}$ b) $\begin{cases} 2x+y-z=9 \\ 3x-4y+z=3 \\ x-3y+2z=-4 \end{cases}$

Solution:

a) $D=\begin{vmatrix} 3 & 2 & 1 \\ 1 & -1 & 3 \\ 2 & 3 & 1 \end{vmatrix}=3\begin{vmatrix} -1 & 3 \\ 3 & 2 \end{vmatrix}-2\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}+\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}=-15$

$$D_x=\begin{vmatrix} -5 & 2 & 1 \\ -5 & -1 & 3 \\ 0 & 3 & 1 \end{vmatrix}=45,\ D_y=\begin{vmatrix} 3 & -5 & 1 \\ 1 & -5 & 3 \\ 2 & 0 & 1 \end{vmatrix}=-30,$$

[illegible]

$$x = \frac{D_x}{D} = \frac{-30}{-10} = 3, \quad y = \frac{D_y}{D} = \frac{10}{-10} = -1, \quad z = \frac{D_z}{D} = \frac{20}{-10} = -2$$

S.S = {(3, −1, −2)}

5. Identify the system has [illegible] solution, no solution, infinite many solution.

a) $\begin{cases} x + 3y = 6 \\ 2x + 6y = 18 \end{cases}$ b) [illegible] c) [illegible]

Solution:

a) $D = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6 - 6 = 0$, $D_x = \begin{vmatrix} 6 & 3 \\ 18 & 6 \end{vmatrix} = 36 - 54 \neq 0$

Since $D = 0$, $D_x \neq 0$, $D_y \neq 0$ ∴ No solution [illegible]

b) $D = \begin{vmatrix} 2 & -5 \\ 3 & 1 \end{vmatrix} = 2 + 15 = 17$

Since $D \neq 0$ ∴ It has one solution [illegible]

c) $D = \begin{vmatrix} 3 & 6 \\ 1 & 2 \end{vmatrix} = 6 - 6 = 0$, $D_x = \begin{vmatrix} 18 & 6 \\ 6 & 2 \end{vmatrix} = 36 - 36 = 0$

$$D_z=\begin{vmatrix}3 & 2 & -5\\ 1 & -1 & -5\\ 2 & 3 & 0\end{vmatrix}=0,$$

$$S.S=\{(x,y,z)\}=\left\{\left(\frac{D_x}{D},\frac{D_y}{D},\frac{D_z}{D}\right)\right\}=\left\{\left(\frac{45}{-15},\frac{-30}{-15},\frac{0}{-15}\right)\right\}$$
$$=\{(-3,2,0)\}$$

$$D=\begin{vmatrix}2 & 1 & -1\\ 3 & -4 & 1\\ 1 & -3 & 2\end{vmatrix}=-10,\quad D_x=\begin{vmatrix}9 & 1 & -1\\ 3 & -4 & 1\\ -4 & -3 & 2\end{vmatrix}=-30$$

$$D_y=\begin{vmatrix}2 & 9 & -1\\ 3 & 3 & 1\\ 1 & -4 & 2\end{vmatrix}=-10,\quad D_z=\begin{vmatrix}2 & 1 & 9\\ 3 & -4 & 3\\ 1 & -3 & -4\end{vmatrix}=20$$

$$\therefore x=\frac{D_x}{D}=\frac{-30}{-10}=3,\ y=\frac{D_y}{D}=\frac{-10}{-10}=1,\ z=\frac{D_z}{D}=\frac{20}{-10}=-2$$

$\therefore$ S.S = {(3,1,-2)}

3. Identify, the system has: unique solution, no solution, infinite many solution.

a) $\begin{cases}x+3y=6\\ 2x+6y=-18\end{cases}$ b) $\begin{cases}2x-5y=11\\ 3x+y=8\end{cases}$ c) $\begin{cases}8x-4y=16\\ 2x-y=4\end{cases}$

Solution:

a) $D=\begin{vmatrix}1 & 3\\ 2 & 6\end{vmatrix}=6-6=0,\ D_x=\begin{vmatrix}6 & 3\\ -18 & 6\end{vmatrix}=36+54\neq 0,$

Since $D=0$, $D_x\neq 0$, $D_y\neq 0$ $\therefore$ No Solution (inconsistence)

b) $D=\begin{vmatrix}2 & -5\\ 3 & 1\end{vmatrix}=2+15=17$

Since $D\neq 0$, $\therefore$ It has one Solution (consistence)

c) $D=\begin{vmatrix}8 & -4\\ 2 & -1\end{vmatrix}=-8+8=0,\ D_x=\begin{vmatrix}16 & -4\\ 4 & -1\end{vmatrix}=-16+16=0$

Since $D = 0$, $D_x = 0$, $D_y = 0$,

$\therefore$ It has infinite Solution (dependent)

54. Determine the value of a and b for which the system

$$\begin{cases} 3x - ay = 1 \\ 6x + 4y = b \end{cases} \text{ have}$$

a) unique solution b) no solution **c) infinite sol**

Solution: First $D = \begin{vmatrix} 3 & -a \\ 6 & 4 \end{vmatrix} = 12 + 6a$

$$D_x = \begin{vmatrix} 1 & -a \\ b & 4 \end{vmatrix} = 4 + ab$$

$$D_y = \begin{vmatrix} 3 & 1 \\ 6 & b \end{vmatrix} = 3b - 6$$

i) System has unique solution, if $D \neq 0 \Rightarrow 12 + 6a \neq 0$

$\therefore a \neq -2,\ b \in R$

ii) has no solution if $D = 0$, $D_x = 0$, $D_y \neq 0$

$\Rightarrow a = -2,\ ab + 4 \neq 0 \Rightarrow ab \neq -4, \Rightarrow b \neq 2$

$\therefore a = -2$ and $b \neq 2$

55. For what values of k does the system

$$\begin{cases} kx + 4y = 5 \\ 9x + ky = 2 \end{cases} \text{ have}$$

a) Unique Solution

b) No Solution

c) Infinite number of Solution

Solution: First, $D = \begin{vmatrix} k & 4 \\ 9 & k \end{vmatrix} = k^2 - 36$

$$D_x = \begin{vmatrix} 5 & 4 \\ 2 & k \end{vmatrix} = 5k - 8,\quad D_y = \begin{vmatrix} k & 5 \\ 9 & 2 \end{vmatrix} = 2k - 45$$

The system have:

a) Unique Solution if $D \neq 0 \Rightarrow k^2 - 36 \neq 0 \Rightarrow k \neq 6$

b) No Solution if $D = 0$, $D_x \neq 0$ and $D_y \neq 0$

$\Rightarrow k = 6$ or $k = -6$, $5k - 8 \neq 0 \Rightarrow 5k \neq 8 \Rightarrow k \neq \frac{8}{5}$

$\therefore k = 6$ or $k = -6$, $k \neq \frac{8}{5}$, $k \neq \frac{45}{2}$

c) Infinite Solution if $D = 0$, $D_x = 0$, $D_y = 0$

$\therefore k = 6$ or $k = -6$, $k = \frac{8}{5}$, $k = \frac{45}{2}$

56. Determine the value of a and b for which the system

a) $\begin{cases} 3x - 2y + z = b \\ 5z - 8y + 9z = 3 \\ 2x + y + az = -1 \end{cases}$

i) unique Solution ii) No Solution iii) infinite many solve

Solution: First we find determinant of coefficient matrix (D)

$$D = \begin{vmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{vmatrix} = 3\begin{vmatrix} -8 & 9 \\ 1 & a \end{vmatrix} - (2)\begin{vmatrix} 5 & 9 \\ 2 & a \end{vmatrix} + 1\begin{vmatrix} 5 & -8 \\ 2 & 1 \end{vmatrix} = -14a - 42$$

$$D_x = \begin{vmatrix} b & -2 & 1 \\ 3 & -8 & 9 \\ -1 & 1 & a \end{vmatrix} = b\begin{vmatrix} -8 & 9 \\ 1 & a \end{vmatrix} - (-2)\begin{vmatrix} 3 & 9 \\ -1 & a \end{vmatrix} + 1\begin{vmatrix} 3 & -8 \\ -1 & 1 \end{vmatrix}$$

$= -8ab - 9b + 6a + 13$

The system have:

a) unique Solution if $D \neq 0 \Rightarrow -14a \neq 42$ $\therefore a \neq -3$

b) No Solution if $D = 0$, $D_x \neq 0$, $D_y \neq 0$, $D_z \neq 0$

$\Rightarrow a = -3$ and $D_x \neq 0 \Rightarrow -8(-3)b - 9b + -18 + 13 \neq 0$

$\Rightarrow 15b - 5 \neq 0$ $b \neq \frac{5}{15}$, $b \neq \frac{1}{3}$

$\therefore a = -3$ and $b \neq \frac{1}{3}$

c) Infinite many Solution if $D = 0$, $D_x = D_y = D_z = 0$

$\therefore a = -3$ and $b = \frac{1}{3}$

Homogeneous linear equation

The system $\begin{cases} a_1x + b_1y = 0 \\ a_2x + b_2y = 0 \end{cases}$ is called homogeneous linear equation

- A system of homogeneous equation always has the trivial Solution
  ($x = 0, y = 0, z = 0$), ($D \neq 0, D_x = 0, D_y = 0$)
- Some time Non-trivial (non zero) Solution exist (infinitely many Solution) ($D = 0$} $D_x = 0, D_y = 0, D_z = 0$)

Illustrative Example

57. Solve each of the following homogeneous system of equation (identify type of Solution)
    i) trivial ii) non-trivial

    a) $\begin{cases} 3x + 5y = 0 \\ 2x - 4y = 0 \end{cases}$ b) $\begin{cases} 2x - 6y = 0 \\ -x + 3y = 0 \end{cases}$

Solution: a) $D = \begin{vmatrix} 3 & 5 \\ 2 & -4 \end{vmatrix} = -12 - 10 = -22 \neq 0$

$$D_x = \begin{vmatrix} 0 & 5 \\ 0 & -4 \end{vmatrix} = 0, \qquad D_y = \begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix} = 0$$

$$\therefore\ x = \frac{D_x}{D} = 0,\ y = 0$$

$\therefore$ S.S = {(0, 0)} $\longleftarrow$ trivial Solution

b) $D = \begin{vmatrix} 2 & -6 \\ -1 & 3 \end{vmatrix} = 6 - 6 = 0 \longleftarrow$ Non trivial

$\therefore$ S.S = {(x, y): x = 3y} $\Leftrightarrow$ {(3a, a)}

58. For what values of k does the homogeneous system has
    i) Trival Solution
    ii) non-zero Solution (non trival Solution)

    a) $\begin{cases} (k-1)x + y = 0 \\ 3x + (k+1)y = 0 \end{cases}$ b) $\begin{cases} x + 2y + kz = 0 \\ 2x + ky + 2z = 0 \\ 3x + y + z = 0 \end{cases}$

Solution;

i) trivial Solution (x=0, y= 0, z = 0) is obtained when D = 0

ii) non-zero Solution are obtained when $D \neq 0$

a) $D = \begin{vmatrix} k-1 & 1 \\ 3 & k+1 \end{vmatrix} = k^2 - 1 - 3 \Rightarrow k^2 - 4$

i) trivial Solution, $k^2 - 4 \neq 0 \Rightarrow k \neq \pm 2$

ii) non trivial Solution, when $D = 0 \Rightarrow k^2 - 4 = 0 \Rightarrow k = 2$ or $k = -2$

b) $D = \begin{vmatrix} 1 & 2 & k \\ 2 & k & 2 \\ 3 & 1 & 1 \end{vmatrix} = 3k^2 - 3k - 6$

i) trivial Solution: $D \neq 0 \Rightarrow 3k^2 - 3k - 6 \Leftrightarrow (k+1)(k-2) \neq 0$
$\therefore k \neq -1$ or $k \neq 2$

ii) Non-zero Solution, $k = -1$ or $k = 2$

Conceptual Example

59. If the determinant $|A| = 3$ and $|B| = 2$, find $|2(AB)^{-1}|$ for 4×4 matrices A and B.

A. $\frac{1}{3}$ B. $\frac{2}{3}$ C. $\frac{4}{3}$ D. $\frac{8}{3}$

Solution:

Recall that $|A^{-1}| = \frac{1}{|A|}$

$|2(AB)^{-1}| = 2^4 \times \frac{1}{|A||B|} = \frac{16}{(3)(2)} = \frac{8}{3}$ **Answer: D**

60. For what values of x and y, $\begin{pmatrix} 2 & 1 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 3x \\ 5y+8 \end{pmatrix}$

Solution: $\begin{pmatrix} 2x+y \\ 3x-4y \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 3x \\ 5y+8 \end{pmatrix}$

$$\Rightarrow \begin{cases} 2x - 3x + y = -4 \\ 3x - 4y + 2 - 5y = 8 \end{cases} \Rightarrow 3\begin{cases} -x + y = -4 \\ 3x - 9y = 6 \end{cases}$$

$$\Rightarrow \begin{cases} -3x+3y=-12 \\ + \; 3x-9y=6 \end{cases}$$

$$-6y=-6, \qquad \therefore y=1,\ x=5$$

61. If $A=\begin{pmatrix}2 & 7\\ 1 & 3\end{pmatrix}$ and $B^{-1}=\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}$ then

What is the value of $(2AB)^{-1}$?

Solution:

Recall that $(2AB)^{-1}=\frac{1}{2}B^{-1}A^{-1}$

If $A=\begin{pmatrix}a & c\\ b & d\end{pmatrix}$ then $A^{-1}=\frac{1}{|A|}\begin{pmatrix}d & -c\\ -b & a\end{pmatrix}$

If $A=\begin{pmatrix}2 & 7\\ 1 & 3\end{pmatrix}$ then $A^{-1}=\frac{1}{-1}\begin{pmatrix}3 & -7\\ -1 & 2\end{pmatrix}=\begin{pmatrix}-3 & 7\\ 1 & -2\end{pmatrix}$

$\therefore (2AB)^{-1}=\frac{1}{2}B^{-1}A^{-1}=\frac{1}{2}\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}-3 & 7\\ 1 & -2\end{pmatrix}$

$$=\frac{1}{2}\begin{pmatrix}1(-3)+1\times 1 & 1\times 7+1(-2)\\ 0(-3)+2(1) & 0\times 7+2(-2)\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-2 & 5\\ 2 & -4\end{pmatrix}=\begin{pmatrix}-1 & 2.5\\ 1 & -2\end{pmatrix}$$

62. Solve a) $\begin{vmatrix}-2x & -3y\\ 4x & -y\end{vmatrix}=7y$ b) $\begin{vmatrix}2x-1 & x+1\\ x+2 & x-2\end{vmatrix}=-12$

Solution: a) $\begin{vmatrix}-2x & -3y\\ 4x & -y\end{vmatrix}=7y \Rightarrow 2xy+12xy=7y$

$$\Rightarrow 14xy=7y \Leftrightarrow 14x=7 \quad \therefore x=\frac{1}{2}$$

b) $\begin{vmatrix}2x-1 & x+1\\ x+2 & x-2\end{vmatrix}=-12 \Rightarrow x^2-8x=-12 \Rightarrow x^2-8x+12$

$$\Rightarrow (x-6)(x-2)=0$$

$$\therefore x=6 \text{ or } x=2$$

The inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \\ 4 & 3 & 1 \end{bmatrix}$ is the matrix

$$A^{-1} = \frac{1}{6}\begin{bmatrix} -7 & 8 & m \\ 8 & -10 & -4 \\ 4 & k & -2 \end{bmatrix} \text{ where;}$$

A. $m = 3, k = 2$ C. $m = 1, k = -3$
B. $m = 5, k = -2$ D. $m = 2, k = -3$

Solution: $A^{-1}. A = I$

$$A^{-1}.A = \frac{1}{6}\begin{bmatrix} -7 & 8 & m \\ 8 & -10 & -4 \\ 4 & k & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \\ 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

C_{11} = the product of row 1 of A^{-1} and column 1 of A.

$\Rightarrow C_{11} = \frac{1}{6}(-7\times2 + 8\times0 + m\times4) = 1 \Rightarrow -14 + 4m = 6$

$\Rightarrow 4m = 20 \Rightarrow m = 5$

$C_{32} = \frac{1}{6}[4\times1 + (-k)\times1 + (-2)(3)] = 0 \Rightarrow -k - 2 = 0$

$\therefore k = -2$ **Answer: B**

54. Which of the following are the values of y and z respectively if

$$\begin{pmatrix} -2 & x+1 \\ y & 0 \end{pmatrix}\begin{pmatrix} 3 & z-1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -7 \\ -3 & -5 \end{pmatrix} \text{...... UEE}$$

A. 2, -1 B. -1, 6 C. 2, 6 D. 6, -1

Solution:

$$\begin{pmatrix} -2 & x+1 \\ y & 0 \end{pmatrix}\begin{pmatrix} 3 & z-1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -6+2x+2 & -2z+2+x+1 \\ 3y & yz-y \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -7 \\ -3 & -5 \end{pmatrix} \Leftrightarrow 2x - 4 = 0 \Rightarrow x = 2 \Leftrightarrow 3y = -3 \Rightarrow y = -1$$

$\Leftrightarrow yz - y = -5 \Rightarrow -z + 1 = -5 \Rightarrow z = 6$ **Answer: B**

65. Let $A = \begin{bmatrix} 4 & 4 & -2 \\ 2 & 6 & 0 \\ 3x & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 3 & 0 \\ -1 & 5 \end{bmatrix}$ are given

If C = AB and $C_{32} = 11$, then what is the value of x.

A. $\frac{3}{5}$ B. $\frac{-4}{3}$ C. $\frac{-2}{3}$ D. -8

Solution: $C_{32} = \begin{pmatrix} 3x & 1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} = 6x + 0 + 15 = 11$

$\Leftrightarrow 6x + 15 = 11 \Rightarrow 6x = -4 \Rightarrow x = \frac{-4}{6} = \frac{-2}{3}$ **Answer: C**

66. If $\begin{pmatrix} x & -3 \\ -1 & y \end{pmatrix}$ is the inverse of $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ then the value of x and y respectively. (UEE)

A. 1, 3 B. 2, 2 C. 4, 1 D. $\frac{1}{2}, -\frac{3}{2}$

Solution: $\begin{pmatrix} x & -3 \\ -1 & y \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Leftrightarrow$

$\begin{pmatrix} 2x-3 & 3x-6 \\ -2+y & -3+2y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Leftrightarrow 2x-3 = 1 \Rightarrow 2x = 4 \Rightarrow x = 2$ and $-2 + y = 0 \Rightarrow y = 2$ **Answer**

67. What are the value of x and y so that

$\begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} y & y \\ 2 & 1 \end{pmatrix}$?

A. x = 3, y = 2 C. x = 4, y = 5
B. x = 0, y = -1 D. x = -2, y = y

Solution: $\begin{pmatrix} 2x-4 & x-1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} y & y \\ 2 & 1 \end{pmatrix} \Leftrightarrow 2x - 4 = y$

$x - 1 = y$

$\Leftrightarrow 2x - 4 = x - 1 \Rightarrow x = 3$

$\therefore\ y = 3(2) - 4 = 2$

Answer: A

68. If $A = \begin{pmatrix} x & -1 \\ 3x & 1 \end{pmatrix}$ then for what values of x is $\det(2A) = x^2$...(UEE)

A. 16 B. 8 C. 4 D. -8

Solution: $|A| = \begin{vmatrix} x & -1 \\ 3x & 1 \end{vmatrix} = x + 3x = 4x$

And $\det(2A) = 2^2|A| = 4|A| = 4(4x) = x^2$

$\Rightarrow x^2 = 16x \Leftrightarrow x^2 - 16x = 0\ x = 0$ or $x = 16$ **Answer: A**

69. What should be the value of k so that the system $\begin{cases} 2kx - y = 4 \\ -4x + ky = 1 \end{cases}$ has a unique Solution?

A. $k \neq 2, -2,$ C. $k = \pm\sqrt{2}$

B. $k \neq -\sqrt{2}, k \neq \sqrt{2}$ D. $k \neq \pm 1$

Solution: $\begin{vmatrix} 2k & -1 \\ -4 & k \end{vmatrix} \neq 0 \Rightarrow 2k^2 - 4 \neq 0 \Rightarrow k^2 - 2 \neq 0, \Rightarrow$

$k \neq \pm\sqrt{2}$ **Answer: B**

70. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ be 2 × 2 matrix. Then the value of A^{2k+1} for $k \in N$ is equal to:

A. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ B. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ C. $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ D. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

* $A^{2k+1} = A^{2k}.A$, Now, $A^{2k} = (A^2)^k$

$$= \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)^k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$\Rightarrow A^{2k+1} = A^{2k}.A = I_2 \cdot A = A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, **Answer: D**

71. Solve for x and y: $\begin{bmatrix} 2 & 4 & -1 \\ -1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 2x+1 \\ 4 \\ x+y \end{bmatrix} = \begin{bmatrix} 12 \\ 11 \end{bmatrix}$

Solution:
Each row of the first matrix is multiplied by each column of second matrix.
$2(2x + 1) + 4(4) + (-1)(x+y) = 12 \Rightarrow 3x - y = -6$ ...... (1)
$-1(2x + 1)+0+5(x+y) = 11 \Rightarrow 3x + 5y = 12$ .... (2)
Subtracting equation (2) from equation (1)
$-6y = -18 \Rightarrow y = 3$ and $3x - 3 = -6 \Rightarrow x = -1$
Therefore, $x = -1$ and $y = 3$.

72. Determine the set of values x such that the matrix
$A = \begin{pmatrix} x^2-1 & 5 \\ 3 & x^2+1 \end{pmatrix}$ is non singular (invertible)

Solution: A is non singular (Invertible) if $|A| \neq 0$

$$\Rightarrow \begin{vmatrix} x^2-1 & 5 \\ 3 & x^2+1 \end{vmatrix} \neq 0 \Leftrightarrow x^4 - 1 - 15 \neq 0 \Leftrightarrow x^4 - 16 \neq 0$$

$$\Rightarrow x^2 - 4)(x^2 + 4) \neq 0 \Rightarrow (x^2 - 4) \neq 0$$
$$\therefore x \neq 2 \text{ or } x \neq -2$$
$$\therefore S.S = R\backslash\{2, -2\}$$

73. What should be the value of α and β respectively so that $\begin{pmatrix} \alpha & 1 \\ 1 & \beta \end{pmatrix}$ is the inverse of $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

A. 0, 0 B. 1, 1 C. -1, 0 D. Not inverti

Solution: $\begin{pmatrix} \alpha & 1 \\ 1 & \beta \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & \alpha+1 \\ \beta & \beta+1 \end{pmatrix} =$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \alpha + 1 = 0 \Rightarrow \alpha = -1$ and $\beta = 0$, **Answer: C**

Given matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & b & b \\ 1 & b & c \end{bmatrix}$, then the value of $|A|$

A. $(1 - b)(c - b)$ C. $(b - 1)(c - b)$
B. $(1 - c)(b - c)$ D. $(c - 1)(b - c)$

Solution: $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & b & b \\ 1 & b & c \end{vmatrix} = 1\begin{vmatrix} b & b \\ b & c \end{vmatrix} - 1\begin{vmatrix} 1 & b \\ 1 & c \end{vmatrix} + 1\begin{vmatrix} 1 & b \\ 1 & b \end{vmatrix}$

$\Rightarrow bc - b^2 - 1(c- b) = b(c - b) - 1(c - b) = (b - 1)(c - b)$

Answer: C

Given the system of equation: find its Solution set

a) $\begin{cases} 2x - 3y = 6 \\ -4x + 6y = -12 \end{cases}$ b) $\begin{cases} 3x - y = 2 \\ -6x + 2y = -4 \end{cases}$ c) $\begin{cases} 5x - y = 1 \\ -5x + y = 4 \end{cases}$

Solution: a) Using row operation:

$$\begin{pmatrix} 2 & -3 & 6 \\ -4 & 6 & -12 \end{pmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{pmatrix} 2 & -3 & 6 \\ 0 & 0 & 0 \end{pmatrix} \leftarrow \text{it has infinite}$$

Solution.

$\therefore$ S.S = {(x, y): 2x - 3y = 6}

* Also we can express as, Let x = C, then $y = \frac{2c}{3} - 2$

$$\therefore S.S = \left\{\left(c, \frac{2c}{3} - 2\right)\right\} = \left\{(0,-2) + \left(1, \frac{2}{3}\right)t, t \in R\right\}$$

b) $\begin{pmatrix} 3 & -1 & 2 \\ -6 & 2 & -4 \end{pmatrix} \xrightarrow{R_2 \to 2R_1 + R_2} \begin{pmatrix} 3 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \leftarrow$ It has infinit many ***Solution*** $\Rightarrow 0 = 0$ (True)

$\therefore$ S.S = {(x,y): 3x − y = 2} or we can express as let x = c, then 3c − y = 2 $\Rightarrow$ y = 3c − 2

$\therefore$ S.S = {(c, 3c -2)}

$\therefore$ S.S = {(0, -2) + (1, 3)t, t $\in$ R} $\leftarrow$ using vector equation of line.

c) $\begin{pmatrix} 5 & -1 & 1 \\ -5 & 1 & 4 \end{pmatrix} \xrightarrow{R_2 \to R_1 + R_2} \begin{pmatrix} 5 & -1 & 1 \\ 0 & 0 & 5 \end{pmatrix} \Rightarrow 0 = 5$ is ...

$\therefore$ It has no Solution.

$\therefore$ S.S $= \varnothing$

76. Let $A = \begin{pmatrix} 0 & \alpha & \beta \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 8 \\ 0 \\ 0 \end{pmatrix}$, and $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

If det (A) = 4, then what is the solution set of the system AX = ...

Solution: Since det (A) = $4 \neq 0$, so that we can use cramer's ...

$$\therefore D_x = \begin{vmatrix} 8 & \alpha & \beta \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 8\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 0 + 0 = 24$$

$$\therefore D_y = \begin{vmatrix} 0 & 8 & \beta \\ 2 & 0 & 1 \\ 3 & 0 & 2 \end{vmatrix} = 0 - 8\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 0 = -8$$

$$\therefore D_z = \begin{vmatrix} 0 & \alpha & 8 \\ 2 & 2 & 0 \\ 3 & 1 & 0 \end{vmatrix} = 0 + 0 + 8\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} = -32$$

$$\therefore \{(x, y, z)^T\} = \left\{\left(\frac{D_x}{D}, \frac{D_y}{D}, \frac{D_z}{D}\right)\right\} = \left\{\left(\frac{24}{4}, \frac{-8}{4}, \frac{-32}{4}\right)^T\right\}$$

$\therefore \{(x, y, z)^T\} = \{(6, -2, -8)^T\}$

77. Given the system of equations: $\begin{cases} x - 3y + z = 0 \\ 4x + z = 3 \\ -3x + y - z = -2 \end{cases}$ its solution

Set is

A. $\{(1, 0, -1) + (1, 1, 4)t, t \in \mathbb{R}\}$ C. $\{(1, 0, -1)\}$

B. $\{(0, 1, 3) + (1, -1, -4)t, t \in \mathbb{R}\}$ D. $\varnothing$

Solution: Here

$$D=\begin{vmatrix}1 & -3 & 1\\ 4 & 0 & 1\\ -3 & 1 & -1\end{vmatrix}=\begin{vmatrix}0 & 1\\ 1 & -1\end{vmatrix}+3\begin{vmatrix}4 & 1\\ -3 & -1\end{vmatrix}+\begin{vmatrix}4 & 0\\ -3 & 1\end{vmatrix}$$

$$= -1 + 3(-4+3) + 4 = 0$$

$$D_x=\begin{vmatrix}0 & -3 & 1\\ 3 & 0 & 1\\ -2 & 1 & -1\end{vmatrix}=3\begin{vmatrix}3 & 1\\ -2 & -1\end{vmatrix}+1\begin{vmatrix}3 & 0\\ -2 & 1\end{vmatrix}=-3+3=0$$

Since $D = 0$, $D_x = 0$, $D_y = 0$, $D_z = 0 \Rightarrow$ system has infinite solution

: from second equation: $4x + z = 3 \Rightarrow z = 3 - 4x$

: From 1st equation: $x - 3y + z = 0 \Rightarrow 3y = x + z$.

: Put $z = 3 - 4x$ in $3y = x + z \Rightarrow 3y = x + 3 - 4x = -3x + 3$

$\Rightarrow y = -x + 1 \Leftrightarrow y = 1 - x$

$\therefore$ S.S $= \{(x, y, z)\} = \{(x, 1 - x, 3 - 4x)\}$, Let $x = 0$

$$= \{(0,1,3)+(t,-t,-4t)\}$$

$$= \{(0,1,3)+(1,-1,-4)t \quad t\in R\}$$

Answer: B

78. Let $A=\begin{pmatrix}-2 & 0 & 5\\ x & -1 & 1\\ 2x+1 & 2 & -x\end{pmatrix}$ if the minor of an element 0 is -16, then the possible value of x is

A. 7, -3 B. 1, 3 C. 3, – 5 D. -3, 5

Solution: 0 is located in 1st row and 2nd column of A, so that we write a_{12}. Thus, minor of $a_{12} = \begin{vmatrix}x & 1\\ 2x+1 & -x\end{vmatrix} = -16 \Leftrightarrow -x^2 - 2x - 1 = -16$

$\Rightarrow x^2 + 2x - 15 = 0 \Leftrightarrow (x - 3)(x + 5) = 0$

$\therefore x = 3$ or $x = -5$ $\therefore$ Answer C

79. From Question # 78, if the cofactor of an element 5 is 25, then the value of x is

A. -1 B. 0 C. 6 D. No x-value

Solution: 5 is located 1st row and 3rd column of matrix A, so that cofactor A_{13} is given by

$$A_{13}=(-1)^{1+3}M_{13}=\begin{vmatrix}x & -1\\ 2x+1 & 2\end{vmatrix}=25$$

$\Rightarrow 2x - (-2x - 1) = 25 \Leftrightarrow 4x + 1 = 25 \Rightarrow 4x = 24 \Rightarrow x = 6$

$\therefore$ Answer: C

80. Solve each of the following.

a) $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ -10 \\ 5 \end{pmatrix}$ b) $\begin{pmatrix} a-3 & a+1 \\ a+1 & a+3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

Solution: We can solve by using Cramer's rule.

a) $$D = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 3\begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} = 24 - 8 - 4 = 12$$

$$D_x = \begin{vmatrix} -4 & 2 & 1 \\ -10 & 4 & 0 \\ 5 & 0 & 2 \end{vmatrix} = -4\begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} - 2\begin{vmatrix} -10 & 0 \\ 5 & 2 \end{vmatrix} + 1\begin{vmatrix} -10 & 4 \\ 5 & 0 \end{vmatrix}$$

$$= -32 + 40 - 20 = -12$$

$D_y = -24, \quad D_z = 36$

$$\therefore x = \frac{D_x}{D} = \frac{-12}{12} = -1, \quad y = \frac{D_y}{D} = \frac{-24}{12} = -2,$$

$$z = \frac{D_z}{D} = \frac{36}{12} = 3$$

b) $$D = \begin{vmatrix} a-3 & a+1 \\ a+1 & a+3 \end{vmatrix} = a^2 - 9 - (a^2 + 2a + 1)$$

$$= -2a - 10, \ a \neq -5$$

$$D_x = \begin{vmatrix} 4 & a+1 \\ 2 & a+3 \end{vmatrix} = 4a + 12 - 2a - 2 = 2a + 10, \ a \neq -5$$

$$D_y = \begin{vmatrix} a-3 & 4 \\ a+1 & 2 \end{vmatrix} = 2a - 6 - 4a - 4 = -2a - 10$$

$$\therefore x = \frac{D_x}{D} = \frac{2a+10}{-2a-10} = \frac{2a+10}{-(2a+10)} = -1,$$

$$x=\frac{D_x}{D}=\frac{-2a\cdot 10}{-2a\cdot 10}=1$$

8) If $A=(a_{ij})_{3\times 3}$ given by $a_{ij}=\begin{cases}0 & \text{if } i>j\\ i+j & \text{if } i\le j\end{cases}$ then find a) $|A|$ b) A^{-1} c) $|A^{-1}|$

Solution

$$A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}=\begin{pmatrix}1+1 & 1+2 & 1+3\\ 0 & 2+2 & 2+3\\ 0 & 0 & 3+3\end{pmatrix}=\begin{pmatrix}2 & 3 & 4\\ 0 & 4 & 5\\ 0 & 0 & 6\end{pmatrix}$$

Since it is an upper triangular $|A|=2\times 4\times 6=48$

b) To find adj A first determine A' then cofactor of A' is adj

$$A'=\begin{pmatrix}2 & 0 & 0\\ 3 & 4 & 0\\ 4 & 5 & 6\end{pmatrix},\ \text{adj } A=\begin{pmatrix}\begin{vmatrix}4 & 0\\ 5 & 6\end{vmatrix} & -\begin{vmatrix}3 & 0\\ 4 & 6\end{vmatrix} & \begin{vmatrix}3 & 4\\ 4 & 5\end{vmatrix}\\ -\begin{vmatrix}0 & 0\\ 5 & 6\end{vmatrix} & \begin{vmatrix}2 & 0\\ 4 & 6\end{vmatrix} & -\begin{vmatrix}2 & 0\\ 4 & 5\end{vmatrix}\\ \begin{vmatrix}0 & 0\\ 4 & 0\end{vmatrix} & -\begin{vmatrix}2 & 0\\ 3 & 0\end{vmatrix} & \begin{vmatrix}2 & 0\\ 3 & 4\end{vmatrix}\end{pmatrix}$$

$$=\begin{pmatrix}24 & -18 & -1\\ 0 & 12 & -10\\ 0 & 0 & 8\end{pmatrix}$$

$$A^{-1}=\frac{\text{adj } A}{|A|}=\frac{1}{48}\begin{pmatrix}24 & -18 & -1\\ 0 & 12 & -10\\ 0 & 0 & 8\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & -\frac{3}{8} & -\frac{1}{48}\\ 0 & \frac{1}{4} & -\frac{5}{24}\\ 0 & 0 & \frac{1}{6}\end{pmatrix}$$

c) $|A^{-1}|=\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{6}=\frac{1}{48}$

$$y=\frac{D_y}{D}=\frac{-2a-10}{-2a-10}=1$$

Let $A=(a_{ij})_{3\times 3}$ given by $a_{ij}=\begin{cases}0 & if\ i>j\\ i+j & if\ i\leq j\end{cases}$ then find a) $|A|$

b) A^{-1} c) $|A^{-1}|$ d) $A.adjA$

Solution:

a) $\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}=\begin{pmatrix}1+1 & 1+2 & 1+3\\ 0 & 2+2 & 2+3\\ 0 & 0 & 3+3\end{pmatrix}=\begin{pmatrix}2 & 3 & 4\\ 0 & 4 & 5\\ 0 & 0 & 6\end{pmatrix}$

Since it is an upper triangular, $\therefore\ |A|=2\times 4\times 6=48$

b) To find adj.A, first determine, A^t then cofactor of (A^t) is adj.

$$A^t=\begin{pmatrix}2 & 0 & 0\\ 3 & 4 & 0\\ 4 & 5 & 6\end{pmatrix},\ adj.A=\begin{pmatrix}\begin{vmatrix}4 & 0\\ 5 & 6\end{vmatrix} & -\begin{vmatrix}3 & 0\\ 4 & 6\end{vmatrix} & \begin{vmatrix}3 & 4\\ 4 & 5\end{vmatrix}\\ -\begin{vmatrix}0 & 0\\ 5 & 6\end{vmatrix} & \begin{vmatrix}2 & 0\\ 4 & 6\end{vmatrix} & -\begin{vmatrix}2 & 0\\ 4 & 5\end{vmatrix}\\ \begin{vmatrix}0 & 0\\ 4 & 0\end{vmatrix} & -\begin{vmatrix}2 & 0\\ 3 & 0\end{vmatrix} & \begin{vmatrix}2 & 0\\ 3 & 4\end{vmatrix}\end{pmatrix}$$

$$=\begin{pmatrix}24 & -18 & -1\\ 0 & 12 & -10\\ 0 & 0 & 8\end{pmatrix}$$

$$\therefore\ A^{-1}=\frac{adj.A}{|A|}=\frac{1}{48}\begin{pmatrix}24 & -18 & -1\\ 0 & 12 & -10\\ 0 & 0 & 8\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & \frac{-3}{8} & \frac{-1}{48}\\ 0 & \frac{1}{4} & \frac{-5}{24}\\ 0 & 0 & \frac{1}{6}\end{pmatrix}$$

c) $|A^{-1}|=\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{6}=\frac{1}{48}$

Explanation: $|A^{-1}| = \frac{1}{|A|}$

d) $A.adj.A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} \cdot \begin{pmatrix} 24 & -18 & -1 \\ 0 & 12 & -10 \\ 0 & 0 & 8 \end{pmatrix} = \begin{pmatrix} 48 & 0 & \\ 0 & 48 & \\ 0 & 0 & \end{pmatrix}$

explanation: (A)(adj.A) = |A|I

82. Find A^{-1} for each case

a) $A = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}$ b) $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ c) $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$

Solution: a) $A^{-1} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$ b) $A^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

c) $A^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{pmatrix}$ for abc $\neq 0$

83. For what values of y is the matrix

$A = \begin{pmatrix} -1 & 1 & y \\ 0 & 6 & 1 \\ -y & 1 & 0 \end{pmatrix}$ non-singular: A $\frac{1}{12}$ B. $\frac{1}{2}, -\frac{1}{3}$

C. $\mathbb{R}$ D. None

Solution: A is non singular (invertible) if $|A| \neq 0$

$$\Rightarrow |A| = \begin{vmatrix} -1 & 1 & y \\ 0 & 6 & 1 \\ -y & 1 & 0 \end{vmatrix} = -1\begin{vmatrix} 6 & 1 \\ 1 & 0 \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ -y & 0 \end{vmatrix} + y\begin{vmatrix} 0 & 6 \\ -y & 1 \end{vmatrix} = 1 - y + 6y^2$$

For all real number y, $6y^2 - y + 1 \neq 0$, since $b^2 - 4ac < c$

$\Rightarrow 1 - 4(6)(1) = -23 < 0$ $\therefore$ Answer C

84. Let $A = \begin{pmatrix} -2 & 0 & x \\ 2y & x+y & -4 \end{pmatrix}$ $and\ B = \begin{pmatrix} 1 & -y \\ 0 & 3 \\ 1-x & 2 \end{pmatrix}$ such that $A + 2B^T = 0$, then the value of y? . . . . UEE 2004/12

A. 0 B. $\dfrac{-13}{2}$ C. -8 D. any real number

Solution:

$$B^T = \begin{pmatrix} 1 & 0 & 1-x \\ -y & 3 & 2 \end{pmatrix} \Rightarrow 2B^T = \begin{pmatrix} 2 & 0 & 2-2x \\ -2y & 6 & 4 \end{pmatrix}$$

$$\Rightarrow A+2B^T = \begin{pmatrix} -2 & 0 & x \\ 2y & x+y & -4 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 2-2x \\ -2y & 6 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow 2 - 2x + x = 0 \Rightarrow x = 2$ and $x+y+6 = 0 \Rightarrow 2 + y + 6 = 0$

$\Rightarrow y = -8$ **Answer: C**

85. Let A and B be 3×3 matrices such that

$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & -1 & \frac{1}{2} \end{pmatrix}$ and $|B| = \dfrac{1}{10}$, *what is* $|2AB^T|$?

A. 1 B. 4 C. 100 D. 400

Solution: $|2AB^T| = 2^3|A||B^T| = 8(2\times 5\times \frac{1}{2})\left(\frac{1}{10}\right) = 4$

86. The Solution set of system of equation $\begin{cases} x + y + 2z = 1 \\ x + 2y + z = 2 \\ -2x - 2y - 4z = -2 \end{cases}$

A. $\{(0, 1, 0)\}$ B. $(-\infty, \infty)$

C. $\{(-3k,k+1,k)\}$ D. $\{(3k,k-1,k)|k\in\mathbb{R}\}$

Solution: $D=\begin{vmatrix}1 & 1 & 2\\ 1 & 2 & 1\\ -2 & -2 & -4\end{vmatrix}=-2\begin{vmatrix}1 & 1 & 2\\ 1 & 2 & 1\\ 1 & 1 & 2\end{vmatrix}=0$

$D_x=\begin{vmatrix}1 & 1 & 2\\ 2 & 2 & 1\\ -2 & -2 & -4\end{vmatrix}=-2\begin{vmatrix}1.. & 1 & 2\\ 2 & .2 & 1\\ 1 & 1 & 2\end{vmatrix}=0,\ D_y=0,\ D_z=0$

$\Rightarrow$ The system has infinite many solution.

$$-\begin{cases}x+y+2z=1\\ x+2y+z=2\end{cases}$$

$$\overline{0+-y+z=-1\Rightarrow y=z+1}$$

Substitute $y=z+1$ in equation $x+y+2z=1$ and solve for x

$\Rightarrow x+z+1+2z=1\Rightarrow x=-3z$

$\Rightarrow$S.S $=\{(x,y,z)\}=\{(-3z,z+1,z)\}=\{(-3k,k+1,k)\}$ $K\in\mathbb{R}$ $\therefore$

Answer: C

87. If $\begin{pmatrix}\alpha & 2 & \beta\\ 2 & 1 & 3\\ -1 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}-5\\ 5\\ 0\end{pmatrix}$ and the determinant of coefficient matrix is -5, then the value of x is equal to:

A) 5 B) $\alpha+\beta$ C) -5α D) 3

Solution: We have $D=-5\neq 0$

$\Rightarrow D_x=\begin{vmatrix}-5 & 2 & \beta\\ 5 & 1 & 3\\ 0 & 0 & 1\end{vmatrix}=-5\begin{vmatrix}1 & 3\\ 0 & 1\end{vmatrix}-2\begin{vmatrix}5 & 3\\ 0 & 1\end{vmatrix}+\beta\begin{vmatrix}5 & 1\\ 0 & 0\end{vmatrix}=-15$

$\therefore x=\dfrac{D_x}{D}=\dfrac{-15}{-5}=3$

Answer: D

88. What is the solution set of the system $\begin{cases}x+y+2z=1\\ x+2y+z=1\\ 3x+4y+5z=3\end{cases}$

A. $\{(1,0,0)\}$ C. $\{(-3k+1,k,k|k\in\mathbb{R}\}$

B. $\{(k + 1, -k, k) | k \in \mathbb{R}\}$ D. ϕ

Explanation:

$$D = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 3 & 4 & 5 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 6 - 2 - 4 = 0$$

$$D_y = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 3 & 3 & 5 \end{vmatrix} = 0,\ D_z = 0 \text{ and } D_x = 0$$

Since, $D = 0, D_x = 0, D_y = 0, D_z = 0$
therefore, the system has infinite solution.

$$\begin{cases} x + y + 2z = 1 \ldots\ldots..(1) \\ x + 2y + z = 1 \ldots\ldots..(2) \\ 3x + 4y + 5z = 3 \ldots..(3) \end{cases}$$

Eliminate x from equation (1) and (2)
by subtracting the two equation.

$$-\begin{cases} x + y + 2z = 1 \\ x + 2y + z = 1 \end{cases}$$

$$0 - y + z = 0 \Rightarrow y = z$$

Substitute $y = z$ in equation (1): $x + y + 2z = 1 \Rightarrow x + z + 2z = 1$
$\Rightarrow x = -3z + 1$. Let $z = k \in \mathbb{R}$.
$\therefore S.S = \{(-3k + 1, k, k) | k \in \mathbb{R}\}$

Answer: C

89. If $A^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix}$. Find a matrix B

such that $AB = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}$.

Solution: Here at
We know that $A^{-1}A = i$ (identity matrix).
Multiply both side of a matrix by A^{-1}, we get

$$\Rightarrow B = A^{-1}(AB) = A^{-1}\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow B = \begin{bmatrix} 1+0+3 & -1+1+0 & 2+(-1)+0 \\ 2+0+5 & -2+0+0 & 4+0+0 \\ -1+0+0 & 1+1+0 & -2+1+0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 1 \\ 7 & -2 & 4 \\ -1 & 2 & -1 \end{bmatrix}$$

Review Exercise on unit six

1. **Conceptual exercise**

a) What is a matrix?

Explanation:

A matrix is a rectangular array of number.

b) What is the difference between row and column matrix?

Explanation:

A row runs horizontally and a column runs vertically.

c) What is the order of a matrix?

Explanation:

The number of rows and column.

d) What is the goal of row/column operation (Gauss - Jordan elimination).

Explanation:

The goal is to get 1 on the diagonal.

e) What is a determinant?

Explanation:

A real number associated with a square matrix.

f) Which systems can be solved using cramer's rule?

Explanation:

Cramer's rule works on systems that have exactly one solution.

g) What is a minor?

Explanation:

A minor for an element in a 3 × 3 matrix is the determinant of a 2 × 2 matrix.

Solve for x, y and z.

a) $\begin{bmatrix} 2x+3 & 8 \\ -5 & -1 \end{bmatrix} - \begin{bmatrix} 10 & 6 \\ 6y-1 & 7 \end{bmatrix} = \begin{bmatrix} -5 & 4z+8 \\ 8 & -8 \end{bmatrix}$

b) $\begin{bmatrix} 2x+3 \\ 5x-2 \\ y \end{bmatrix} + \begin{bmatrix} 3y \\ 2y+4 \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ -5 \end{bmatrix}$

c) $\begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \\ 0 \end{bmatrix}$

d) $\begin{bmatrix} 5x-6 & 7 \\ 2y+2 & z \end{bmatrix} - \begin{bmatrix} x & -2 \\ y & 3z+1 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 5y & 3z \end{bmatrix}$

3. Find x and y if $\begin{pmatrix} x & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 5 & -4 \\ -2 & 7 \end{pmatrix} = \begin{pmatrix} y & y \\ 14 & 5 \end{pmatrix}$

4. Solve for x and y.

a) $\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2x \\ 8x & 1 \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ -4 & 2y \\ -15 & 10 \end{bmatrix}$

b) $\begin{bmatrix} -2 & 1 & 2 \\ 3 & 2 & 4 \\ 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 6x & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 19 & 9 \\ 8 & y \end{bmatrix}$

5. Evaluate the determined of the matrix.

a) $\begin{pmatrix} -4 & 2 \\ -8 & 8 \end{pmatrix}$ b) $\begin{pmatrix} 7 & -4 \\ 3 & -1 \end{pmatrix}$ c) $\begin{pmatrix} 12 & 5 \\ 4 & 2 \end{pmatrix}$

6. Evaluate the determinant of the matrix.

a) $\begin{pmatrix} 5 & 0 & 1 \\ 1 & 4 & -1 \\ 3 & 2 & 0 \end{pmatrix}$ b) $\begin{pmatrix} -3 & 3 & -1 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 3 & 4 & 2 \\ -1 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}$

7. Find the inverse of the matrix.

a) $\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$ b) $\begin{pmatrix} -4 & 9 \\ 3 & -7 \end{pmatrix}$ c) $\begin{pmatrix} 9 & -2 \\ 5 & -1 \end{pmatrix}$

8. For what values of x and y the matrix $\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} x & -2 \\ y & 1 \end{pmatrix}$ are inverse of each other.

9. Let $A = \begin{bmatrix} 11 & 2 & 4m \\ 4 & 1 & -3 \\ -8 & -1 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 & 2 \\ 0 & 2 & 1 \\ 4 & n & 3 \end{bmatrix}$. For what values of m and n the two matrices, A and B, to be inverse of each other.

10. Let $A = \begin{bmatrix} 11 & 5 \\ 2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$.

What is the solution set of the system AX = B.

11. If $A = \begin{bmatrix} -7 & 5 \\ 3 & -2 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$, then $(AB)^{-1}$ is equal to ____________.

12. Solve the matrix equation:

a) $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$ b) $\begin{bmatrix} 6 & 4 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$

13. Let $A = \begin{pmatrix} 0 & \alpha & \beta \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 9 \\ 0 \\ 0 \end{pmatrix}$, and $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

If det (A) = 6, then what is the solution set of the system AX = b

14. If $A = \begin{pmatrix} 4 & -3 & 8 \\ 0 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix}$, then $\det(A^TA)$ is equal to __________

15. What elementary row operation does $-3R_1 + R_3$ denote? Which row can it replace? Explain

16. Use an augmented matrix to solve the linear system.

a) $\begin{cases} 2x + 4y + 5z = 5 \\ x + 3y + 3z = 2 \\ 2x + 4y + 4z = 2 \end{cases}$ b) $\begin{cases} 3x + 3y + z = 1 \\ 2x + 3y + z = -1 \\ 2x + 4y + z = -2 \end{cases}$

17. What is the solution set of the system?

a) $\begin{cases} 2x + y + 3z = -4 \\ 2x + 3y + z = -4 \\ 6x + 5y + 7z = -12 \end{cases}$ b) $\begin{cases} x + y - 2z = 5 \\ x + 2y + z = 8 \\ 2x + 3y - z = 13 \end{cases}$

Unit Seven

Vectors and Scalars

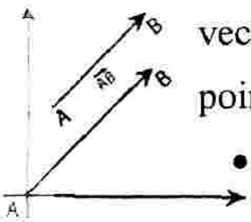
Quantities are categorized in to two.

i) Scalar **quantities** such as temperature area, volute, work, mass etc and have magnitude only.

ii) Vector **quantities**, such as velocity force, acceleration, magnetic field etc, have magnitude and direction.

Definition: A **vector** in the plane is a directed line segment. The directed line segment $\overrightarrow{AB}$ has **initial point A** and **terminal point B.**

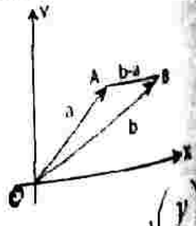
- The length (magnitude) of directed line segment is denoted by $\left|\overrightarrow{AB}\right|$.



- The component form by directed line segment of the vector $\overrightarrow{AB}$ with initial point $A = (x_1, y_1)$ and terminal point $B = (x_2, y_2)$ is $\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1)$
- The length of (magnitude) $\overrightarrow{AB}$ is

$$\left|\overrightarrow{AB}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- **Position vector** If the initial point of vector is fixed to origin (0,0) then it is called position vector, i.e. If the initial point is origin and the terminal point P is vector every point in plane is called **Position vector** $\overrightarrow{OP}$
- Any vector $\overrightarrow{AB}$ we can expressed interms of positioned vector
  - $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$



Note: $\overrightarrow{AB}$ = position vector of B relative to A

$\overrightarrow{OB}$ = position vector of B elative to O

- Direction of position vector $\overrightarrow{AB}$ is $\tan\theta = \dfrac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

Illustrative Example

1. Find the position vector $\overrightarrow{AB}$, the direction and the magnitude of $\overrightarrow{AB}$.

a) A = (1, 6), B = (-3, -2) c) A = (3,1); B = (5, 3)
b) A = (-2, 7), B = (1, -2) d) A = (2,1); B = (7, -4)

Solution: Let **O(0, 0) be origin**

a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (-3,-2) - (1,6) = (-4, -8) \leftarrow$ position vector

Direction of $\overrightarrow{AB}$ is $\tan\theta = \frac{y}{x} = \frac{-8}{-4} = 2 \Rightarrow \theta = \tan^{-1}(2)$

$$\left|\overrightarrow{AB}\right| = \sqrt{(-4)^2 + (-8)^2} = \sqrt{16+64} = \sqrt{80} = 4\sqrt{5} \text{ unit.}$$

b) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (1, -2) - (-2,7) = (3,-9) \leftarrow$ position vector

Direction of $\overrightarrow{AB}$ is $\tan\theta = \frac{y}{x} = \frac{-9}{3} = -3 \Rightarrow \theta = \tan^{-1}(-3)$

$\left|\overrightarrow{AB}\right| = \sqrt{(3)^2 + (-9)^2} = \sqrt{9+81} = \sqrt{90} = 3\sqrt{10} \leftarrow$ magnitude

c) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (5, 3) - (3, 1) = (2, 2) \leftarrow$ position vector

Direction of $\overrightarrow{AB}$ is $\tan\theta = \frac{y}{x} = \frac{2}{2} = 1 \Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$

$$\left|\overrightarrow{AB}\right| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2} \text{ unit.}$$

d) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (7, -4) - (2, 1) = (5, -5)$

Direction of $\overrightarrow{AB}$ is $\tan\theta = \frac{y}{x} = \frac{-5}{5} = -1 \Rightarrow \theta = \tan^{-1}(-1) = \frac{7\pi}{4}$

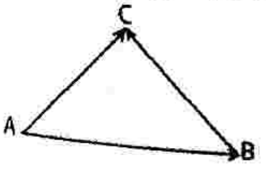
$\left|\overrightarrow{AB}\right| = \sqrt{5^2 + (-5)^2} = \sqrt{50} = 5\sqrt{2}$ unit.

Operation of Vector

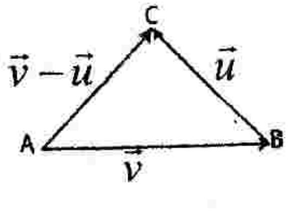
I. **Vector Addition**

Triangle method

a)



b)



- $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$
- $\overrightarrow{AB}-\overrightarrow{CB}=\overrightarrow{AC}$
- $\overrightarrow{BC}+\overrightarrow{CA}=\overrightarrow{BA}$
- $\overrightarrow{BC}-\overrightarrow{BA}=\overrightarrow{AC}$
- $-\overrightarrow{CB}+\overrightarrow{AB}=\overrightarrow{AC}$

Note: $\overrightarrow{AB}=-\overrightarrow{BA}$

2. ABCD is parallelogram, find a single vector of the following

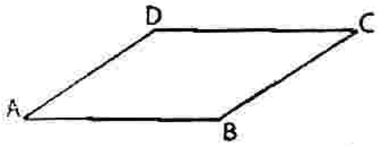
a) $\overrightarrow{AB}-\overrightarrow{CB}$ c) $\overrightarrow{DC}-\overrightarrow{DA}$

b) $\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{DC}$ d) $\overrightarrow{DB}+\overrightarrow{BA}+\overrightarrow{DA}$

Solution: a) From triangle law of addition $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

$\Rightarrow \overrightarrow{AB}-\overrightarrow{CB}=\overrightarrow{AC}$ → Explanation $\overrightarrow{BC}=-\overrightarrow{CB}$

b) $\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{DC}$



$\Rightarrow \overrightarrow{AC}-\overrightarrow{DC}\Rightarrow \overrightarrow{AC}+\overrightarrow{CD}=\overrightarrow{AD}$

$\therefore \overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{DC}=\overrightarrow{AD}$

c) $\overrightarrow{DC}-\overrightarrow{DA}\Rightarrow \overrightarrow{DC}+\overrightarrow{AD}=\overrightarrow{AD}+\overrightarrow{DC}=\overrightarrow{AC}$

$\therefore \overrightarrow{DC}-\overrightarrow{DA}=\overrightarrow{AC}=\overrightarrow{AD}-\overrightarrow{CD}$

d) $\overrightarrow{DB}+\overrightarrow{BA}+\overrightarrow{DA}\Rightarrow \overrightarrow{DA}+\overrightarrow{DA}=2\overrightarrow{DA}$

II. Multiplication of vectors by scalars

Definition: If $\vec{u}=(a, b)$, $\vec{v}=(c, d)$, k is scalar then

i) $k\vec{u}=k(a, b)=(ka, kb)$

ii) $k\vec{v}=k(c, d)=(kc, kd)$

iii) $|k\vec{u}|=|k||\vec{u}|=|k|\sqrt{a^2+b^2}$

iv) $|k\vec{v}|=|k||\vec{v}|=|k|\sqrt{c^2+d^2}$

4. Let $\vec{u}=(1, -2)$ and $\vec{v}=(-2, 4)$, then find

a) $2\vec{u}$ c) $\vec{u}+\vec{v}$ e) $3\vec{u}+\frac{1}{2}\vec{v}$

b) $\frac{-1}{2}\vec{v}$ d) $\left|\frac{-1}{2}\vec{v}\right|$ f) $\left|3\vec{u}+\frac{1}{2}\vec{v}\right|$

Solution:

a) $2\vec{u}=2(1, -2)=(2, -4)$

b) $\frac{-1}{2}\vec{v} = \frac{-1}{2}(-2, 4) = (1, -2)$

c) $\vec{u} + \vec{v} = (1, -2) + (-2, 4) = (-1, 2)$

d) $\left|\frac{-1}{2}\vec{v}\right| = \left|\frac{-1}{2}\right| |\vec{v}| = \frac{1}{2}\sqrt{(-2)^2 + (4)^2} = \frac{1}{2}\sqrt{4+16} = \sqrt{5}$

e) $3\vec{u} + \frac{1}{2}\vec{v} = 3(1,-2) + \frac{1}{2}(-2,4) = (3, -6) + (-1, 2) = (2, -4)$

f) $\left|3\vec{u} + \frac{1}{2}\vec{v}\right| = \sqrt{(2)^2 + (-4)^2} = \sqrt{4+16} = 2\sqrt{5}$

5. If the magnitude of $\vec{v}$ is 5 unit then find the magnitude of:

a) $4\vec{v}$ b) $-3\vec{v}$ c) $\frac{-1}{2}\vec{v}$ d) $\frac{2}{5}\vec{v}$

Solution: Given $|\vec{v}| = 5$ unit

a) $|4\vec{v}| = 4|\vec{v}| = 4(5) = 20$ unit

b) $|-3\vec{v}| = 3|\vec{v}| = 3(5) = 15$ unit

c) $\frac{-1}{2}\vec{v} = \frac{1}{2}|\vec{v}| = \frac{1}{2}(5) = \frac{5}{2} unit$

d) exercise

Parallel or collinear vector

If two vectors are collinear or parallel then any one of them can be express as scalar multiple of the other.

i.e. $\vec{u} = k\vec{v} \Rightarrow \vec{u} \parallel \vec{v}$

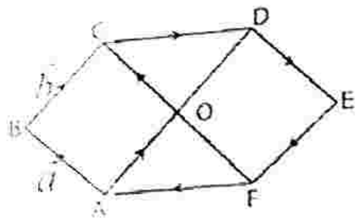
4. ABCDEF are vertices of regular hexagon. In which $\overrightarrow{AB} = \vec{a}$=(1,2) and $\overrightarrow{BC} = \vec{b}$ = (-2, 3). Express the other vector interms of the vector $\vec{a}$ *and* $\vec{b}$. (other vector are $\overrightarrow{CD}$, $\overrightarrow{DE}$, $\overrightarrow{EF}$ and $\overrightarrow{AF}$)

Solution: • Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$

• since $\overrightarrow{AB} \parallel \overrightarrow{OC} \Rightarrow \overrightarrow{AB} = \vec{a}$ and $\overrightarrow{OC} = \vec{a}$

• $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \vec{b} - \vec{a}$

$CD = (-2, 3) - (1, 2) = (-3, 1)$



- $\overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD}$ and $\overrightarrow{OE} \parallel \overrightarrow{CD}$

$= \vec{b} - \vec{a} - (\vec{b}) = -\vec{a}$

$\Rightarrow \overrightarrow{DE} = -(\vec{a}) = (1, 2) = (-1, -2).$

- $\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = -\vec{a} - (\vec{b} - \vec{a}) = -\vec{b}$

$\Rightarrow \overrightarrow{EF} = -\vec{b} = (-2, 3) = (2, -3).$

- $\overrightarrow{FA} = \overrightarrow{OA} - \overrightarrow{OF} = -\vec{b} - (-\vec{a}) = \vec{a} - \vec{b}$

$\Rightarrow \overrightarrow{FA} = \vec{a} - \vec{b} = (1, 2) - (-1, 3) = (3, -1)$

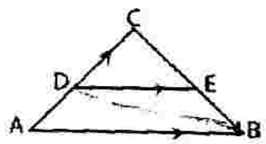
$\therefore \overrightarrow{CD} = \vec{b} - \vec{a}$, $\overrightarrow{DE} = -\vec{a}$, $\overrightarrow{EF} = -\vec{b}$ and $\overrightarrow{FA} = \vec{a} - \vec{b}$

Mid – point theorem

5. Using vectors show that, the segment joining the mid-points of two side of a triangle is parallel to the third side and equal to half of it.

Solution:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$



$$\overrightarrow{DB} + \overrightarrow{BE} = \overrightarrow{DE}$$

$$\overrightarrow{DE} = \frac{1}{2}AB,\ \overrightarrow{BE} = \frac{1}{2}BC$$

$$\Rightarrow \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{BC}\right) = \overrightarrow{DE} \Leftrightarrow \overrightarrow{DE} = \frac{1}{2}\overrightarrow{AC}$$

6. If ABCD is parallelogram in Cartesian coordinate planes with $A = (1, 1)$, $B = (7, 5)$, and $C = (10, 10)$. What are the coordinates of $D(x, y)$

A. (4, 6) B. (9, 9) C. (6, 3) D. (4, 5)

Solution: Since $\overrightarrow{AB} \parallel \overrightarrow{BC}$ and $\overrightarrow{AD} \parallel \overrightarrow{BC}$

$\Rightarrow AD = (x, y) - (1, 1) = (x - 1, y - 1)$

$\Rightarrow \overrightarrow{BC} = (10, 10) - (7, 5) = (3, 5)$

$\Rightarrow (x - 1, y - 1) = (3, 5) \Rightarrow x - 1 = 3$ and $y - 1 = 5$

$\therefore x = 3 + 1 = 4$ and $y = 5 + 1 = 6$

$\therefore D(x, y) = (4, 6)$

Unit vector

- A vector whose magnitude is one is called a unit vector
- The standard unit vectors are i = (1, 0) and j = (0, 1)
- Any vector can express as linear combination of unit vector i and j

  Let $\vec{v}$ = (a, b) expressed as

  $\vec{v} = (a, b) = a(1,0) + b(0, 1) = ai + bj$

  $\vec{v} = (2, 3) = 2(1,0) + 3(0, 1) = 2i + 3j$

Which of the following is unit vector

a) $\vec{v} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ b) $\vec{v} = \left(\frac{3}{5}, \frac{-4}{5}\right)$ c) (-1, 1)

Solution:

a) $|v| = \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = \sqrt{\frac{4}{5} + \frac{1}{5}} = \frac{5}{5} = 1 \leftarrow$ unit vector

$\Rightarrow v = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ is unit vector

b) $\vec{v} = \left(\frac{3}{5}, \frac{-4}{5}\right) \Rightarrow |\vec{v}| = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1 \leftarrow$unit vector

c) $\vec{v}\,(-1,1) \Rightarrow |\vec{v}| = \sqrt{(-1)^2 + (1)^2} = \sqrt{01+1} = \sqrt{2} \leftarrow$ not unit vector

Unit vector in the direction of $\vec{v}$

- If $\vec{v}$ is a non-zero vector in the plane, then the vector $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$ has length 1 and the same direction of $\vec{v}$

8. Find unit vector:

i) in the direction of the vector $\vec{v}$

ii) in the direction opposite of $\vec{v}$ if given that

a) $\vec{v} = (-3, 4)$ b) $\vec{v} = (2, -4)$c) $\vec{v} = (x, y)$

Solution: i) Unit vector in the direction of $\vec{v}$ given by

a) i) $\frac{\vec{v}}{|\vec{v}|} = \frac{(-3,4)}{\sqrt{(-3)^2 + 4^2}} = \frac{(-3,4)}{\sqrt{25}} = \frac{1}{5}(-3,4) = \left(\frac{-3}{5}, \frac{4}{5}\right)$

ii) Unit vector in the direction opposite to v is

$$\frac{-v}{|v|} \Rightarrow \left(\frac{3}{5}, \frac{-4}{5}\right)$$

b) i) $\frac{v}{|v|} = \frac{1}{\sqrt{2^2+(-4)^2}}(2,-4) = \frac{1}{2\sqrt{5}}(2,-4) = \left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$

unit vector in the same direction

ii) Unit vector in the direction opposite of $\vec{v} = (2, -4)$ is

$$\frac{-v}{|v|} = \left(\frac{-2}{2\sqrt{5}}, \frac{4}{2\sqrt{5}}\right) = \left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

c) i) $\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$ ii) $\left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}\right)$

9. If $\vec{v} = 2i + 3j$ and $\vec{u} = 3i - j$ then

Find a) $\vec{v} + \vec{u}$ b) $|\vec{v} + \vec{u}|$ c) $|2\vec{v} - 3\vec{u}|$

d) Unit vector in the direction of $\vec{u} + \vec{v}$

e) Unit vector in the direction opposite to vector $\vec{u} + \vec{v}$

Solution:

a) $\vec{v} + \vec{u} = 2i + 3j + 3i - j = (2+3)i + (3-1)j = 5i+2j$

b) $|\vec{v} + \vec{u}| = |5i + 2j| = \sqrt{5^2 + 2^2} = \sqrt{29}$

c) $|2\vec{v} - 3\vec{u}| = |-5i + 9j| = \sqrt{(-5)^2 + (9)^2} = \sqrt{106}$

d) Unit vector in the direction of $\vec{v} + \vec{u} = \frac{\vec{v}+\vec{u}}{|\vec{v}+\vec{u}|} = \frac{5i}{\sqrt{29}} + \frac{2j}{\sqrt{29}}$

e) Unit vector in the direction opposite to vector

$$\vec{v} + \vec{u} = \frac{-(\vec{v}+\vec{u})}{|\vec{v}+\vec{u}|} = \frac{-1}{\sqrt{29}}(5i+2j)$$

10. If $\vec{a} = 4i + j$, $\vec{b} = 3i - 2j$ and $\vec{c} = -i - 2j$

- Find the unit vector parallel to $2\vec{a} - \vec{b} - \vec{c}$ but in opposite direction

Solution: $2\vec{a} - \vec{b} - \vec{c} = 2(4i + j) - (3i - 2j) - (-i - 2j) = 6i + 6j$

Unit vector in the direction opposite of $2\vec{a} - \vec{b} - \vec{c}$

$$\Rightarrow \frac{-(2\vec{a}-\vec{b}-\vec{c})}{|2\vec{a}-\vec{b}-\vec{c}|} = \frac{-1}{|6i+6j|}(6i+6j)$$

$$= \frac{-6i}{6\sqrt{2}} + \frac{-6j}{6\sqrt{2}} = \frac{-i}{\sqrt{2}} - \frac{j}{\sqrt{2}}$$

11. If A = (1, 2) and B = (3, 5) be two points, find the unit vector along a) $\overrightarrow{AB}$ b) $\overrightarrow{BA}$

Solution:

$\overrightarrow{AB} = (3, 5) - (1, 2) = (2, 3) \longleftarrow$ position vector of $\overrightarrow{AB}$

$\Rightarrow$ Unit vector along $\overrightarrow{AB}$ is $\dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \dfrac{(2,3)}{\sqrt{4+9}} = \left(\dfrac{2}{\sqrt{13}}, \dfrac{3}{\sqrt{13}}\right)$

b) $\overrightarrow{BA} = (1,2) - (3,5) = (-2, -3)$

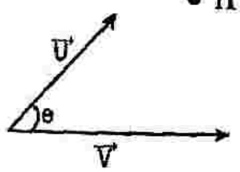
$\Rightarrow$ Unit vector along $\overrightarrow{BA} = \dfrac{\overrightarrow{BA}}{|\overrightarrow{BA}|} = \left(\dfrac{-2}{\sqrt{13}}, \dfrac{-3}{\sqrt{13}}\right)$

Scalar (dot or Inner) product of vector

Definition: If $\vec{u} = (a_1, a_2)$, $\vec{v} = (b_1, b_2)$ and θ is the angle between $\vec{u}$ and $\vec{v}$ then the dot product of $\vec{u}$ and $\vec{v}$ is denoted by $\vec{u} \cdot \vec{v}$ is defined by

i) $\vec{u} \cdot \vec{v} = (a_1, a_2)\cdot(b_1, b_2) = a_1b_1 + a_2b_2$

ii) $\vec{u} \cdot \vec{v} = |\vec{u}|.|\vec{v}| \cos\theta$



- if θ is acute, then $\vec{u} \cdot \vec{v} > 0$
- if θ is obtuse, then $\vec{u} \cdot \vec{v} < 0$
- if θ is $\dfrac{\pi}{2}$, then $\vec{u} \cdot \vec{v} = 0$ (($\vec{u} \perp \vec{v}$)
- if $\theta = 0$, then $\vec{u} \parallel \vec{v}$
- if $\theta = \pi$, then $\vec{u} \parallel \vec{v}$ but opposite in direction

Properties of dot product

- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $k(\vec{u} \cdot \vec{v}) = k\vec{u} \cdot \vec{v} = \vec{u} \cdot k\vec{v}$
- $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w})$

- $|\vec{u}|^2 = \vec{u}\cdot\vec{u} = |\vec{u}||\vec{u}|$
- $|\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = |u|^2 + 2\vec{u}\cdot\vec{v} + |\vec{v}|^2$
- $|\vec{u}-\vec{v}|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = |u|^2 - 2\vec{u}\cdot\vec{v} + |\vec{v}|^2$

$\Rightarrow |\vec{u}+\vec{v}| = \sqrt{(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})} = \sqrt{|\vec{u}|^2 + 2\vec{u}\cdot\vec{v} + |\vec{v}|^2}$

$\Rightarrow |\vec{u}-\vec{v}| = \sqrt{(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})} = \sqrt{|\vec{u}|^2 - 2\vec{u}\cdot\vec{v} + |\vec{v}|^2}$

12. Find the dot product of the vector $\vec{u}$ and $\vec{v}$ and state, angle between acute or obtuse

a) $\vec{u} = (-2, 5)$ and $\vec{v} = (3, 4)$ c) $\vec{v} = 4i - j$ and $\vec{u} = (-3i + 2j)$

b) $\vec{u} = (4, -3)$ and $\vec{v} = (-2, 3)$ d) $\vec{v} = 8i - 3j$ and $\vec{u} = 2i + 7j$

Solution:

a) $\vec{u}\cdot\vec{v} = (-2,5)\cdot(3,4) = -6+20 = 14 \leftarrow$ Angle b/n $\vec{u}$ and $\vec{v}$ is acute

b) $\vec{u}\cdot\vec{v} = (4,-3)\cdot(-2,3) = -8-9 = -17 \leftarrow$ Angle b/n $\vec{u}$ and $\vec{v}$ is obtuse

c) $\vec{v}\cdot\vec{v} = (4i - j)\cdot(2i+7j) = 8 - 7 = 1 \longleftarrow$ Acute angle

d) $\vec{v}\cdot\vec{u} = (8i - 3j)\cdot(2i+7j) = 16 - 21 = -5 \longleftarrow$ Obtuse angle

13. Determine the value of k so that the angle between the vectors $\vec{u} = (3, -2)$ and $\vec{v} = (4, 3k)$ is

a) acute b) obtuse c) right angle (or thogional)

Solution:

a) Acute $\Rightarrow \vec{u}\cdot\vec{v} > 0, \Rightarrow (3, -2)\cdot(4, 3k) > 0$

$\Rightarrow 12 - 6k > 0 \Rightarrow 12 > 6k \Rightarrow k < 2$

b) obtuse $\Rightarrow \vec{u}\cdot\vec{v} < 0 \Rightarrow (3, -2)\cdot(4, 3k) < 0 \Rightarrow 12 - 6k < 0$

$\Rightarrow 12 < 6k \Rightarrow k > 2$

c) Right angle) $\Rightarrow \vec{u}\cdot\vec{v} = 0 \Rightarrow 12 - 6k = 0 \Rightarrow k = 2 \leftarrow \vec{u} \perp \vec{v}$

14. If $\vec{v} = (3, -1)$ and $\vec{u} = (1, 3)$. Find $(\vec{v}\cdot\vec{u})\vec{u}$

Solution:

$(\vec{v}\cdot\vec{u})\vec{u} \Rightarrow ((3, -1)\cdot(1, 3))(1, 3) = (3 + (-3))(1, 3) = (0, 0)$

15. Let $\vec{u} = (2, -2)$, $\vec{v} = (5, 8)$ and $\vec{w} = (-4, 3)$

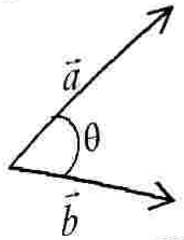
Find:

a) $\vec{u}\cdot\vec{v} = (2, -2)\cdot(5, 8) = 10 - 16 = -6$

b) $(\vec{u}\cdot\vec{v})\vec{w} = -6(-4, 3) = (24, -18)$.

c) $|\vec{w}|^2 = \vec{w}\cdot\vec{w} = (-4, 3)\cdot(-4,3) = 16 + 9 = 25$

Angle between two vector



$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \Rightarrow \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|a||b|}\right)$$

$$0 \le \theta \le \pi$$

16. Find the consin of the angle between $\vec{v}$ and $\vec{u}$ if
   a) $\vec{v} = (-1, -1)$ and $\vec{u} = (3, 0)$ c) $\vec{v} = (-2, 1)$ and $\vec{u} = (0, 3)$
   b) $\vec{v} = (1, 2)$ and $\vec{u} = (-1, -2)$d) $\vec{v} = (4, 3)$ and $\vec{u} = (1, 7)$

Solution: Let θ be the angle between $\vec{v}$ and $\vec{u}$

$$\vec{v}\cdot\vec{u} = |v||\vec{u}|\cos\theta \Rightarrow \cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$$

a) $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} \Rightarrow \dfrac{(-1,-1)\cdot(3,0)}{\sqrt{1^2+1^2}\sqrt{3^2+0^2}} = \dfrac{-3}{3\sqrt{2}} = \dfrac{-1}{\sqrt{2}} \Rightarrow \theta = \dfrac{3\pi}{4}$

b) $\cos\theta = \dfrac{(1,2)\cdot(-1,-2)}{(\sqrt{1^2+2^2})(\sqrt{1^2+2^2}} = \dfrac{-1-4}{\sqrt{5}\cdot\sqrt{5}} = \dfrac{-5}{5} = -1 \Rightarrow \theta = \cos^{-1}(-1) = \pi$

c) $\cos\theta = \dfrac{(-2,1)\cdot(0,3)}{\sqrt{4+1}\sqrt{3^2}} = \dfrac{3}{3\sqrt{5}} = \dfrac{1}{\sqrt{5}} \Rightarrow \theta = \cos^{-1}\left(\dfrac{1}{\sqrt{5}}\right)$

d) $\cos\theta = \dfrac{(4,3)\cdot(1,7)}{(\sqrt{16+9})\sqrt{1+49}} = \dfrac{4+21}{5\sqrt{50}} = \dfrac{25}{25\sqrt{2}} = \dfrac{1}{\sqrt{2}} \Rightarrow \theta = \cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{\pi}{4}$

17. Let $\vec{v} = (1,7)$ and $\vec{u} = (4, 3)$. Find
   a) the angle between $\vec{v}$ and $\vec{u} - \vec{v}$
   b) the angle between $\vec{u}$ and $2\vec{u} - \vec{v}$

Solution: $\vec{v}\cdot(\vec{u} - \vec{v}) = |\vec{v}||\vec{u} - \vec{v}|\cos\theta$

Let θ be the angle between $\vec{v}$ and $\vec{u} - \vec{v}$

a) $\cos\theta = \dfrac{\vec{v}\cdot(\vec{u}-\vec{v})}{|\vec{v}||\vec{u}-\vec{v}|} \Rightarrow$

$$\frac{(1,7)\cdot(3,-4)}{(\sqrt{1^2+7^2})(\sqrt{3^2+4^2}} = \frac{3-28}{\sqrt{50}\sqrt{25}} = \frac{-25}{25\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$$

b) $\cos\theta = \dfrac{\vec{u}\cdot(2\vec{u}-\vec{v})}{|\vec{u}||2\vec{u}-\vec{v}|} \Rightarrow$

$$\frac{(4,3)\cdot(7,-1)}{\sqrt{4^2+3^2}\sqrt{7^2+1^2}} = \frac{28-3}{\sqrt{25}\sqrt{50}} = \frac{25}{25\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

Theorem Two vector $\vec{a}$ *and* $\vec{b}$ are

i) Orthogonal if and only if $\vec{a}\cdot\vec{b} = 0$

ii) Parallel if and only if $\dfrac{\vec{a}\cdot\vec{b}}{|a||b|} = \pm 1$

18. Determine **m** such that the two vectors are orthogonal.

a) $\vec{v} = 3i - 2j$, $\vec{u} = 4i + 5mj$ c) $\vec{v} = 3i + 4j$, $\vec{u} = mi + 2j$
b) $\vec{v} = 5mi + 3j$, $\vec{u} = 2i + 7j$ d) $\vec{v} = mi - 3j$, $\vec{u} = mi + 2mj$

Solution: a) $\vec{v}\cdot\vec{u} = 0 \Rightarrow (3i - 2j)\cdot(4i + 5mj) = 0$

$$\Rightarrow 12 - 10m = 0 \Rightarrow m = \frac{12}{10} = \frac{6}{5}$$

b) $\vec{v}\cdot\vec{u} = 0 \Rightarrow (5mi + 3j)\cdot(2i + 7j) = 0 \Rightarrow 10m + 21 = 0$

$\Rightarrow -10m = 21 \Rightarrow m = -2\cdot 1$

c) $\vec{v}\cdot\vec{u} = 0 \Rightarrow (3i + 4j)\cdot(mi + 2j) = 3m + 8 = 0 \Rightarrow m = \dfrac{-8}{3}$

d) $\vec{v}\cdot\vec{u} = 0 \Rightarrow (mi - 3j)\cdot(mi + 2mj) = 0 \Rightarrow m^2 - 6m = 0$

$\Rightarrow m(m - 6) = 0$

$\Rightarrow m = 0$ or $m = 6$

19. If $\vec{a} = i + 2j$, $\vec{b} = -i + 2j$ and $\vec{c} = 3i + j$. Find **t** such that $\vec{a} + t\vec{b}$ is perpendicular to C.

Solution: $(\vec{a} + t\vec{b})\cdot\vec{c} = 0 \Rightarrow (i + 2j + -ti + 2tj)\cdot(3i + j) = 0$

$\Rightarrow [(1 - t)i + (2 + 2t)j]\cdot(3i + j) = 0 \Rightarrow 3 - 3t + 2 + 2t = 0$

$\Rightarrow 5-t=0 \Rightarrow t=5$

20. Let $\vec{a}=2i-4j$ an $\vec{b}=\left(-\frac{1}{2}i+j\right)$ then show that $\vec{a}$ is parallel to $\vec{b}$

Solution:

$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|a||b|}=\frac{(2i-4j)\cdot\left(\frac{-1}{2}i+j\right)}{\sqrt{4+16}\sqrt{\frac{1}{4}+1}}=\frac{-1-4}{2\sqrt{5}\cdot\frac{\sqrt{5}}{2}}=-1$$

$$\Rightarrow \theta=\cos^{-1}(-1)=\pi \Rightarrow \vec{a}\parallel\vec{b}$$

21. Vector $\vec{a}$ and $\vec{b}$ makes an angle of $\theta=\frac{2}{3}\pi$.

If $|\vec{a}|=3$ and $|\vec{b}|=4$, then find

a) $\vec{a}\cdot\vec{b}$ b) $(\vec{a}-\vec{b})\cdot(2\vec{a}+\vec{b})$ c) $|\vec{a}+\vec{b}|$

Solution:

a) $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta \Rightarrow (3)(4)\cos\frac{2\pi}{3}=12\left(\frac{-1}{2}\right)=-6$

b) $(\vec{a}-\vec{b})\cdot(2\vec{a}+\vec{b})=2\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}-\vec{b}\cdot 2\vec{a}-\vec{b}\cdot\vec{b}$

$=2|a|^2-\vec{a}\cdot\vec{b}-|b|^2$

$=2(3)^2-(-6)-(4)^2=8$

c) $|\vec{a}+\vec{b}|=\sqrt{(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})}=\sqrt{|a|^2+2\vec{a}\cdot\vec{b}+|b|^2}$

$=\sqrt{9-12+16}=\sqrt{13}$

22. Vectors $\vec{a}$ and $\vec{b}$ makes an angle $\theta=\frac{\pi}{4}$,

if $|\vec{a}|=\sqrt{2}$ and $|\vec{b}|=1$ then find

a) $|\vec{a}-\vec{b}|$ b) $|\vec{a}+2\vec{b}|$

c) angle between $\vec{a}-\vec{b}$ and $\vec{a}+2\vec{b}$

Solution: a) $|\vec{a}-\vec{b}|=\sqrt{(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})}$

$$=\sqrt{|\vec{a}|^2-2|\vec{a}||\vec{b}|\cos\theta+|\vec{b}|^2}$$

$$\therefore |\vec{a}-\vec{b}| = \sqrt{\sqrt{2}^2+2(\sqrt{2})(1)\frac{\sqrt{2}}{2}+1} = \sqrt{2-2+1} = 1$$

b) $|\vec{a}+\vec{b}| = \sqrt{(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})} = \sqrt{|a|^2+2|a||b|\cos\frac{\pi}{4}+|b|^2}$

$$\therefore |\vec{a}+\vec{b}| = \sqrt{\sqrt{2}^2+2(\sqrt{2})(1)\frac{\sqrt{2}}{2}+1} = \sqrt{2+2+1} = \sqrt{5}$$

c) Let θ be angle between $\vec{a}-\vec{b}$ and $\vec{a}+2\vec{b}$ is given by

$$\cos\theta = \frac{(\vec{a}-\vec{b})\cdot(\vec{a}+2\vec{b})}{|\vec{a}-\vec{b}||\vec{a}+2\vec{b}|} \Rightarrow \frac{|a|^2+|a||b|\cos\theta-2|b|^2}{|\vec{a}-\vec{b}||\vec{a}+2\vec{b}|}$$

$$\Rightarrow \cos\theta = \frac{\sqrt{2}^2+(\sqrt{2})(1)-2(1)^2}{(1)(\sqrt{5})} = \frac{2+1-2}{\sqrt{5}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

23. Let $|\vec{a}| = 7$, $|\vec{b}| = 3$, and $|\vec{a}+\vec{b}| = 5$

Find a) $\vec{a}\cdot\vec{b}$ b) $|\vec{a}-\vec{b}|$ c) $|2\vec{a}+3\vec{b}|$

Solution: To find $\vec{a}\cdot\vec{b}$ first interpret $|\vec{a}+\vec{b}|$

$$\Rightarrow |\vec{a}+\vec{b}| = \sqrt{(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})} = \sqrt{|a|^2+2\vec{a}\cdot\vec{b}+|b|^2}$$

$$\Rightarrow 5 = \sqrt{7^2+2\vec{a}\cdot\vec{b}+3^2} = \sqrt{49+9+2a\cdot b}$$

$$\Rightarrow 5^2 = 49+9+2\vec{a}\cdot\vec{b} \Rightarrow 2\vec{a}\cdot\vec{b} = 25-58 = -33$$

$$\Rightarrow \vec{a}\cdot\vec{b} = \frac{-33}{2}$$

b) $|\vec{a}-\vec{b}| = \sqrt{(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})} = \sqrt{|a|^2-2\vec{a}\cdot\vec{b}+|b|^2}$

$$\Rightarrow |\vec{a}-\vec{b}| = \sqrt{49-(2)\left(\frac{-33}{2}\right)+9} = \sqrt{91}$$

c) $|2\vec{a}+3\vec{b}| = \sqrt{4|\vec{a}|^2+12\vec{a}\cdot\vec{b})+9|b|^2}$

$$= \sqrt{4(49) + 12\left(\frac{-33}{2}\right) + 81}$$

$$\Rightarrow |2\vec{a} + 3\vec{b}| = \sqrt{196 - 198 + 81} = \sqrt{79}$$

24. Let $|\vec{a}| = 13$, $|\vec{b}| = 19$ and $|\vec{a} + \vec{b}| = 24$, then find

a) $\vec{a} \cdot \vec{b}$ b) Angle between $\vec{a}$ and $\vec{b}$

c) $|a - b|$ d) $|3\vec{a} - 4\vec{b}|$ e) $|3\vec{a} + 4\vec{b}|$

Solution: From $|\vec{a} + \vec{b}|$ we have

$$|\vec{a} + \vec{b}| = \sqrt{|\vec{a}|^2 + 2a \cdot b + b^2} = \sqrt{169 + 2a \cdot b + 361}$$

$$\Rightarrow (24) = \sqrt{2\vec{a} \cdot \vec{b} + 530} \Rightarrow 576 = 2\vec{a} \cdot \vec{b} + 530$$

$$\Rightarrow 2\vec{a} \cdot \vec{b} = 576 - 530 = 46 \Rightarrow \vec{a} \cdot \vec{b} = 23$$

b) Let θ be angle between $\vec{a}$ and $\vec{b}$, then

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{23}{|a||b|} = \frac{23}{(13)(19)} = \frac{23}{247} \Rightarrow \theta = \cos^{-1}\left(\frac{23}{247}\right)$$

c) $|\vec{a} - \vec{b}| = \sqrt{|a|^2 - 2\vec{a} \cdot \vec{b} + |b|^2}$

$$= \sqrt{169 - 2(23) + 361} = \sqrt{484} = 22$$

d) $|3\vec{a} - 4\vec{b}| = \sqrt{9|a|^2 - 24\vec{a} \cdot \vec{b} + 16|b|^2}$

$$= \sqrt{1521 - 552 + 5776} = \sqrt{6745}$$

e) $|3\vec{a} + 4\vec{b}| = \sqrt{9|a|^2 + 24\vec{a} \cdot \vec{b} + 16|b|^2}$

$$= \sqrt{1521 + 552 + 5776} = \sqrt{7849} \approx 88.6$$

23. Let $|\vec{a}| = 1$, $|\vec{b}| = 1$ and angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$, then find

a) $|\vec{a} + \vec{b}|$ b) angle between $\vec{a}$ and $\vec{a} + \vec{b}$

Solution: a) $|\vec{a} + \vec{b}| = \sqrt{|a|^2 + 2|\vec{a}||\vec{b}|\cos\frac{\pi}{3} + |b|^2}$

$$= \sqrt{1+2(1)(1)\frac{1}{2}+1} = \sqrt{3}$$

b) Let β be angle between $\vec{a}$ and $\vec{a}+\vec{b}$ is given by

$$\cos\beta = \frac{\vec{a}\cdot(\vec{a}+\vec{b})}{|a||\vec{a}+\vec{b}|} = \frac{|a|^2+|a||b|\cos\frac{\pi}{3}}{(1)(\sqrt{3})} = \frac{1+\frac{1}{2}}{\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \beta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

26. The cosine of the angle between the vector $\vec{u}$ = 2i + bj and $\vec{v}$ = bi + j is $\frac{3}{\sqrt{10}}$ find b

Solution:

$$\Rightarrow \frac{\vec{u}\cdot\vec{v}}{|u||v|} = \frac{(2i+bj)\cdot(bi+j)}{\sqrt{4+b^2}\sqrt{b^2+1}} = \frac{2b+b}{\sqrt{b^4+5b^2+4}}$$

$$\Rightarrow \frac{3}{\sqrt{10}} = \frac{3b}{\sqrt{b^4+5b^2+4}} \Rightarrow \frac{9}{10} = \frac{9b^2}{b^4+5b^2+4}$$

$\Rightarrow 10b^2 = b^4 + 5b^2 + 4$

$\Rightarrow b^4 - 5b^2 + 4 = 0 \Rightarrow (b^2-4)(b^2-1) = 0 \Rightarrow b = \pm 2, \pm 1$

27. If $\vec{u}$ and $\vec{v}$ are unit vector and angle between them is 30^0. Find

a) |3u – v| b) $(\vec{u}+\vec{v})\cdot(2\vec{u}-\vec{v})$

Solution: since $\vec{u}$ and $\vec{v}$ are unit vector ⇒ |u| = 1, |v| = 1

a) $|3\vec{u}-\vec{v}| = \sqrt{9|\vec{u}|^2 - 6|\vec{u}||v|\cos 30^0 + |v|^2} = \sqrt{9-\frac{6\sqrt{3}}{2}}$

$\Rightarrow |3\vec{u}-\vec{v}| = \sqrt{10-3\sqrt{3}}$

b) $(\vec{u}+\vec{v})\cdot(2\vec{u}-\vec{v}) = 2|u|^2 + |u||v|\cos 30^0 - |v|^2$

$$= (2)(1) + (1)(1)\frac{\sqrt{3}}{2} - 1 = 1 + \frac{\sqrt{3}}{2}$$

28. If $\vec{a}+\vec{b}+\vec{c} = 0$, and $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$

Solution: $\vec{a} + \vec{b} + \vec{c} = 0 \Rightarrow \vec{a} + \vec{b} = -\vec{c} \Rightarrow |\vec{a} + \vec{b}|^2 = |-\vec{c}|^2$

$\Rightarrow |a|^2 + 2|a||b| \cos\theta + |b|^2 = |-\vec{c}|^2 \leftarrow \theta$ is angle between $\vec{a}$ and $\vec{b}$

$\Rightarrow (3)^2 + 2(3)(5)\cos\theta + 5^2 = 7^2 \Rightarrow \cos\theta = \dfrac{49-9-25}{30} = \dfrac{1}{2}$

$\Rightarrow \theta = \cos^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3}$

29. If $|2\vec{a} + \vec{b}| = |2\vec{a} - \vec{b}|$. Then find angle between $\vec{a}$ and $\vec{b}$.

Solution: $\Rightarrow |2a + b|^2 = 4|a|^2 + 4|\vec{a}||\vec{b}|\cos\theta + |\vec{b}|^2$

$\Rightarrow |2a - b|^2 = 4|\vec{a}|^2 - 4|\vec{a}||\vec{b}|\cos\theta + |\vec{b}|^2$

$\Rightarrow |2\vec{a} + \vec{b}|^2 = |2\vec{a} - \vec{b}|^2$

$\Rightarrow 4|\vec{a}|^2 + 4|\vec{a}||\vec{b}|\cos\theta + |b|^2 = 4|\vec{a}|^2 - 4|\vec{a}||\vec{b}|\cos\theta + |b|^2$

$\Rightarrow 8|a||b|\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \dfrac{\pi}{2}$

30. If $\vec{u} = \vec{v}$ then find angle between $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$

Solution: Let θ be the angle between $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$

$$\Rightarrow \cos\theta = \frac{(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v})}{|\vec{u}+\vec{v}||u-\vec{v}|} = \frac{|\vec{u}|^2 - |\vec{v}|^2}{|\vec{u}+\vec{v}||\vec{u}-\vec{v}|}$$

$$= \frac{0}{|\vec{u}+\vec{v}||\vec{u}-\vec{v}|} = 0$$

$\Rightarrow \theta = \dfrac{\pi}{2}, \Rightarrow \vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are perpendicular (orthogonal)

31. Let the angle between $\vec{v} = -2\vec{i} - \vec{j} + 2\vec{k}$ and $\overline{AB}$ be 60°, where A and B are points in space. If $\vec{V}.\overline{AB} = 2$ then what is the distance between A and B? (EUEE)

A. $\dfrac{3}{4}$ B. $\dfrac{4}{5}$ C. $\dfrac{4}{3}$ D. $\dfrac{5}{4}$

Solution: $\vec{V}.\overline{AB} = |V||AB|\cos 60° = 2$

$$\Rightarrow |\overline{AB}| = \frac{2}{|V|\cos 60°} = \frac{2}{\sqrt{4+1+4}}(2) = \frac{4}{3}$$

Application of Vector

Definition: The **work w** done by a constant force F as its point of application moves along a vector $\vec{a}$ is $W = \vec{F} \cdot \vec{a} = |\vec{F}||\vec{a}|\cos\theta$.

32. Three forces $F_1 = 3i + 2j$, $F_2 = 3i - 2j$, $F_3 = 2i + j$ act on a particle which moves a point A to point B and then to point C if $A = i + 2j$, $B = (-2i - 5j)$ and $C = 3i + j$ then find the work done by
    a) each force in moving from B to C
    b) each force in moving from A to B.
    c) the combined force from A to C.
    d) the combined force from A to B.

Solution: $\overrightarrow{BC} = 3i + j - (2i - 5j) = i + 6j$

$\overrightarrow{AB} = 2i - 5j - (i + 2j) = i - 7j$

$\overrightarrow{AC} = 3i + j - (i + 2j) = 2i - j$

a) $\vec{F_1} \cdot \overrightarrow{BC} = (3i + 2j)\cdot(i + 6j) = 3 + 12 = 15J$

$\vec{F_2} \cdot \overrightarrow{BC} = (3i - 2j)\cdot(i + 6j) = 6 - 12 = -6J$

$\vec{F_3} \cdot \overrightarrow{BC} = (2i + j)\cdot(i + 6j) = 2 + 6 = 8J$

b) $\vec{F_1} \cdot \overrightarrow{AB} = (3i+2j)\cdot(i-7j) = 3 - 14 = -11J \leftarrow$ work done by F_1

$\Rightarrow \vec{F_2} \cdot \overrightarrow{AB} = (3i-2j)\cdot(i-7j) = 3+ 14 = 17J \leftarrow$ work done by F_2

$\Rightarrow \vec{F_3} \cdot \overrightarrow{AB} = (2i+j)\cdot(i-7j) = 2- 7 = -52J \leftarrow$ work done by F_3

c) $(\vec{F_1} + \vec{F_2} + \vec{F_3}) \cdot (\overrightarrow{AC}) = (8i + j)\cdot(2i-j) = 16 - 1 = 15J$

d) $(F_1 + F_2 + F_3) \cdot \overrightarrow{AB} = (8i + j)\cdot(i-7j) = 8 - 7 = 1J \leftarrow$ work done by resultant

33. Two force $F_1 = 3i + 4j$ and $F_2 = -2i + j$ act on a particle which moves a point A(2, -1) to point B(-3, 1)
    Find the work done by a) F_1 b) F_2 c) resultant

Solution: Determent $\overrightarrow{AB} = (-3, 1) - (2, -1) = (-5, 2) = -5i + 2j$

a) work done by $F_1 = \vec{F_1} \cdot \overrightarrow{AB} = (3i+4j)\cdot(-5i + 2j) = -15 + 8 = -7J$

b) work done by $F_2 = \vec{F_2} \cdot \overrightarrow{AB} = (-2i + j)\cdot(-5i+2j = 10 + 2 = 12J$

c) work done by resultant $(\vec{F}_1 + \vec{F}_2) = (3i+4j+-2i+j)\cdot(-5i + 2j)$

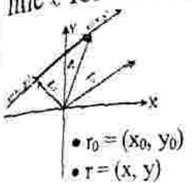
$= (i + 5j)\cdot(-5i+2j) = 5j$

Vectors and Lines

In plane line is determined by point on the line and given slope. On the other hand lines determined by a point and non zero vector. (WOW! It is surprising). The non zero vector $\vec{v}$ is the direction vector which is parallel to the line ℓ.

Vector equation of the line ℓ

Suppose that ℓ is a line in plane passing through a point $P_0(x_0, y_0)$ parallel to a non zero vector $\vec{v} = (a, b)$. Then $P(x, y)$ is set of all point on line ℓ for which $\overrightarrow{P_0P}$ is parallel to $\vec{v}$.



$\Rightarrow \overrightarrow{P_0P} = t\vec{v}, -\infty < t < \infty$

$\Rightarrow (x, y) - (x_0, y_0) = t(a, b) \Rightarrow r = r_0 + t\vec{v}$

$\Rightarrow$ The set of all point (x, y) on ℓ is given by

$\Rightarrow (x, y) = (x_0, y_0) + t(a, b)$

If the line ℓ passing through $P_0(x_0, y_0)$ parallel to vector $\vec{v} = (a, b)$ then

i) $(x, y) = (x_0, y_0) + t(a, b)$ is vector equation of the line

ii) $\left.\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \end{aligned}\right\}$ is called parametric equation of the line ℓ.

solving for t will get:

iii) $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b}$ is standard (rectangular) equation of line

35. Find vector, parametric, and standard (rectangular) equation of the line. If the line ℓ passing through the point

a) $P_0(1, 5)$ and $P(-2, -1)$
b) $P_0(-2, 4)$ and $P(4, -4)$
c) $P_0(-2, 4)$ and parallel to the vector $\vec{v} = (-2, 3)$
d) $P_0(1, -2)$ and parallel to the vector $\vec{v} = (2, 4)$
e) $P_0(3, -1)$ and perpendicular to the vector $\vec{v} = (4, 2)$
f) $P_0(0, 3)$ that is parallel to the line $x = -5 + t$, $y = 1 - 2t$

Solution: To find direction vector $\vec{v}$ for the line passing through P_0 and P is given by $\vec{v} = \overrightarrow{P_0P} \Rightarrow (x, y) = (x_0, y_0) + t\,\overrightarrow{P_0P}$

$$\Rightarrow (x, y) = P_0 + t\overrightarrow{P_0P}$$

a) $\vec{v} = \overrightarrow{P_0P} = (-2, -1) - (1, 5) = (-3, -6) \longleftarrow$ parallel vector

$\Rightarrow (x, y) = (1, 5) + t(-3, -6) \longleftarrow$ vector equation of the line

$\Rightarrow x = 1 - 3t,\ y = 5 - 6t \longleftarrow$ parametric equation of the line

$\Rightarrow \dfrac{x-1}{-3} = \dfrac{y-5}{-6} \leftarrow$ standard (rectangular) equation of the line

$\Rightarrow 6x - 3y = -9$

b) $\vec{v} = \overrightarrow{P_0P} = (4, -4) - (-2, 4) = (6, -8) \longleftarrow$ parallel vector

$\Rightarrow (x, y) = (-2, 4) + t(6, -8) \longleftarrow$ vector equation of the line

$\Rightarrow x = -21 + 6t,\ y = 4 - 8t \longleftarrow$ parametric equation of the line

$\Rightarrow \dfrac{x+2}{6} = \dfrac{y-4}{-8} \leftarrow$ standard (rectangular) equation of the line

$\Rightarrow 8x + 6y = 8$

c) direction (parallel) vector $\vec{v} = (-2, 3)$, and the line passing through $P_0 = (-2, 4)$

The set all (x, y) in ℓ is given by

$\Rightarrow (x, y) = (-2, 4) + t(-2, 3) \longleftarrow$ vector equation of the line

$\Rightarrow x = -2 - 2t,\ y = 4 + 3t \longleftarrow$ parametric equation of the line

$\Rightarrow$ solving for, t: $\dfrac{x+2}{-2} = \dfrac{y-4}{3} \longleftarrow$ standard equation

$\Rightarrow 3x + 2y = 2$

d) $P_0 = (1, -2),\ \vec{v} = (2, 4)$

$\Rightarrow (x, y) = (1, -2) + t(2, 4) \longleftarrow$ vector equation of the line

$\Rightarrow x = 1 + 2t,\ y = -2 + 4t \longleftarrow$ parametric equation of the line

$\Rightarrow$ solving for t: $\dfrac{x-1}{2} = \dfrac{y+2}{4} \longleftarrow$ standard equation

$\Rightarrow 4x - 2y = 8$

e) Let $P(x, y)$ and $P_0(3, -1)$ be on the lien ℓ

Perpendicular vector $\vec{v} = (4, 2)$

$\Rightarrow \overrightarrow{P_0P} = (x,y) - (3, -1) = (x - 3, y + 1)$

Since $\overrightarrow{P_0P} \perp \vec{v} \Rightarrow \overrightarrow{P_0P} \cdot \vec{v} = 0$

$\Rightarrow (x-3, y+1)\cdot(4, 2) = 4x - 12 + 2y + 2 = 0$

$\Rightarrow 4x + 2y = 10 \Rightarrow 2x + y = 5$

To find the direction vector $\vec{v}$, set $2x + y = 0$

$\Rightarrow y = -2x$

$\Rightarrow$ The set of all parallel vector is given by

$\vec{v} = (x, y) = (x, -2x) = x(1, -2) = t(1, -2)$

We have $P_0 = (3, -1)$

$\Rightarrow (x, y\_ = (3, -1) + t(1, -2) \longleftarrow$ vector equation of ℓ.

$\Rightarrow x = 3 + t, y = -1 - 2t \longleftarrow$ parametric equation of ℓ.

f) solving for $t = \dfrac{x+5}{1} = \dfrac{y-1}{-2} \Rightarrow \vec{v} = (1,-2)$

$\Rightarrow (x, y) = (0, 3) + t(1, -2)$

$\Rightarrow x = 0 + t, y = 3 - 2t.$

36. The line L passes through the point (1, -4) and parallel to the vector $\vec{v} = (-2, 5)$. Determine which point does not lie on L, and find t

A. (-1, 1) B. (3, -9) C. $\left(2, \dfrac{-13}{2}\right)$ D. $(-3, 9)$

Solution: The set of all point, P(x, y) on ℓ is determined by vector equation of the line L:

$\Rightarrow (x, y) = (1, -4) + t(-2, 5) = (1 - 2t, -4 + 5t)$

$\Rightarrow$ Parametric equation of L: $x = 1 - 2t$ and $y = -4 + 5t$

Solving for t

$\Rightarrow \dfrac{x-1}{-2} = \dfrac{y+4}{5} \Rightarrow 5x - 5 = -2y - 8 \Rightarrow 5x + 2y = -3$

A. (-1, 1): $5(-1) + 2(1) = -3 \Rightarrow$ (-1, 1) on L; t = 1

B. (3, -9): $5(3) + 2(-9) = 15 - 18 = -3 \Rightarrow$ (3, -9) on L, t = -1

C. $\left(2, \dfrac{-13}{2}\right)$: $5(2) + 2\left(\dfrac{-13}{2}\right) = 10 - 13 = -3 \Rightarrow \left(2, \dfrac{-13}{2}\right)$ on L, $t = \dfrac{-1}{2}$

D. $(-3, 9)$: $5(-3) + 2(9) = -15 + 18 = 3 \Rightarrow$ (-3, 9) not on L. (has not the same t)

$P = P_0 + tv$ $v = P_0$

$\therefore$ (-1, 1), (3, -9), $\left(2, \frac{-13}{2}\right)$ are lie on ℓ and collinear. But (-3, ...

is not on ℓ.

37. Are the following points collinear.

a) A(1, -1), B(-2, 5), C(3, -5)

b) A(2, 5) B(-1, -4), C(-2, -5)

Solution: The set of all point P(x, y) on L is determined by:

$P(x, y) = A + t\overrightarrow{AB} \Rightarrow (x, y) = (1, -1) + t(-3, 6)$

$\Rightarrow (x, y) = (1 - 3t, -1 + 6t)$

A. $(1, -1) = (1 - 3t, -1 + 6t) \Rightarrow 1 - 3t = 1$ and $-1 + 6t = -1$

$\Rightarrow -3t = 0$ and $6t = 0 \Rightarrow t = 0 \Rightarrow (1, -1)$ on ℓ,

B. $(-2, 5) = (1 - 3t, -1 + 6t) \Rightarrow 1 - 3t = -2$ and $-1 + 6t = 5$,

$\Rightarrow -3t = -3 \Rightarrow t = 1$ and $-1 + 6t = 5 \Rightarrow t = 1$

$\Rightarrow (-2, 5)$ lie on ℓ,

C. $(3, -5) = (1 - 3t, -1 + 6t) \Rightarrow 1 - 3t = 3$ and $-1 + 6t = -5$

$\Rightarrow -3t = 2 \Rightarrow t = \frac{-2}{3}$ and $6t = -5 + 1 = -4 \Rightarrow t = \frac{-4}{6} = \frac{-2}{3}$

$\Rightarrow (3, -5)$ lie on ℓ,

$\therefore$ (1, -1),(-2, 5) and (3, -5) are collinear.

d. $P(x, y) = A + t\overrightarrow{AB} \Rightarrow (x, y) = (2, 5) + t(-3, -9)$

$\Rightarrow (x, y) = (2 - 3t, 5 - 9t)$

$\Rightarrow A(2, 5) = (2 - 3t, 5 - 9t)$ when $t = 0$

$B(-1, -4) = (2 - 3t, 5 - 9t)$ when $t = 1$

$C(-2, -5) = (2 - 3t, 5 - 9t)$

$\Rightarrow 2 - 3t = -2 \Rightarrow t = \frac{4}{3}$ and $5 - 9t = -5 \Rightarrow t = \frac{10}{9}$

Since t is not the same

$\Rightarrow$ A, B and C are not collinear

38. Find Parametric equation of the line L

a) L: $3x + 4y = 10$ b) L: $\frac{x-1}{5} = \frac{y+6}{7}$

Solution: To find parallel (direction) vector, $\vec{v}$

a) set: $3x + 4y = 0 \Rightarrow y = \dfrac{-3}{4}x$ the set of all direction vector parallel to the Line

L: $3x + 4y = 10$ is $y = \dfrac{-3}{4}x$

$$\Rightarrow \vec{v} = (x, y) = t(a, b) = t\left(x, \frac{-3}{4}x\right) = t\left(1, \frac{-3}{4}\right) = t(4, -3)$$

We can choice any point on line $3x + 4y = 10$

Let $x = 2, \Rightarrow y = 1 \Rightarrow P_0 = (2, 1)$

$\Rightarrow (x, y) = (2, 1) + t(4, -3) \longleftarrow$ vector equation of ℓ,

$\Rightarrow x = 2 + 4t, y = 1 - 3t \longleftarrow$ parametric equation

b) L: $\dfrac{x-1}{5} = \dfrac{y+6}{7} \Rightarrow \vec{v} = (5,7)$, $P_0 = (1, -6)$

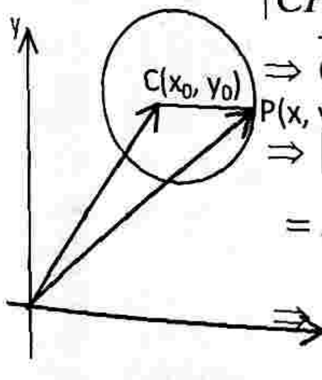
$\Rightarrow (x, y) = (1, -6) + t(5, 7) \longleftarrow$ vector equation of ℓ,

$\Rightarrow x = 1 + 5t$ and $y = -6 + 7t \longleftarrow$ parametric equation

Vector and circle

39. Using vector method find the general equation of the circle with center $C(x_0, y_0)$ and radius, R

Let P(x, y) be set of point in circle such that



$$|\overrightarrow{CP}| = R$$

$$\Rightarrow \overrightarrow{CP} = (x, y) - (x_0, y_0) = (x - x_0, y - y_0)$$

$$\Rightarrow |\overrightarrow{CP}| = \sqrt{\overrightarrow{CP} \cdot \overrightarrow{CP}}$$

$$= \sqrt{(x - x_0, y - y_0) \cdot (x - x_0, y - y_0)}$$

$$\Rightarrow \sqrt{(x - x_0)^2 + (y - y_0)^2} = R$$

Squaring both side

$$\Rightarrow (x - x_0)^2 + (y - y_0)^2 = R^2$$

40. Using vector method write the equation of the circle center (4,-3) radius 5

Solution: Let P(x, y) be point on the circle and C(4, -3) is center

$$\Rightarrow |\overrightarrow{CP}| = R \Rightarrow |(x - 4, y + 3| = 5$$

$\Rightarrow (x-4, y+3)\cdot(x-4, y+3) = 25$

$\Rightarrow (x-4)^2 + (y+3)^2 = 25$

41. Find the equation of circle with diameter with end points A(2, 3) and B(6, 9)

Solution: Center $C\left(\dfrac{2+6}{2}, \dfrac{3+9}{2}\right) = C(4,6)$

Radius R = $\sqrt{(6-4)^2 + (9-6)} = \sqrt{4+9} = \sqrt{13}$

Let P(x, y) be point on circle

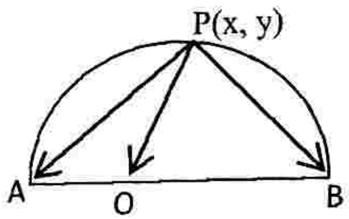
$\Rightarrow |\overrightarrow{CP}| = \sqrt{13} \Rightarrow |(x-4, y-6| = \sqrt{13}$

$\Rightarrow (x-4, y-6)\cdot(x-4, y-6) = 13$

$\Rightarrow (x-4)^2 + (y-6)^2 = 13$

42. Show that the angle in a semicircle is right angle

Solution:



- $|\overrightarrow{AO}| = |\overrightarrow{OB}| = |\overrightarrow{OP}| \longleftarrow$ radius
- $\overrightarrow{PA} = \overrightarrow{PO} + \overrightarrow{OA}$
- $\overrightarrow{PB} = \overrightarrow{PO} + \overrightarrow{OB} = \overrightarrow{PO} - \overrightarrow{OA}$

[$\overrightarrow{OB}$ *and* $\overrightarrow{OA}$ are equal magnitude but opposite direction

$\Rightarrow \overrightarrow{PA}\cdot\overrightarrow{PB} = (\overrightarrow{PO} + \overrightarrow{OA})\cdot(\overrightarrow{PO} - \overrightarrow{OA})$

$\Rightarrow |\overrightarrow{PO}|^2 - |\overrightarrow{OA}|^2 = 0$ since $|\overrightarrow{PO}| = |\overrightarrow{OA}|$

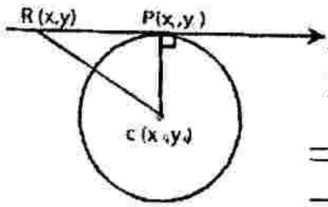
$\Rightarrow \overrightarrow{PA}$ is right angle to $\overrightarrow{PB}$

Equation of tangent line to Circle

Let R(x, y) = Point on tangent line

P = (x_1, y_1) = Point of tangency on circle

C(x_0, y_0) = center of circle.



$\Rightarrow \overrightarrow{RP}\cdot\overrightarrow{PC} = 0$. The radius of circle is perpendicular to the tangent line at point of tangency at P(x_1, y_1)

$\Rightarrow (x_1 - x, y_1 - y)\cdot(x_0 - x_1, y_0 - y_1) = 0$

$\Rightarrow \boxed{x_1x + y_1y = r^2}$ if center origin.

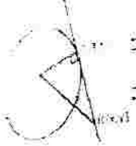
43. Find the equation of tangent line to the circle at given point P.

a) $x^2 + y^2 = 10$ at point $P(3, 1)$
b) $x^2 + y^2 = 5$ at point $P(-1, 2)$
c) $(x - 3)^2 + (y + 1)^2 = 8$ at point $P(1, 1)$
d) $x^2 + y^2 - 8x + 10y + 21 = 0$ at point $P(2, -1)$

Solution: Let $R(x, y)$ point on tangent line.

a) $x^2 + y^2 = 10$, $P(3, 1)$ and center $C(0, 0)$

$\Rightarrow \overrightarrow{RP} = (3-x, 1-y)$ and $\overrightarrow{PC} = (0-3, 0-1) = (-3,1)$



$\Rightarrow \overrightarrow{RP} \cdot \overrightarrow{PC} = 0 \Rightarrow (3-x, 1-y)\cdot(-3, -1) = 0$

$\Rightarrow -9 + 3x - 1 + y = 0 \Rightarrow 3x + y = 10$

$\therefore 3x + y = 10$ is equation of tangent line.

b) $x^2 + y^2 = 5$ at $P(-1, 2)$, center $(0, 0)$, Let $R(x, y)$.

$\Rightarrow \overrightarrow{RP} = (-1-x, 2-y)$ and $\overrightarrow{PC} = (0,0) - (-1,2) = (1,-2)$

$\Rightarrow \overrightarrow{RP} \cdot \overrightarrow{PC} = 0 \Rightarrow (-1-x, 2-y)\cdot(1, -2) = 0$

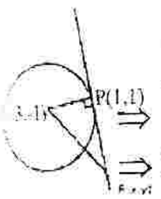
$\Rightarrow -1-x-4+2y = 0 \Rightarrow 2y - x = 5$

$\therefore -x + 2y = 5$ is equation of tangent line.

c) $(x - 3)^2 + (y + 1)^2 = 8$, at $P(1, 1)$, center, $C(3, -1)$, $R(x, y)$

$\Rightarrow \overrightarrow{RP} = (1, 1) - (x, y) = (1 - x, 1 - y)$

$\Rightarrow \overrightarrow{PC} = (3,-1) - (1,1) = (2,-2)$



$\Rightarrow \overrightarrow{RP} \cdot \overrightarrow{PC} = 0 \Rightarrow (1-x, 1-y)\cdot(2, -2) = 0$

$\Rightarrow 2 - 2x + -2 + 2y = 0 \Rightarrow -2x + 2y = 0$

$\therefore -2x + 2y = 0$ is equation of tangent line.

d) $x^2 + y^2 - 8x + 10y + 21 = 0$ Let $R= (x, y)$ on line

$\Rightarrow (x-4)^2 + (y+5)^2 = 20$, at $P(2, -1)$, center $(4, -5)$

$\Rightarrow \overrightarrow{RP} = (2, -1) - (x, y) = (2 - x, -1 - y)$

$\Rightarrow \overrightarrow{PC} = (4,-5) - (2,-1) = (2,-4)$

$\Rightarrow \overrightarrow{RP} \cdot \overrightarrow{PC} = 0 \Rightarrow (2-x, -1-y)\cdot(2, -4) = 0$

$\Rightarrow 4 - 2x + 4 + 4y = 0 \Rightarrow -2x + 4y = -8$

$\Rightarrow -x + 2y = -4$

$\therefore -x + 2y = -4$ is equation of tangent line.

II. Vector in space (three dimensional vector)

Unit vector (i, j, k)

43. If $A = (1, 2, 0)$ and $B(3, 2, -1)$ be two points, find the unit vector in the direction of

a) $\overrightarrow{AB}$ b) $\overrightarrow{BA}$

Solution: $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (3, 2, -1) - (1, 2, 0) = (2, 0, -1)$

$\Rightarrow$ Unit vector along

$$\hat{AB} = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \left(\frac{2}{\sqrt{5}}, 0, \frac{-1}{\sqrt{5}}\right) = \frac{2}{\sqrt{5}}i - \frac{k}{\sqrt{5}}$$

$\Rightarrow$ unit vector along $\overrightarrow{BA} = \dfrac{\overrightarrow{BA}}{|\overrightarrow{BA}|} = \left(\dfrac{-2}{\sqrt{5}}, 0, \dfrac{1}{\sqrt{5}}\right) = \dfrac{-2}{\sqrt{5}}i + \dfrac{k}{\sqrt{5}}$

14. Determine the value of m, so that $\vec{a} = 2i + mj + k$ and $\vec{b} = 4i - 2j - 2k$ are perpendicular

Solution: $\vec{a} \cdot \vec{b} = 0 \Rightarrow (2i + mj + k)\cdot(4i - 2j - 2k) = 0$
$\Rightarrow 8 - 2m - 2 = 0 \Rightarrow -2m = -6 \Rightarrow m = 3$

45. Find the scalar (dot) product of each of the following pair of vector.

a) $\vec{v} = (2, -3, 1)$ and $\vec{u} = (3, 1, -2)$

b) $\vec{v} = (-1, 1, 2)$ and $\vec{u} = (2, -2, -1)$

Solution: a) $\vec{v} \cdot \vec{u} = (2, -3, 1)\cdot(3, 1, -2) = 6 + -3 - 2 = 1$

b) $\vec{v} \cdot \vec{u} = (-1, 1, 2)\cdot(2, -2, -1) = -2 - 2 - 2 = -6$

46. If $\vec{a} = 2i + j - k$, $\vec{b} = i + 2j + k$, $\vec{c} = 3i - 2j + 2k$.

Find a) unit vector parallel to $\vec{a}$ - 2b + c

b) unit vector in direction opposite to $\vec{a}$ - 2b + c

Solution: Firs, $\vec{a} - 2\vec{b} + \vec{c} = 2i + j - k - 2(i + 2j + k) + 3i - 2j +$
$2k \Rightarrow \vec{a} - 2\vec{b} + \vec{c} = 3i - 5j - k$

Unit vector along $\vec{a} - 2\vec{b} + \vec{c} = \dfrac{\vec{a} - 2\vec{b} + \vec{c}}{|\vec{a} - 2\vec{b} + \vec{c}|}$

$$\Rightarrow \frac{3i - 5j - k}{\sqrt{3^2 + 5^2 + 1}} = \frac{3i - 5j - k}{\sqrt{35}} = \left(\frac{3}{\sqrt{35}}, \frac{-5}{\sqrt{35}}, \frac{-1}{\sqrt{35}}\right)$$

$$= \frac{3i}{\sqrt{35}} - \frac{5j}{\sqrt{35}} - \frac{k}{\sqrt{35}}$$

b) Unit vector in direction opposite is $= \left(\frac{-3}{\sqrt{35}}, \frac{5}{\sqrt{35}}, \frac{1}{\sqrt{35}}\right)$

$$= \frac{-3i}{\sqrt{35}} + \frac{5j}{\sqrt{35}} + \frac{k}{\sqrt{35}}$$

47. Find the angle between the vector.

a) $\vec{v} = (-4, 0, 2)$ and $\vec{u} = (2, 0, -1)$

b) $\vec{v} = (2, 0, 2)$ and $\vec{u} = (2, 2, 0)$

c) $\vec{v} = 2i + 3j + k$ and $\vec{u} = -3i - j + 0k$

Solution: Let θ be the angle between vector.

a) $\cos\theta = \frac{\vec{v}\cdot\vec{u}}{|\vec{v}||\vec{u}|} = \frac{-8+0-2}{\sqrt{16+0+4}\sqrt{4+0+1}} = \frac{-1}{\sqrt{100}} = \frac{-10}{10} = -1$

$\Rightarrow \cos\theta = -1 \Rightarrow \theta = \cos^{-1}(-1) = \pi$

b) $\cos\theta = \frac{\vec{v}\cdot\vec{u}}{|\vec{v}||\vec{u}|} = \frac{4+0+0}{\sqrt{4+4}\sqrt{4+4}} = \frac{4}{\sqrt{64}} = \frac{4}{8} = \frac{1}{2}$

$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

c) $\cos\theta = \frac{\vec{v}\cdot\vec{u}}{|\vec{v}||\vec{u}|} = \frac{(2,3,1)\cdot(-3,-1,2)}{\sqrt{4+9+1}\sqrt{9+1+4}} = \frac{-6-3+2}{\sqrt{14}\sqrt{14}} = \frac{-7}{14} = \frac{-1}{2}$

$\Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \cos^{-1}\left(\frac{-1}{2}\right) = \pi - \cos^{-1}\left(\frac{-1}{2}\right) = \pi - \frac{\pi}{3}$

$$= \frac{2\pi}{3}$$

III. Transformation of the plane.

When a point P is given a new position, the point P is said to have a transformation.

There are two categories of transformation (motion)

i) Non-Rigid motion

ii) Rigid motion

i) **Non-Rigid motion-** A transformation that does not preserve distance.

$$\overline{AB} \neq \overline{A'B'}$$

ii) **Rigid Motion-** A transformation that preserves distance.

- segment moved into congruent segment

- angle in to congruent angle.
- Triangle into congruent triangle.
- $\overline{AB} = \overline{A'B'}$

iii) Identity **transformation:** A transformation the image of every point is itself.

Identity transformation is a rigid motion.

There are three types of **Rigid Motion** transformation)

1. **Translation** 2. **Reflection** 3. **Rotation**

1. **Translation**

Translation: A translation if every point in a line or a plane figure moves the same distance in the same direction.

Translation vector. If point A is translated to A', the vector $\overrightarrow{AA'}$ is called **translation vector.**

If $\vec{v} = (h, k)$ is translation vector then the image of

$$(x, y) \xrightarrow{\vec{v}=(h,k)} (x+h, y+k)$$

If the point P(4, 6) has been translated to p'(9, 8)

$$P(4, 6) \longrightarrow P'(9, 8)$$

Translating vector $\overrightarrow{PP'} = \vec{v} = (9-4, 8-6) = (5, 2)$

$\Rightarrow$ P(4, 6) moved 5 unit to right and 2 unit up.

T(x, y) = (x + 5, y + 2)

Illustrative Example

48. Which one of the following transformation does not describe Rigid motion?

A. T((x, y)) = (2y, 2x) C. T((x, y)) = (x + 1, y+1)

B. T((x, y)) = (-x, y) D. T((x, y)) = (-y, x)

Solution: Let A = (a, b), B = (c, d)

$\Rightarrow$ Distance $\overline{AB} = \sqrt{(a-c)^2 + (b-d)^2}$

A. A' = T((a, b)) = (2b, 2a), B' = T((c, d)) = (2d, 2c)

$\Rightarrow$ Distance $\overline{A'B'} = \sqrt{4(b-d)^2 + 4(a-c)^2} \Rightarrow \overline{AB} = 2\overline{A'B'}$

$\Rightarrow \overline{AB} \neq \overline{A'B'} \Rightarrow$ Not rigid.

B. A' = T((a, b) = (-a, b), B' = T((c, d)) = (-c, d)

$\Rightarrow A'B' = \sqrt{(a-c)^2 + (b-d)} \Rightarrow \overline{AB} = \overline{A'B'} \Rightarrow$ Rigid motion

C. A' =T((a, b)) = (a+1, b+1) and B' = T((c, d)) = (c +1, d+

$\Rightarrow A'B' = \sqrt{((a+1-(c+1))^2 + [(b+1-d-1)]^2}$

$= \sqrt{(a-c)^2 + (b-d)^2}$

$\Rightarrow \overline{AB} = A'B' \Rightarrow$ rigid motion (Preserve distance)

d) $A' = T((a, b)) = (-b, a)$ and $B' = T((c, d)) = (-d, c)$

$\Rightarrow A'B' = \sqrt{(d-b)^2 + (a-c)^2} \Rightarrow \overline{AB} = A'B' \Rightarrow$ Rigid motion

In choice A-distance $\overline{AB} \neq A'B$

Answer: A

49. Suppose T is transformation. Let A = (3, 1) and B = (2, 4) then which one is rigid and which is not rigid.

a) $T((x, y)) = (x, y-1)$ b) $T((x, y)) = (3x, y+2)$

c) $T((x, y)) = (x-3, y-2)$

Solution: Distance $AB = \sqrt{(3-2)^2 + (1-4)^2} = \sqrt{10}$

a) $A' = T((3, 1)) = (3, 1-1) = (3, 0)$ and $B' = T((2, 4)) = (2, 3)$

$\Rightarrow A'B' = \sqrt{(3-2)^2 + (0-3)^2} = \sqrt{1+9} = \sqrt{10} \Rightarrow \overline{AB} = \overline{A'B'}$

$\therefore T((x, y)) = (x, y-1)$ is rigid motion

b) $A' = T((3, 1)) = (9, 1+2) = (9, 3)$ and $B' = T((2, 4)) = (6, 6)$

$\Rightarrow A'B' = \sqrt{(9-6)^2 + (3-6)^2} = \sqrt{18} \Rightarrow AB \neq A'B'$

$\therefore T((x, y)) = (3x, y+2)$ is not rigid.

c) $A' = T((3, 1)) = (3-3, 1-2) = (0, -1)$ and $B' = T((2, 4))$
$= (2-3, 4-2) = (-1, 2)$

$\Rightarrow A'B' = \sqrt{(0+1)^2 + (-1-2)^2} = \sqrt{10} \Rightarrow AB = A'B'$

$\therefore T((x, y)) = (x-3, y-2)$ is rigid motion

50. If translation T takes the point (7, 2) to the point (5, 3) then find the image of

a) P(3, 4)
b) the triangle with vertices A(2, 3), B(3, -4) and C(2, -1)
c) the line ℓ: $y = 2x - 3$
d) the line ℓ: $3x - 4y = 7$
e) the circle: $x2 + y2 = 5$
f) the circle: $(x-1)2 + y2 = 4$
g) the circle: $x2 + y2 - 4x + 6y + 7 = 0$
h) the parabola: $y = 2x2 + 4$
i) the parabola: $y2 = 3x$

j) the ellipse: $3x2 + 4y2 - 12 = 0$

k) $y = \sin x$

Solution: Translation vector $\vec{v} = (5, 3) - (7, 2) = (-2, 1)$

$\Rightarrow$ image of$(x, y) = (x - 2, y + 1)$

$\Rightarrow$ the movement 2 unit to left, 1 unit up

a) $T((3, 4)) = (3 - 2, 4 + 1) = (1, 5)$

b) $T((2, 3)) = (2 - 2, 3 + 4) = (0, 4)$

$T((2, -1)) = (2 - 2, -1 + 1) = (0, 0)$

$\Rightarrow A'(0, 4), B'(1, -3)$ and $(0, 0)$

c) Under translation line maps in to parallel line

$\Rightarrow$ vector $\vec{v} = (-2, 1)$ the line moves 2 unit left and 1 unit up $\Rightarrow$ put $x + 2$ in x and $y - 1$ in y.

$\Rightarrow \ell': y - 1 = 2(x + 2) - 3 \Rightarrow \ell': y = 2x + 2$

d) $\ell': 3(x + 2) - 4(y - 1) = 7 \Rightarrow \ell': 3x - 4y = -3$

e) $(x + 2)^2 + (y - 1)^2 = 5$, putting $x + 2$ in x and $y - 1$ in y

f) the image of circle $(x-1)^2 + y^2 = 4$ is $(x+2-1)^2 + (y-1)^2 = 4$

$\Rightarrow (x + 1)^2 + (y - 1)^2 = 4$

g) the image of circle $x^2 + y^2 - 4x + 6y + 7 = 0$

$\Rightarrow (x + 2)^2 + (y - 1)^2 - 4(x+2) + 6(y-1) + 7 = 0$

$\Rightarrow x^2 + 4x + 4 + y^2 - 2y + 1 - 4x - 8 + 6y - 6 + 7 = 0$

$\Rightarrow x^2 + y^2 + 4y + 4 = 0$

h) Put $x + 2$ in x and $y - 1$ in y

the image of parabola $y = 2x^2 + 4$ will be

$\Rightarrow y - 1 = 2(x+2)^2 + 4 \Rightarrow y = 2(x^2 + 4x + 4) + 4 + 1$

$\Rightarrow y = 2x^2 + 8x + 13$

i) put $x + 2$ in x and $y - 1$ in y

the image of parabola $y^2 = 3x$ is:

$\Rightarrow (y - 1)^2 = 3(x + 2) \Rightarrow y^2 - 2y + 1 = 3x + 6$

$\Rightarrow y^2 - 2y - 3x - 5 = 0$

j) Put $x + 2$ in x and $y - 1$ in y

the image of ellipse: $3x^2 + 4y^2 - 12 = 0$ is

$\Rightarrow 3(x + 2)^2 + 4(y - 1)^2 - 12 = 0$

$\Rightarrow 3(x^2 + 4x + 4) + 4(y^2 - 2y + 1) - 12 = 0$

$\Rightarrow 3x^2 + 4y^2 + 12x + -8y + 4 = 0$

k) Put $x + 2$ in x and $y - 1$ in y

the image of $y = \sin x \Rightarrow y - 1 = \sin(x + 2)$

$\Rightarrow y = \sin(x + 2) + 1$

51. If the point A(3, 7) is transulated to the point A'(0, 2) then find the equation of the image of a) (0, 0) d) $6x + 5y = 9$
b) (2, 1) c) $y = 3x + 4$ e) $x^2 - y^2 = 8$

Solution: $\vec{v} = (h, k) = \overrightarrow{AA'} = (0, 2) - (3, 7) = (-3, -5)$

a) $T((0, 0)) = (0 + -3, 0 + -5) = (-3, -5)$
b) $T((2, 1)) = (2 + -3, -5 + 1) = (-1, -4)$
c) The line move 3 unit to left, 5 unit down so that substitute $x + 3$ in x and $y + 5$ in y for $y = 3x + 4$ image will be
$\Rightarrow y + 5 = 3(x + 3) + 4 \Rightarrow y = 3x + 8$
d) Image of $6x + 5y = 9$ will be $6(x + 3) + 5(y + 5) = 9$
$\Rightarrow 6x + 5y = -34$
e) $(x + 3)^2 - (y + 5)^2 = 8$

52. A Translation T takes the point (1, 2) to the point (5, -1) A second translation S takes the point (1, 2) to the point (-2, 3) then find a) T followed by S taken the point (1, 2)
b) S followed by T takes the point (1, 2)
c) What do you conclude?

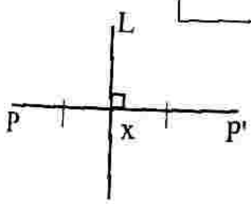
Solution: $T((1, 2)) \xrightarrow{\vec{v}=(4,-3)} (5, -1)$

$S((1, 2)) \xrightarrow{\vec{u}=(-3,1)} (-2, 3)$

a) $S(T(1, 2)) = S(5, -1) = (5 + -3, -1 + 1) = (2, 0)$
b) $T(S(1, 2) = T(-2, 3) = (4 + -2, 3 + -3) = (2, 0)$
c) $S(T(x, y)) = T(S(x, y))$ and we observed
$S(T(1,. 2)) = T(S(1, 2)) = (2, 0)$

Reflections

> If a point **P** is reflected in a mirror so that its image is P', the mirror (or line of reflection) L is the perpendicular bisector of $\overline{PP'}$

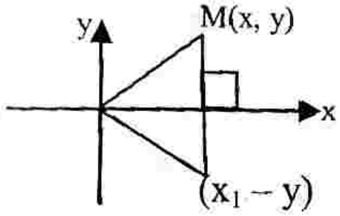


i) $\overline{PP'} \perp L$, L is line of reflection
ii) $\overline{PX} = \overline{P'X}$
iii) if P is on L, its image P' in L is P itself

A. Reflection in the axes

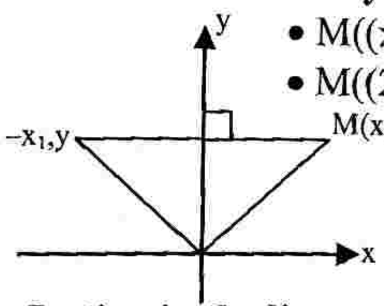
i) Reflection in the x-axis ($y = 0$)

$M((x, y)) = (x, -y)$



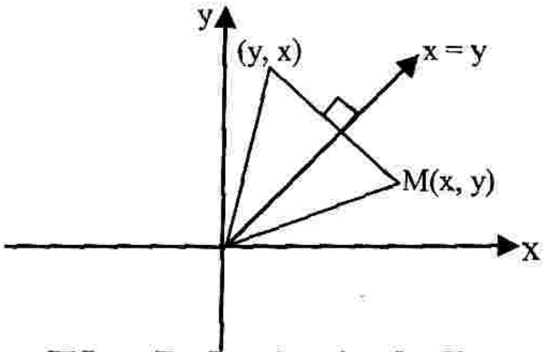
- $M(3, 4) = (3, -4)$
- $M(-4, 8) = (-4, -8)$

ii) Reflection in the y-axis ($x = 0$)



- $M((x, y)) = (-x, y)$
- $M((2, 4)) = (-2, 4)$
- $M((x, a^x)) = (x, a^{-x})$

iii) Reflection in the line $y = x$



- $M((x, y)) = (y, x)$
- $M((2, 5)) = (5, 2)$
- $M((x, a^x)) = (x, \log_a^x)$

IV) Reflection in the line $y = -x$

- $M((x, y)) = (-y, -x)$

Example:
- $M((-2, 4)) = (-4, 2)$
- $M((3, 1)) = (-1, -3)$

V) Reflection in the line $x = C$

- $M((a, b)) \quad \underline{\underline{x = C}} \quad (2c - a, b)$

Example: $M((3, 4)) \quad \underline{\underline{x = 5}} \quad (2(5) - 3, 4) = (7, 4)$

Illustrative Example

53. Find the image in the line of reflection $x = 3$

a) $M(4, 1)$ b) $y = 2x - 5$ c) $(x - 2)^2 + (y + 1)$

Solution: $M((a, b)) = (2(3) - a, b) = (6 - a, b)$

a) $M(4, 1) = (2(3) - 4, 1) = (2, 1)$

b) take two point arbitrarly on $y = 2x - 5$

Let $(1, -3)$ and $(2, -1)$

$\Rightarrow M((1, -3)) \xrightarrow{x=3} (2(3) - 1, -3) = (5, -3)$

$M((2, -1)) \xrightarrow{x=3} (2(3) - 2, -1) = (4, -1)$

...uation of the line passing through $M'(5,-3)$ and $M'(4,-1)$ is given

$y_1 = m(x - x_1) \Rightarrow m = \dfrac{-3-(-1)}{5-4} = -2$

$\Rightarrow y - (-3) = -2(x - 5) \Rightarrow y + 3 = -2x + 10 \Rightarrow y = -2x + 7$

$\therefore$ image of $y = 2x - 5$ is $y = -2x + 7$

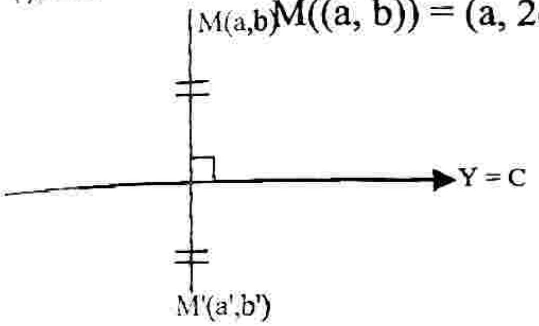
c) Take center $(2, -1)$

$\Rightarrow M((2, -1)) \xrightarrow{x=3} (2(3) - 2, -1) = (4, -1)$

$\therefore$ image of $(x - 2)^2 + (y + 1)^2 = 7$ is $(x - 4)^2 + (y + 1)^2 = 7$

VI) Reflection on the line y = c

$M((a, b)) = (a, 2c - b)$



54. Find the image in the line of reflection $y = 2$

a) $M((3, 4))$ b) $y = 3x + 1$ c) $x^2 + y^2 - 2y - 2 = 0$

Solution: $M((a, b)) \xrightarrow{y=c} (a, 2c - b)$

a) $M((3, 4)) \xrightarrow{y=2} (3, 2(2) - 4) = (3, 0)$

b) First, take two point arbitrarily on $y = 3x + 1$

Let $(1, 4)$ and $(3, 10)$

$\Rightarrow M((1, 4)) \xrightarrow{y=2} (1, 2(2) - 4) = (1, 0)$

$\Rightarrow M((3, 10)) \xrightarrow{y=2} (3, 2(2) - 10) = (3, -6)$

The image of the given line is the equation of the line passing through $(1, 0)$ and $(3, -6)$ is given by $y - y_1 = m(x - x_1)$,

$m = \dfrac{0+6}{1-3} = -3$

$\Rightarrow y - 0 = -3(x - 1) \Rightarrow y = -3x + 3$

c) center of circle is $(0, 1) \Rightarrow M(0, 1) = (0, 2(2) - 1)) = (0, 3)$

Thus, the image will be $x^2 + y^2 - 6y + 6 = 0$

55. Find the image in the line $y = 0$ (x – axis)

a) M((3, 5)) b) $y = 2^x$ c) $y = \log_2^x$ d) $y = \sin x$

Solution: M((a, b)) = (a, -b)

a) M((3, 5)) = (3, -5),

b) $y = -2^x$

c) the image of $y = \log_2^x$ in the line of reflection

$$x - \text{axis } (y = 0) \text{ is } y = -\log_2^x = \log_2^{\frac{x}{1}} = \log_{\frac{1}{2}}^x$$

d) $y = -\sin x$ or $y = \sin(-x)$

56. Find the image in the line, $x = 0$ (y – axis)

a) $y = x$, b) $y = 3^x$ c) $y = \log_3^x$ d) $y = 3x + 4$

Solution: M((a, b)) = (-a, b)

a) $y = -x$ b) $y = 3^{-x}$ c) $y = \log_3^{-x}$

d) First take two point on $y = 3x + 4$

Let (1, 7) and (-2, -2)

$\Rightarrow$ M((1, 7)) = (-1, 7) and M(-2, -2) = (2, -2)

Equation of the line passing through (-1, 7) and (2, -2) is given by $y - y_1 = m(x - x_1)$

$$\Rightarrow m = \frac{7-(-2)}{-1-2} = \frac{9}{-3} = -3 \Rightarrow y - 7 = -3(x+1)$$

$\Rightarrow$ the image of the line $y = 3x + 4$ is $y = -3x + 4$

57. Find the images after reflection in the line $y = x$

a) (3, -2) b) $y = 4^x$ c) $y = \ln x$ d) $y = \log_a^x$

Solution: M(a, b) = (b, a)

a) M(3, -2) = (-2, 3) b) $y = \log_4^x$ c) $y = e^x$ d) $y = a^x$

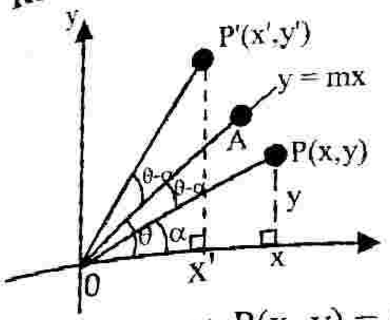
58. If a point P has the coordinates (5, 2), find its reflection in the

a) x – axis b) y-axis c) $y = -x$ d) $y = x$

Solution: a) M((5, 2)) = (5, -2) b) M(5, 2) = (-5, 2)

c) $M((5, 2)) \xrightarrow{y=-x} (-2, -5)$ d) $M(5, 2) \xrightarrow{y=x} (2, 5)$

B. Reflection in the line y = mx, where m = $\tan\theta$



- Let $P'(x', y')$ be the image of $P(x, y)$
- Let $OP = r \Rightarrow OP' = r$
- $\cos\alpha = \frac{x}{r} \Rightarrow x = r\cos\alpha$
- $\sin\alpha = \frac{y}{r} \Rightarrow y = r\sin\alpha$

$\Rightarrow P(x, y) = (r\cos\alpha, r\sin\alpha)$

$m(\angle P'OX') = \theta - \alpha + \theta = 2\theta - \alpha$

- $\cos(2\theta - \alpha) = \frac{x'}{r} \Rightarrow x' = r\cos(2\theta - \alpha)$

$\Rightarrow x' = r\cos(2\theta - \alpha) = r\cos2\theta\cos\alpha + r\sin2\theta\sin\alpha$

$\Rightarrow \boxed{x' = x\cos2\theta + y\sin2\theta}$

- $\sin(2\theta - \alpha) = \frac{y'}{r} \Rightarrow y' = r\sin(2\theta - \alpha)$

$\Rightarrow y' = r\sin2\theta\cos\alpha - r\cos2\theta\sin\alpha$

$\Rightarrow y' = x\sin2\theta - y\cos 2\theta$

The image of P(x, y), in the line of reflection y = mx where m = $\tan\theta$ is $x' = x\cos2\theta + y\sin2\theta$. $y' = x\sin2\theta - y\cos2\theta$

Illustrative Example

59. Find the image of (4, 5) n the line of reflection

a) $y = \sqrt{3}x$ b) $y = 3x$

Solution: a) $y = \sqrt{3}x \Rightarrow m = \sqrt{3} \Rightarrow \tan\theta = \sqrt{3}$

$$\Rightarrow \theta = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

$$x' = x\cos2\theta + y\sin2\theta \Rightarrow x' = 4\cos\frac{2\pi}{3} + 5\sin\frac{2\pi}{3}$$

$$\Rightarrow x' = 4\left(\frac{-1}{2}\right) + 5\frac{\sqrt{3}}{2} = -2 + \frac{5\sqrt{3}}{2}$$

$$y' = x\sin 2\theta - y\cos 2\theta \Rightarrow y' = 4\sin\frac{2\pi}{3} - 5\cos\frac{2\pi}{3}$$

$$\Rightarrow y' = 2\sqrt{3} - 5\left(\frac{-1}{2}\right) = 2\sqrt{5} + \frac{5}{2}$$

$$\therefore P(4, 5) \xrightarrow{y=\sqrt{3}x} \left(-2 + \frac{5\sqrt{3}}{2}, 2\sqrt{3} + \frac{5}{2}\right)$$

b) $y = 3x \Rightarrow m = 3 \Rightarrow \tan\theta = 3 \Rightarrow \sin\theta = \frac{3}{\sqrt{3^2+1}} = \frac{3}{\sqrt{10}}$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{3^2+1}} = \frac{1}{\sqrt{10}}$$

$$\Rightarrow \cos 2\theta = \cos(\theta + \theta) = \cos^2\theta - \sin^2\theta = \left(\frac{1}{\sqrt{10}}\right)^2 - \left(\frac{3}{\sqrt{10}}\right)^2 = \frac{-4}{5}$$

$$\Rightarrow \sin 2\theta = 2\cos\theta\sin\theta = 2\left(\frac{1}{\sqrt{10}}\right)\left(\frac{3}{\sqrt{10}}\right) = \frac{3}{5}$$

$$\Rightarrow x' = x\cos 2\theta + y\sin 2\theta \Rightarrow x' = \frac{-4}{5}x + \frac{3}{5}y$$

$$\Rightarrow y' = x\sin 2\theta - y\cos 2\theta \Rightarrow y' = \frac{3}{5}x - \left(\frac{-4}{5}\right)y = \frac{3}{5}x + \frac{4}{5}y$$

Then the image of (4, 5) will be

$$\Rightarrow x' = \frac{-4}{5}x + \frac{3}{5}y \Rightarrow \frac{-4}{5}(4) + \frac{3}{5}(5) = \frac{-1}{5}$$

$$\Rightarrow y' = \frac{3}{5}x + \frac{4}{5}y \Rightarrow \frac{3}{5}(4) + \frac{4}{5}(5) = \frac{37}{5}$$

$$\Rightarrow M((x, y)) = \left(\frac{-4}{5}x + \frac{3}{5}y, \frac{3}{5}x + \frac{4}{5}y\right)$$

$$\Rightarrow M((4, 5)) = \left(\frac{-4}{5}(4) + \frac{3}{5}(5), \frac{3}{5}(4) + \frac{4}{5}(5)\right) = \left(\frac{-1}{5}, \frac{37}{5}\right)$$

60. The image of the point (2, 2) reflected in line passing through the origin and making 30^0 with the x-axis …

A. (2, -2)
C. $\left(\sqrt{3}-1, 1+\sqrt{3}\right)$
B. $\left(1+\sqrt{3}, \sqrt{3}-1\right)$
D. $\left(\sqrt{3}, -\sqrt{3}\right)$

Solution: we use $x' = x\cos 2\theta + y\sin 2\theta$

$y' = x\sin 2\theta - y\cos 2\theta$

$\theta = 30^0, x = 2, y = 2$

$$\Rightarrow x' = 2\cos 60^0 + 2\sin 60^0 = 2\left(\frac{1}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right) = 1+\sqrt{3}$$

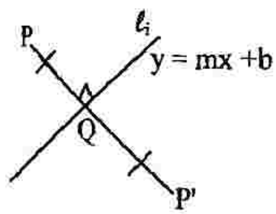
$$y' = 2\sin 60^0 - 2\cos 60^0 = 2\left(\frac{\sqrt{3}}{2}\right) - 2\left(\frac{1}{2}\right) = \sqrt{3}-1$$

$M((2, 2)) = (1 + \sqrt{3}, \sqrt{3} - 1)$

$\therefore$ Answer B

61. From question #60, find the image of (2, 2) if $\theta = 60^0, 150^0, \ldots$
Exercise left for you (Tip!)

C. Reflection in the line y = mx + b



To find the image of P(a, b) in the line of reflection ℓ: y – mx + b

1. Determine the slope of the line
$\Rightarrow$ slope of $\ell = m$
2. Determine the slope of the line $\overline{PP'}$
$\Rightarrow$ slope of $\overline{PP'}$ or $\ell = \dfrac{-1}{m}$
Since $\ell \perp PP'$
3. Determine the intersection point of the line ℓ and PP' or ℓ'
4. Q is the mid-point of PP', find coordinate of P'

Illustrative Example

62. Find the image of the point P(1, -2) after reflection in the line $y = 3x + 1$

Solution: Step 1: Slope of ℓ is $m = 3 \Rightarrow$ slope of PP' or ℓ' is $\frac{-1}{3}$

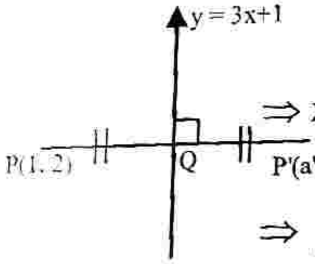
Step 2: Equation of ℓ' passing through P(1,-2) with slope $\frac{-1}{3}$

$$\Rightarrow y-(-2) = \frac{-1}{3}(x-1) \Rightarrow y+2 = \frac{-1}{3}(x-1)$$

$$\Rightarrow 3y + 6 = -x + 1 \Rightarrow x + 3y = -5$$

Step 3: The intersection of ℓ and ℓ'

$$\Rightarrow \begin{cases} y = 3x+1 \\ x+3y = -5 \end{cases}$$



$$\Rightarrow x + 3(3x+1) = -5 \Rightarrow 10x = -8 \Rightarrow x = \frac{-4}{5}$$

$$\Rightarrow y = 3\left(\frac{-4}{5}\right)+1 = \frac{-7}{5}$$

$\therefore$ the intersection point $Q\left(\frac{-4}{5}, \frac{-7}{5}\right)$

Step 4: $Q\left(\frac{-4}{5}, \frac{-7}{5}\right)$ is the mid-point of PP', Let the image of P(1, -2) is P'(a', b')

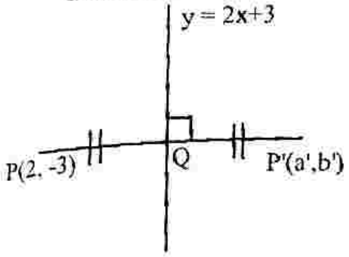
$$\Rightarrow \frac{1+a'}{2} = \frac{-4}{5} \text{ and } \frac{-2+b'}{2} = \frac{-7}{5}$$

$$\Rightarrow a' + 1 = \frac{-8}{5} \Rightarrow a' = \frac{-13}{5} \text{ and } b' = \frac{-14}{5} + 2 = \frac{-4}{5}$$

$\therefore$ The image of $M((1, -2)) = \left(\frac{-13}{5}, \frac{-4}{5}\right)$

63. Find the image of the circle $(x-2)^2 + (y+3)^2 = 4$ after reflection in the line $y = 2x + 3$.

Solution: The center of the given circle is (2, -3)



• slope of ℓ is $m = 2$

$\Rightarrow$ slope of ℓ' is $m' = \dfrac{-1}{2}$

• Equation of the line ℓ' passing through (2,-3) with slope $\dfrac{-1}{2}$ is:

$$\Rightarrow y + 3 = \frac{-1}{2}(x-2) \Rightarrow 2y + 6 = -x + 2$$

$$\Rightarrow x + 2y = -4$$

The intersection of ℓ and ℓ' is at mid pt of PP': $\begin{cases} y = 2x+3 \\ x+2y = -4 \end{cases}$

$\Rightarrow x + 2(2x+3) = -4 \Rightarrow x = -2$ and $y = -1$

Let the image of P(2, -3) is P' = (a', b')

• $Q(-2,-1) = \left(\dfrac{2+a'}{2}, \dfrac{-3+b'}{2}\right)$

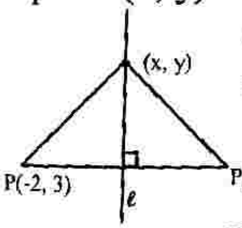
$\Rightarrow \dfrac{a'+2}{2} = -2 \Rightarrow a' = -6$ and $\dfrac{-3+b'}{2} = -1 \Rightarrow b' = 1$

$\therefore$ The image of M((2, -3)) = (-6, 1)

$\Rightarrow$ image of circle is $(x+6)^2 + (y-1)^2 = 4$

64. In a reflection, the image of the point P(-2, 3) is P'(6, 7). Find the equation of line of reflection.

Solution; Since ℓ is the perpendicular bisector of PP', the set of all point (x, y) in ℓ are equidistant to P(-2, 3) and P'(6, 7)



$$\Rightarrow (x+2)^2 + (y-3)^2 = (x-6)^2 + (y-7)^2$$

$$\Rightarrow x^2 + 4x + 4 + y^2 - 6y + 9$$

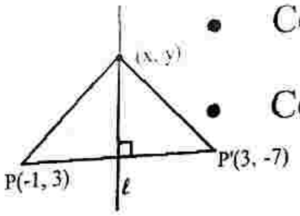
$$= x^2 - 12x + 36 + y^2 - 14y + 49$$

$$16x + 8y = 36 + 49 - 13 = 72$$

$\Rightarrow 2x + y = 9 \leftarrow$ equation of line of reflection

65. The image of the circle $x^2 + y^2 + 2x - 6y = 0$ when it is reflected about the line ℓ is $x^2 + y^2 - 6x + 14y = 0$. Find the equation of line of reflection:

Solution: First we determine the center of each circle



- Center of $x^2 + y^2 + 2x - 6y = 0$ is $(-1, 3)$
- Center of $x^2 + y^2 - 6x + 14y = 0$ is $\left(\frac{-6}{-2}, \frac{-14}{2}\right) = (3, -7)$

$\Rightarrow \ell$ is perpendicular bisector of PP', the set of all point (x, y) is ℓ are equidistant from P(-1, 3) and P'(3, -7)

$\Rightarrow (x+1)^2 + (y-3)^2 = (x-3)^2 + (y+7)^2$

$\Rightarrow x^2 + 2x + 1 + y^2 - 6y + 9 = x^2 - 6x + 9 + y^2 + 14y + 49$

$\Rightarrow 8x - 20y = 48 \Rightarrow 2x - 5y = 12$

$\therefore$ The equation of the line of reflection is $4x - 10y = 25$

66. Find the image of the line L: $y = 2x + 1$ after it has been reflected along the line ℓ: $y = 2x + 6$

Solution: since $L \| \ell$

$\Rightarrow L'$: $y = 2x + b$

Take (0, 1) on $y = 2x + 1$, the image of (0, 1) in $y = 2x + 6$

(-4, 3) i.e M((0, 1)) = (-4, 3) lies on L'

$\Rightarrow 3 = 2(-4) + b \Rightarrow b = 3 + 8 = 11$

$\Rightarrow L'$: $y = 2x + 11$ is the image of L: $y = 2x + 1$

67. Find the image of the line L: $y = 3x - 6$ after it has been reflected along the line ℓ: $y = x - 4$

Solution: Since L: $y = 3x - 6$ and ℓ: $y = x - 4$ are non-parallel

$$\text{The intersection of } \begin{cases} y = 3x - 6 \\ y = x - 4 \end{cases} \text{ is } (1, -3)$$

Then take point on L: $y = 3x - 6$ say (2, 0)

The image of (2, 0) on line of reflection $y = x - 4$

$\Rightarrow$ M((2, 0)) = (4, -2)

$\Rightarrow$ The equation of line L' passing through (4, -2) and (1, -3) is given by $y - y_1 = m(x, x_1)$

$$m = \frac{-2-(-3)}{4-1} = \frac{1}{3}$$

$$\Rightarrow y-(-2)=\frac{1}{3}(x-4)\Rightarrow y+2=\frac{1}{3}x-\frac{4}{3}\Rightarrow y=\frac{1}{3}x-\frac{10}{3}$$

$\therefore$ The image of $y = 3x - 6$ is $y=\frac{1}{3}x-\frac{10}{3}$

Double Transformation: A transformation. Translation followed by reflection or reflection followed by translation i.e. T(M(x, y)) or M(T(x,y)) or T(R(x, y)) or R(M(x, y)) ....etc.

- T(M(x, y)) ≡ Reflection of point followed by translation T.
- M(T(x, y)) ≡ Translation of point followed by reflection M.
- M(R(x, y)) ≡ Rotation R followed by reflection M.

Note: T(M(x, y)) ≠ M(T(x, y))

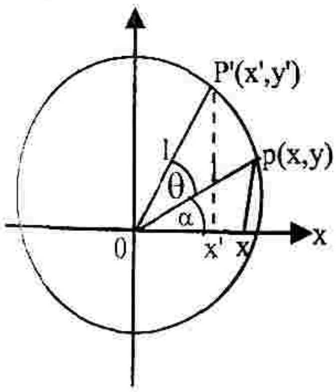
Rotation

A **rotation R** of θ about a point 0, is called center of rotation and θ is called an amount of rotation.

- If θ > 0, then rotation is counter clock wise direction
- If θ < 0, then rotation is clock wise direction

Rotation about the origin

Let R be a rotation through an angle θ about the origin then the image of P(x, y) obtained by formula



x' = xcosθ - ysinθ

y' = xsinθ + ycosθ

To proof: consider the following unit circle

⇒ OP = OP' = 1 (radius of unit circle)

$$\Rightarrow \cos\alpha=\frac{x}{OP}=\frac{x}{1}=x\Rightarrow x=\cos\alpha$$

$$\Rightarrow \sin\alpha=\frac{y}{OP}=\frac{y}{1}=1\Rightarrow y=\sin\alpha$$

In angle P'OA'(α + θ)

$$\cos(\alpha+\theta)=\frac{x'}{OP'}=\frac{x'}{1}=x'$$

x' = cos(α + θ) = cosαcosθ - sinαsinθ = xcosθ - ysinθ

$\Rightarrow \boxed{x' = x\cos\theta - y\sin\theta}$

$$\text{And } \sin(\alpha+\theta)=\frac{y'}{OP'}=\frac{y'}{1}\Rightarrow y'=\sin(\alpha+\theta)$$

⇒ y' = sinαcosθ + cosαsinθ = ycosθ + xsinθ

$\Rightarrow \boxed{y' = x\sin\theta + y\cos\theta}$

The six Basic rotation

i) If R is Rotation through $\theta = \frac{\pi}{2}$ **about (0, 0)**

$$R((x, y)) = (x\cos\frac{\pi}{2} - y\sin\frac{\pi}{2}, x\sin\frac{\pi}{2} + y\cos\frac{\pi}{2})$$

$$R_{\frac{\pi}{2}}(x, y) = (-y, x)$$

ii) **If R is rotation through $\theta = \frac{-\pi}{2}$ about(0, 0)**

$$\Rightarrow R((x, y)) = \left(x\cos\left(\frac{-\pi}{2}\right) - y\sin\left(\frac{-\pi}{2}\right), x\sin\left(\frac{-\pi}{2}\right) + y\cos\left(\frac{-\pi}{2}\right)\right)$$
$$= (y, -x)$$

iii) **If R is rotation through $\theta = \pi$ about (0, 0)**

$$\Rightarrow R((x,y)) = (x\cos\pi - y\sin\pi, x\sin\pi + y\cos\pi)$$
$$= (-x, -y)$$

iv) **If R is rotation through $\theta = \frac{3}{2}\pi$ about (0, 0)**

$$\Rightarrow R((x, y)) = \left(x\cos\frac{3}{2}\pi - y\sin\frac{3}{2}\pi, x\sin\frac{3}{2}\pi + y\cos\frac{3}{2}\pi\right)$$
$$= (0 - y(-1), x(-1) + 0) = (y, -x)$$

v) **If R is rotation through $\theta = \frac{-3}{2}\pi$ about (0, 0)**

$$\Rightarrow R((x,y)) = \left(x\cos\left(\frac{-3}{2}\pi\right) - y\sin\left(\frac{-3}{2}\pi\right), x\sin\left(\frac{-3}{2}\pi\right) + y\cos\left(\frac{-3}{2}\pi\right)\right)$$
$$= (0 - y(1), x(1) + y(0) = (-y, x)$$

Note: i) $R_{90^0} = R_{-270^0}$

ii) $R_{-90^0} = R_{270^0}$

iii) $R_{120^0} = R_{-240^0}$

iv) $R_{180^0} = R_{-180^0}$

v) $R_{90^0} = R_{450^0}$

vi) **If R is rotation through $\theta = 2\pi k, k \in z$, about (0, 0)**
$R((x,y)) = (x, y)$ …….. which is called an identity transformation

1. Find the image of each part, in rotation through the indicated angle about the origin.

a) $(4, 2),\ \theta = \frac{\pi}{3}$ d) $3x - 2y = 8,\ \theta = \frac{3}{2}\pi$

b) $(2, -3),\ \theta = \frac{-\pi}{6}$ e) $3x + 4y = -5$, with $\tan\theta = \frac{-3}{4}$

c) $(-1, 4),\ \theta = \frac{\pi}{4}$ for $\frac{\pi}{2} \le \theta \le \pi$

f) $x^2 + y^2 - 4x + 2y = 0,\ \theta = \frac{-3\pi}{2}$

Solution: use $x' = x\cos\theta - y\sin\theta$, $y' = x\sin\theta + y\cos\theta$

a) $x = 4, y = 2,\ \theta = \frac{\pi}{3}$

$$\Rightarrow x' = 4\cos\frac{\pi}{3} - 2\sin\frac{\pi}{3} = 4\left(\frac{1}{2}\right) - 2\left(\frac{\sqrt{3}}{2}\right) = 2 - \sqrt{3}$$

$$\Rightarrow y' = 4\sin\frac{\pi}{3} + 2\cos\frac{\pi}{3} = 4\left(\frac{\sqrt{3}}{2}\right) + 2\left(\frac{1}{2}\right) = 2\sqrt{3} + 1$$

$$\therefore R((4, 2)) = (2 - \sqrt{3}, 2\sqrt{3} + 1)$$

b) $x = 2, y = -3,\ \theta = \frac{-\pi}{6}$

$$\Rightarrow x' = 2\cos\left(\frac{-\pi}{6}\right) - 3\sin\left(\frac{-\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2}\right) + \frac{3}{2} = \frac{2\sqrt{3} + 3}{2}$$

$$\Rightarrow y' = 2\sin\left(\frac{-\pi}{6}\right) + 3\cos\left(\frac{-\pi}{6}\right) = 2\left(\frac{-1}{2}\right) + 3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3} - 2}{2}$$

$$\therefore R((2, -3)) = \left(\frac{2\sqrt{3} + 3}{2}, \frac{3\sqrt{3} - 2}{2}\right)$$

c) $x = -1, y = 4, \theta = \frac{\pi}{4}$

$$\Rightarrow x' = -1\cos\frac{\pi}{4} - 4\sin\frac{\pi}{4} = \frac{-\sqrt{2}}{2} - \frac{4\sqrt{2}}{2} = \frac{-5\sqrt{2}}{2}$$

$$\Rightarrow y' = -1\sin\frac{\pi}{4} + 4\sin\frac{\pi}{4} = \frac{-\sqrt{2}}{2} + \frac{4\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}$$

$$\therefore R((-1, 4)) = \left(\frac{-5\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$$

d) Take two point arbitrary on $3x - 2y = 8$

let $x = 2$, $y = (-1) \Rightarrow (2, -1)$ and $(0, -4)$, $\theta = \frac{3}{2}\pi$

To find image of (2, -1)

$$\Rightarrow x' = 2\cos\left(\frac{3\pi}{2}\right) - (-1)\sin\left(\frac{3\pi}{2}\right) = 2(0) - (1) = -1$$

$$\Rightarrow y' = 2\sin\left(\frac{3\pi}{2}\right) + (-1)\cos\left(\frac{3\pi}{2}\right) = 2(-1) + 0 = -2$$

$\therefore R((2, -1)) = (-1, -2)$ and

To find image of (0, -4)

$$\Rightarrow x' = 0\cos\frac{3\pi}{2} - (-4)\sin\frac{3\pi}{2} = 0 - 4 = -4$$

$$\Rightarrow y' = 0\sin\frac{3\pi}{2} + (-4)\cos\frac{3\pi}{2} = 0(-1) + (-4)(0) = 0$$

$\therefore ((R(0, -4)) = (-4, 0)$, Thus

Equation of the line passing through $(-1, -2)$ and $(-4, 0)$ is given by

$$y - y' = m(x - x_1) \Rightarrow y - (-2) = \frac{-2-0}{-1-(-4)}(x - (-1))$$

$$\Rightarrow y + 2 = \frac{-2}{3}(x+1) \Rightarrow 3y + 6 = -2x - 2 \Rightarrow 2x + 3y = -8$$

$\therefore$ The image of the line is $2x + 3y = -8$

e) First determine sinθ and cosθ from $\tan\theta = \frac{-3}{4}$

$$\Rightarrow \sin\theta = \frac{3}{5} \text{ and } \cos\theta = \frac{-4}{5}$$

- Take two point arbitrarly on $3x + 4y = -5$

let $x = 5, y = -5 \Rightarrow (5, -5)$ and

let $x = 1, y = -2 \Rightarrow (1, -2)$

To find image of the point (5, -5)

$$\Rightarrow x' = 5\left(\frac{-4}{5}\right) - (-5)\left(\frac{3}{5}\right) = -4 + 3 = 1$$

$$y' = 5\left(\frac{3}{5}\right) + (-5)\left(\frac{-4}{5}\right) = 3 + 4 = 7$$

$$\therefore R((5, -5)) = (-1, 7)$$

To find image of the point (1, -2)

$$\Rightarrow x' = 1\left(\frac{-4}{5}\right) - (-2)\left(\frac{3}{5}\right) = \frac{-4}{5} + \frac{6}{5} = \frac{2}{5}$$

$$\Rightarrow y' = 1\left(\frac{3}{5}\right) + (-2)\left(\frac{-4}{5}\right) = \frac{3}{5} + \frac{8}{5} = \frac{11}{5}$$

$$\therefore R((1, -2)) = \left(\frac{2}{5}, \frac{11}{5}\right)$$

The equation of the line passing through (-1, 7) and $\left(\frac{2}{5}, \frac{11}{5}\right)$ is:

$$m = \frac{\frac{11}{5} - 7}{\frac{2}{5} + 1} = \frac{-24}{7}$$

$$\Rightarrow y - 7 = \frac{-24}{7}(x + 1) \Rightarrow y = \frac{-24}{7}x + \frac{25}{7}$$

$\therefore 24x + 7y = 25$ is the image of $3x + 4y = -8$

f) center of $x^2 + y^2 - 4x + 2y = 0$ is (2, -1), radius = $\sqrt{5}$

$$x = 2, y = -1, \ \theta = \frac{-3\pi}{2}$$

$$\Rightarrow x' = 2\cos\left(\frac{-3\pi}{2}\right) - (-1)\sin\left(\frac{-3\pi}{2}\right) = 0 - 1 = -1$$

$$\Rightarrow y' = 2\sin\left(\frac{-3\pi}{2}\right) + -1\cos\left(\frac{-3\pi}{2}\right) = -2 - 0 = -2$$

$\therefore R((2, -1)) = (-1, -2)$

$\therefore$ The image of circle is $(x+1)^2 + (y+2)^2 = 5$

$\Rightarrow x^2 + y^2 + 2x + 4y = 0$

Rotation about any point (a, b)

If P'(x', y') is the image of a point P(x, y) when rotated by an angle θ about (a, b), then

$$x' = a + (x - a)\cos\theta - (y - b)\sin\theta$$
$$y' = b + (x - a)\sin\theta + (y - b)\cos\theta$$

- To rotate P(x, y) in an angle θ about (a, b) we have the following step,

Step 1: Translate $(a, b) \longrightarrow (0, 0)$

- Translating vector $\vec{v} = (-a, -b)$

Step 2: Translate $(x, y) \xrightarrow{\vec{v}=(-a,-b)} (x - a, y - b)$

Step 3: Rotate, (x-a, y-b) by θ about (0,0) then translate by (-v)=(a,b)

72. Find the image of P(3,1) when it is rotated through $\theta = \frac{\pi}{2}$ about (-2, 3)

Method I

- Translate $(-2, 3) \longrightarrow (0, 0)$, $\vec{v} = (2, -3)$
- Translate $(3, 1) \xrightarrow{\vec{v}=(2,-3)} (5, -2)$
- Rotate (5, -2) by $\theta = \frac{\pi}{2}$ about (0, 0)

$$x = 5, y = -2, \ \theta = \frac{\pi}{2}$$

$$x' = 5\cos\frac{\pi}{2} - (-2)\sin\frac{\pi}{2} = 0 + 2 = 2$$

$y' = 5\sin\frac{\pi}{2} + (-2)\cos\frac{\pi}{2} = 5 + 0 = 5$

R((5, -2)) = (2, 5)

- Translate (2, 5) by $(-\vec{v}) = (-2, 3)$

T((2, 5)) = (2 + (-2), 5 + 3) = (0, 8)

∴ The image of the point (3, 1) about (-2, 3) with an angle $\theta = \frac{\pi}{2}$ is (0, 8)

i.e. R((3, 1)) = (0, 8) about (-2, 3) with $\theta = \frac{\pi}{2}$

Method II

$\Rightarrow x' = a+(x-a)\cos\theta - (y-b)\sin\theta = -2+(3-(-2))\cos\frac{\pi}{2} - (1-3)\sin\frac{\pi}{2} = 0$

$\Rightarrow y' = b+(x-a)\sin\theta + (y-b)\cos\theta = 3+(3+2)\sin\frac{\pi}{2} + (1-3)\cos\frac{\pi}{2} = 8$

∴ R((3, 1)) = (0, 8)

73. Find the equation of the line x + y = 0 after it has been rotated by $\theta = 270^0$ about (3, 2)

Solution:

- Choose any two arbitrary points, say (-1, 1) and (2, -2)
- Center of rotation, (a, b) = (3, 2) and $\theta = 270^0$

To find image of R((-1, 1)) about (a, b) = (3, 2)

$\Rightarrow x' = a + (x - a)\cos\theta - (y - b)\sin\theta$

$= 3 + (-1 - 3)\cos(270^0) - (1 - 2)\sin 270^0 = 3 + (-1) = 2$

$\Rightarrow y' = b + (x - a)\sin\theta + (y - b)\cos\theta$

$= 2 + (-1 - 3)\sin(270^0) + (1 - 2)\cos 270^0 = 6$

∴ R((-1, 1)) = (2, 6)

To find image of (2, -2) about (a, b) = (3, 2)

$x' = 3 + (2 - 3)\cos 270^0 - (-2 - 2)\sin 270^0 = 3 - 4 = -1$

$y' = 2 + (2 - 3)\sin 270^0 + (-2 - 2)\cos 270^0 = 2 + 1 = 3$

∴ R((2, -2)) = (-1, 3) thus,

Equation of line passing through (2, 6) and (-1, 3).

$\Rightarrow m = \frac{6-3}{2-(-1)} = \frac{3}{3} = 1$ use $y - y_1 = m(x - x_1)$

$\Rightarrow y - 6 = 1(x - 2) \Rightarrow y = x + 4 \Leftrightarrow -x + y = 4$

$\therefore$ Image of the line is y − x = 4

Note: In a rotation R, the image of A is A' and the image of B is B' then the intersection point of the perpendicular bisector of $\overline{AA'}$ and $\overline{BB'}$ is the center of rotation.

Illustrative Example

74. In rotation R, the image of A(3, 5) is A'(-2, 4) and the image of B(1, 4) is B'(-1, 2)

Find: a) center of rotation

b) the amount of rotation (θ)

c) the image of (-1, 0)

Solution: To determine center of rotation determine the intersection of the perpendicular bisector of $\overline{AA'}$ and $\overline{BB'}$

a)

- Mid-point of $\overline{AA'} = \left(\frac{3+(-2)}{2}, \frac{5+4}{2}\right) = \left(\frac{1}{2}, \frac{9}{2}\right)$
- Mid-point of $\overline{BB'} = \left(\frac{1+(-1)}{2}, \frac{4+2}{2}\right) = (0,3)$
- Slope of AA' $= \frac{4-5}{-2-3} = \frac{-1}{-5} = \frac{1}{5}$
- Slope of $\perp$ bisector of $\overline{AA'}$ is -5

$\Rightarrow$ Equation of $\perp$ bisector of $\overline{AA'}$ is $y - \frac{9}{2} = -5\left(x - \frac{1}{2}\right)$

$\Rightarrow$ y = -5x + 7 $\Rightarrow$ y + 5x = 7

- Slope of $\overline{BB'} = \frac{1-(-1)}{4-2} = \frac{2}{2} = 1$
- Slope of perpendicular bisector of BB' is -1

$\Rightarrow$ Equation of $\perp$ bisector of $\overline{BB'}$ passing through (0, 3)

$\Rightarrow$ y − 3 = -1(x − 0) $\Rightarrow$ y = -x + 3

Center of rotation is intersection of $\begin{cases} y + 5x = 7 \\ y = -x + 3 \end{cases}$

$\Rightarrow$ -x + 3 + 5x = 7 $\Rightarrow$ 4x = 4 $\Rightarrow$ x = 1 and y = 2

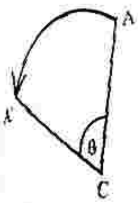
$\therefore$ center of rotation (1, 2)

b) **To determine amount of rotation θ**

We use vector dot product between angle $\overrightarrow{AC}$ and $\overrightarrow{A'C}$ or $\overrightarrow{BC}$ and $\overrightarrow{B'C}$ where C is center of rotation.

$\Rightarrow \overrightarrow{AC} = (1, 2) - (3, 5) = (-2, -3)$

$\Rightarrow \overrightarrow{A'C} = (1, 2) - (-2, 4) = (3, -2)$



$$\cos\theta = \frac{\overrightarrow{AC}\cdot\overrightarrow{A'C}}{|\overrightarrow{AC}||\overrightarrow{A'C}|} = \frac{(-2,-3)\cdot(3,-2)}{\sqrt{4+9}\sqrt{9+4}} = \frac{-6+6}{13} = 0$$

$$\Rightarrow \theta = \cos^{-1}(0) = \frac{\pi}{2}$$

$\therefore$ The amount of rotation is $\theta = \dfrac{\pi}{2}$ about (1, 2)

c) The image of (-1, 0) when we rotate by about (1, 2) is obtained by:

$x' = a + (x - a)\cos\theta - (y - b)\sin\theta$

$\Rightarrow x' = 1 + (-1 - 1)\cos\dfrac{\pi}{2} - (0 - 2)\sin\dfrac{\pi}{2} = 1 + 2(1) = 3$

$y' = b + (x - a)\sin\theta + (y - b)\cos\theta$

$\Rightarrow y' = 2 + (-1 - 1)\sin\dfrac{\pi}{2} + (0 - 2)\cos\dfrac{\pi}{2} = 2 + (-2) = 0$

$\therefore R((-1, 0)) = (3, 0)$

75. If M is reflection in the line y = 2x and R is rotation about the origin through $\theta = -270^0$. Find

a) R(M(3, 2)) b) M(R(3, 2))

c) Does R(M((x, y)) = M(R((x, y)))

Solution: $y = 2x, \Rightarrow \tan\theta = 2$

$$\Rightarrow \sin\theta = \frac{2}{\sqrt{5}}, \cos\theta = \frac{1}{\sqrt{5}}$$

$$\Rightarrow \sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right) = \frac{4}{5}$$

$$\Rightarrow \cos 2\theta = \cos^2\theta - \sin^2\theta = \left(\frac{1}{\sqrt{5}}\right)^2 - \left(\frac{2}{\sqrt{5}}\right)^2 = \frac{1}{5} - \frac{4}{5} = \frac{-3}{5}$$

$\Rightarrow x' = x\cos 2\theta + y\sin 2\theta \Rightarrow 3\left(\frac{-3}{5}\right) + 2\left(\frac{4}{5}\right) = \frac{-1}{5}$

$\Rightarrow y' = x\sin 2\theta - y\cos 2\theta \Rightarrow 3\left(\frac{4}{5}\right) - 2\left(\frac{-3}{5}\right) = \frac{18}{5}$

$\therefore R(M((3, 2)) = R\left(\frac{-1}{5}, \frac{18}{5}\right) = \left(\frac{-18}{5}, \frac{-1}{5}\right)$

b) M(R(3, 2) = M(-2, 3)

$x' = -2\left(\frac{-3}{5}\right) + 3\left(\frac{4}{5}\right) = \frac{6}{5} + \frac{12}{5} = \frac{18}{5}$

$y' = -2\left(\frac{4}{5}\right) + (-3)\left(\frac{-3}{5}\right) = \frac{-8}{5} + \frac{9}{5} = \frac{1}{5}$

$\therefore M(R((3, 2)) = M(-2,3) = \left(\frac{18}{5}, \frac{1}{5}\right)$

c) $R(M(x, y)) \neq M(R)(x, y))$

Solved Problem

76. If $\vec{a} = 2i - 3j$, $\vec{b} = 3i - j$ and $\vec{c} = i - j$ then which of the following is equal to $\vec{a} .(2\vec{b} - \vec{c})$

A) -10B. 7 C. 10 D. 13

Solution: $\vec{a} .(2\vec{b} - \vec{c}) = (2i - 3j)\cdot(5i - j) = 10 + 3 = 13$

77. If A = (2, 3), B = (3, 4) and c = (-1, -2) then the dot product of $\overrightarrow{AB}$ and $\overline{AC}$

A. -10 B. -2 C. 2 D. -8

Solution:

$\overrightarrow{AB} = (3, 4) - (2, 3) = (1,1)$ and $\overrightarrow{AC} = (-1,-2)-(3,4) = (-3,-5)$

$\Rightarrow \overrightarrow{AB} \cdot \overrightarrow{AC} = (1,1)\cdot(-3, -5) = -3 + -5 = -8$

78. Let $\vec{u} = (-1, 10)$ and $\vec{v} = (1, 3)$. A unit vector in the direction $\vec{u} - 2\vec{v}$ is ….(UEE)

A. $\left(\frac{-3}{5}, \frac{-4}{5}\right)$ B. $\left(\frac{-3}{5}, \frac{4}{5}\right)$ C. $\left(\frac{3}{5}, \frac{-4}{5}\right)$ D. $\left(\frac{3}{5}, \ldots\right)$

Solution: $\vec{u} - 2\vec{v} = (-1, 10) - (2, 6) = (-3, 4)$

Unit vector in the direction of (-3, 4) = $\frac{(-3,4)}{|(-3,4)|} = \frac{(-3,4)}{\sqrt{3^2+4^2}}$

$$= \left(\frac{-3}{5}, \frac{4}{5}\right)$$

79. If $(i + 5j) - \vec{x} = (3i + 2j) - 2(i - 3j)$ then the norm of the vector $\vec{x}$ is

A. 11 B. $\sqrt{10}$ C. 3 D. $\sqrt{3}$

Solution: $\vec{x} = i + 5j - 3i - 2j + 2i - 6j = -3j \Rightarrow |\vec{x}| = \sqrt{(-3)^2} = 3$

80. If the magnitude of vector $\vec{v}$ is 15 and its direction is 120^0 from the positive x-axis. What are the vector $\vec{v}$ ..... (UEE)

A. $\left(\frac{-15}{2}, \frac{15\sqrt{3}}{2}\right)$ C. $\left(\frac{-15}{2}, 15\sqrt{2}\right)$

B. $\left(10\sqrt{3}, \frac{15}{2}\right)$ D. $\left(12\sqrt{3}, \frac{-15}{\sqrt{2}}\right)$

Solution: $\cos 120^0 = \frac{a}{15} \Rightarrow a = 15\cos 120^0 = \frac{-15}{2}$

$$\sin 120^0 = \frac{b}{15} \Rightarrow b = 15\sin 120^0 = \frac{15\sqrt{3}}{2}$$

$$\therefore \vec{v} = (a,b) = \left(\frac{-15}{2}, \frac{15\sqrt{3}}{2}\right)$$

Answer: A

81. If $|\vec{a}| = 4$, $|\vec{b}| = \sqrt{3}$ and $\vec{a} \cdot \vec{b} = -3$ then the magnitude of $\vec{a} - \vec{b}$ .....EHEECE

A. 5 B. 7 C. $\sqrt{13}$ D. $\sqrt{22}$

Solution: $|\vec{a} - \vec{b}| = \sqrt{(\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})} = \sqrt{|\vec{a}|^2 - 2\vec{a} \cdot \vec{b} + |\vec{b}|^2}$

$$= \sqrt{4^2 - 2(-3) + \sqrt{3}^2} = \sqrt{16 + 6 + 3} = \sqrt{25} = 5$$

82. If $\vec{u} = -i + 2j$, $\vec{v} = -3i + j$ and $\vec{w} = -3j$ then the cosine of the angle between $\vec{v} + \vec{w}$ and $\vec{v}$

A. $\frac{1}{\sqrt{5}}$ B. $\frac{1}{2}$ C. $\frac{4}{\sqrt{20}}$ D. 0

Solution: $\cos\theta = \frac{(\vec{v}+\vec{w})\cdot(\vec{v})}{|\vec{v}+\vec{w}||\vec{v}|} = \frac{(-i-j)\cdot(-3+j)}{\sqrt{2}\sqrt{10}} = \frac{2}{\sqrt{20}} = \frac{1}{\sqrt{5}}$

83. Let $\vec{b}$ is unit vector, and $|\vec{a}| = 3$. The vector $\vec{a}$ and $\vec{b}$ makes an angle $\theta = \frac{\pi}{3}$. What is $|\vec{a} - 2\vec{b}|$

A. $\sqrt{5}$ B. $\sqrt{7}$ C. $\sqrt{8}$ D. $\sqrt{19}$

Solution: $|\vec{a} - 2\vec{b}| = \sqrt{|a|^2 - 4|\vec{a}||\vec{b}|\cos\theta + 4|b|^2}$, $|\vec{b}| = 1$

$$= \sqrt{9 - 4(3)(1)\left(\frac{1}{2}\right) + 4(1)} = \sqrt{13-6} = \sqrt{7}$$

84. Suppose that a triangle has vertices A = (1, 2), B = (3, 4) and C = (2, 5). What is the interior angle of triangle ABC at vertex B.

A. 30^0 B. 45^0 C. 60^0 D. 90^0

Solution: Position vector of $\overrightarrow{BA} = (1, 2) - (3, 4) = (-2, -2)$ and $\overrightarrow{BC} = (2, 5) - (3, 4) = (-1, 1)$

$$\Rightarrow \cos\beta = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} = \frac{(-2,-2)\cdot(-1,1)}{\sqrt{8}\sqrt{2}} = \frac{2-2}{\sqrt{16}} = \frac{0}{4} = 0$$

$\Rightarrow B = \cos^{-1}(0) = 90^0$,

Answer: D

85. What is vector equation of the line passing through (-1, 4) and parallel to the line y – 3x – 10 = 0…UEE

A. (x,y) = (0, -7) + t(1,3) C) (x,y) = (0,7) + t(1, -3)

B. (x,y) = (1, 10) + t(1, 3) D) (x, y) = (-1, 4) + t(1, 3)

Solution: $y - 3x - 10 = 0 \Rightarrow y = 3x + 10$

Direction vector $\vec{v}$ is obtained by setting y = 3x

$\Rightarrow \vec{v} = (a, b) = (a, 3a) = a(1, 3) = t(1, 3)$

$\Rightarrow (x, y) = (-1, 4) + t(1, 3)$

Answer: D

86. Suppose that the equation of a line ℓ is given by ℓ: $2x-3y-7=0$. What is vector equation of ℓ.

A. $(x, y) = (0, -7) + t(2, -3)$ C. $(x,y) = (-3, -4) + t(1, -6)$

B. $(x, y) = (2, -1) + t(3, 2)$ D. $(x, y) = (-4, 2) + t(1, 5)$

Solution: $2x - 3y + 7 = 0 \Rightarrow 3y = 2x - 7 \Rightarrow y = \frac{2}{3}x - \frac{7}{3}$

$\Rightarrow$ Direction vector $\vec{v}$ parallel to ℓ is $y = \frac{2}{3}x$

$\Rightarrow \vec{v} = (a, b) = \left(a, \frac{2}{3}a\right) = a\left(1, \frac{2}{3}\right) = t(3,2)$

$\therefore$ Take any point on: $y = \frac{2}{3}x - \frac{7}{3}$, say $(2, -1)$

$\Rightarrow (x, y) = (2, -1) + t(3, 2)$ **Answer: B**

87. Let $(x, y) = (2, 5) + t(1, -2)$ is vector equation of ℓ. What is the standard equation of ℓ

A. $y + 2x = 9$ C. $y = 3x - 5$

B. $3y - 4x = 3$ D. $-2y + x = 3$

Solution: $(x, y) = (2, 5) + t(1, -2) \Rightarrow x = 2 + t, y = 5 - 2t$

Solving for, $t = \frac{x-2}{1} = \frac{y-5}{-2} \Rightarrow y + 2x = 9$, **Answer: A**

89. If $\vec{u} = 2i - j$, $|\vec{v}| = 5$ and $\vec{u}$ and $\vec{v}$ are perpendicular vectors then $|\vec{u} + 3v|$ ....

A. $\sqrt{55}$ B. $9\sqrt{5}$ C. $\sqrt{230}$ D. 20

Solution: $|\vec{u} + 3\vec{v}| = \sqrt{(\vec{u} + 3\vec{v}) \cdot (\vec{u} + 3\vec{v})}$

$= \sqrt{|\vec{u}|^2 + 6|\vec{u}||\vec{v}|\cos\theta + 9|\vec{v}|^2}$

$= \sqrt{\sqrt{5}^2 + 6(\sqrt{5})(5)\cos 90^0 + 9(5)^2} = \sqrt{230}$

90. If $A\left(-\sqrt{3},3\right)$, $B(0, 0)$ and $C\left(-2\sqrt{3},0\right)$ are vertices of triangle, then what is the interior angle at C

A. 30^0 B. 45^0 C. 60^0 D. 135^0

Solution: $\overrightarrow{CA} = \left(-\sqrt{3},3\right) - \left(-2\sqrt{3},0\right) = \left(\sqrt{3},3\right)$

$\overrightarrow{CB} = (0,0) - \left(-2\sqrt{3},0\right) = \left(2\sqrt{3},0\right)$

$$\Rightarrow \cos(c) = \frac{\overrightarrow{CA}\cdot\overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|} = \frac{(\sqrt{3},3)\cdot(2\sqrt{3},0)}{\sqrt{3+9}\sqrt{12+0}} = \frac{6}{\sqrt{144}} = \frac{6}{12} = \frac{1}{2}$$

$$\Rightarrow C = \cos^{-1}\left(\frac{1}{2}\right) = 60^0,$$

Answer: C

91. Let ℓ be a line given by $x - 2y = 0$ parametric equation of the line passes through (5, 0) and perpendicular to ℓ is ….. UEE
A. $(x, y) = (5, 0) + t(1, 2)$ C. $(x, y) = (5 + t, -2t), t\in(-\infty,\infty)$
B. $(x,y) = (5,0)+t(1,-2), t\geq 0$ D. $(x,y) = (5 - t, 2t), [0, \infty)$

Solution: $x - 2y = 0 \Rightarrow x = 2y \Rightarrow y = \frac{1}{2}x \Rightarrow \left(x, \frac{1}{2}x\right) = t(2,1)$

Let $\vec{v} = (a, b) \Rightarrow (a, b)\cdot(2, 1) = 0$
$\Rightarrow 2a + b = 0 \Rightarrow b = -2a \Rightarrow (a, b) = (a, -2a)$
$\Rightarrow \vec{v} = t(1, -2)$
$\therefore$ Vector equation $(x, y) = (5, 0) + t(1, -2)$
$\Rightarrow$ Parametric equation $(x, y) = (5+ t, -2t)$

Answer: C

92. A translation takes $x^2 + (y + 1)^2 = 5$ to $(x - 2)^2 + y^2 = 5$ what is the image of (1, 3) under this transformation (UEE)
A. (3, 4) B. (3, 2) C. (-1, 2) D. (-2, 3)

Solution:
Center of $x^2+(y+1)^2 =5$ is (0, -1) center of $(x - 2)^2 +y^2 =5$ is (2, 0)
$T(0, -1) = (2, 0)$, $\vec{v} = (2, 1)$
$T(1, 3) = (2 + 1, 3 + (1)) = (3, 4)$

Answer: A

93. A circle that passes through A = (2, 3) and center at C = (1, 1). The equation of the line tangent to the circle at A is ….(UEE)
A. $3x + y = 9$ C. $3x - 2y = 0$
B. $x + 2y = 8$ D. $x+y = 5$

Solution: Let R(x, y) be point on ℓ

$$\Rightarrow \overrightarrow{CA}\cdot\overrightarrow{AR} = 0 \Rightarrow ((2,3)-(1,1))\cdot(x-2, y-3)$$
$$\Rightarrow (1, 2)\cdot(x - 2, y - 3) = x - 2 + 2y - 6 = x + 2y = 8$$

Answer

94. A translation T takes point P(-1, 0) to point Q(1, 3). What is the image of the ellipse $9(x - 1)^2 + 4(y + 2)^2 = 36$
A. $9(x - 3)^2 + 4(y + 2)^2 = 36$ C. $9(x + 1)^2 + 4y^2 = 36$
B. $9(x + 3)^2 + 4(y - 1)^2 = 36$ D. $9(x - 2)^2 + 4(y - 1)^2 = 36$

Solution: Translating vector $\vec{v} = \overrightarrow{PQ} = (1, 3) - (-1, 0) = (2, 3)$

move 2 unit to right and 3 unit up $\Rightarrow (x - 2, y - 3)$

$\Rightarrow 9(x - 2 - 1)^2 + 4(y - 3 + 2)^2 = 36 \Rightarrow 9(x - 3)^2 + 4(y - 1)^2 = 36$

Answer: A

95. In a translation T the image of P(1, -1) is P'(3, 2). If a line ℓ, its parametric equation is given by $(x, y) = (2, -1) + t(1, 1)$ for are $t \in R$ is translated by T, what is image of ℓ.....(UEE)

A. $y = x + 3$ C. $x - y = 2$

B. $2x + y = 1$ D. $3y - x = 3$

Solution: Translating vector $\vec{v} = \overrightarrow{PP'} = (3 - 1, 2 - (-1)) = (2, 3)$

$\Rightarrow (x, y) = (2 + t, -1 + t) \Rightarrow x - 2 = y + 1 \Rightarrow x - y = 3$

Image, move 2 unit to left and 3 unit up

i.e. $(x - 2, y - 3)$

$\Rightarrow x - 2 - (y - 3) = 3 \Rightarrow x - y + 1 = 3 \Rightarrow x - y = 2$

Answer: C

96. If T is translation that sends (0, 0) to (3, -2) and M is reflection that maps (0, 0) to (2, 4) then what is T(M(1, 3))?

A. (0, -1) C. $\left(\frac{3}{2}, 4\right)$

B. (4, 1) D. (3, 2)

Solution: Slope of MM' $= \dfrac{4-0}{2-0} = 2 \Rightarrow$ slop of $\ell \Rightarrow \dfrac{-1}{2}$

Mid-point of MM' = (1, 2)

$\Rightarrow$ Equation of ℓ $\dfrac{y-2}{x-1} = \dfrac{-1}{2} \Rightarrow y - 2 = -\dfrac{1}{2}x + \dfrac{1}{2}$

$\Rightarrow y = -\dfrac{1}{2}x + \dfrac{5}{2} \longrightarrow$ line of reflection

$M(1, 3) \xrightarrow{y=-\frac{1}{2}x+\frac{5}{2}} (-3, 1)$

$T(M(1, 3)) = (-3, 1) = \left(3 + (-3), 1 + (-2)\right) = (0, -1)$

Answer: A

97. If R is rotation through $\theta = 90^0$ and is followed by reflection on the x-axis what is the image of ellipse given by $\frac{1}{4}(x-1)^2 + (y+2)^2 = 1$ under the two transformation

A. $4(x - 1)^2 + (y + 1)^2 = 4$ C. $(x - 1)^2 + 4(y + 2)^2 = 4$

B. $4(x - 2)^2 + (y - 1)^2 = 4$ D. $(x + 1)^2 + 4(y - 2)^2 = 4$

Solution:

Center $(1, -2) \xrightarrow{R_{90^0}} (2, 1) \longrightarrow$ Rotation with $\theta = 90^0$

$M(2, 1) = (2, -1) \longrightarrow$ reflection on x-axis

x-ellipse become y-ellipse.

$\Rightarrow$ Image will be $4(x - 2)^2 + (y + 1)^2 = 4$.

Answer: A

98. Determine the value of m for which the vectors $\vec{a} = -2i + 2j + mj$ and $\vec{b} = 3i + 6j$ are i) Perpendicular ii) parallel

Solution: a) $\vec{a} \perp \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = 0$

$\Rightarrow (-2i + (2 + m)j) \cdot (3i + 6j) = -6 + 12 + 6m = 0$

$\Rightarrow 6m = -6 \Rightarrow m = -1$

b) $\vec{a} \parallel \vec{b} \Rightarrow \cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|a||b|} = \pm 1$

$$\Rightarrow \frac{(-2i + (2+m)j) \cdot (3i + 6j)}{\sqrt{4 + (2+m)^2}\sqrt{3^2 + 6^2}} = \pm 1$$

$$\Rightarrow \frac{6m + 6}{3\sqrt{5m^2 + 20m + 40}} = \pm 1$$

$$\Rightarrow \frac{2m + 2}{\sqrt{5m^2 + 20m + 40}} = \pm 1$$

$\Rightarrow (2m + 2)^2 = 5m^2 + 20m + 40 \longleftarrow$ squaring and simplifying

$\Rightarrow m^2 + 12m + 36 = 0 \Leftrightarrow (m+6)(m+6) = 0$

$\Rightarrow m = -6$

99. Find unit vector parallel to the line $y = 3x + 2$

Solution: Vector $\vec{v} = (a, b)$ parallel to the line is set $y = 3x$

$\Rightarrow \vec{v} = (\vec{a}, \vec{b}) = (a, 3a) = a(1, 3) = (1, 3)$

Unit vector in the direction of $\vec{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{(1,3)}{\sqrt{10}} = \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$

100. Which of the following is a vector that lies on the line through (0, 0) and (2, 4)…….UEE2004/12

A. $\vec{u} = (2,1)$ C. $\vec{u} = \left(\frac{1}{2}, 2\right)$

B. $\vec{u} = (-1,-2)$ D. $\vec{u} = (-2,-6)$

Solution: Let A = (0, 0) and B = (2, 4)

$\Rightarrow \overrightarrow{AB} = (2, 4) - (0, 0) = (2, 4)$

Vector equation of ℓ: (x, y) = (0, 0) + t(2, 4) = (2t, 4t)

$\Rightarrow (x, y) = (2t, 4t)$ if $t = \frac{-1}{2}, \vec{u} = (-1,-2)$

Answer: B

101. Let i and j be standard unit vector, A = (-1, 0) and B = (2, 2). If $\vec{v} = 3\overrightarrow{AB} - 3\vec{i} + 2\vec{j}$ then unit vector in the direction of $\vec{v}$,...(UEE)

A. $\left(\frac{3}{5}, \frac{4}{5}\right)$ C. $\left(\frac{-3}{5}, \frac{4}{5}\right)$

B. $\left(\frac{-3}{5}, \frac{-4}{5}\right)$ D. $\left(\frac{3}{5}, \frac{-4}{5}\right)$

Solution: $\overrightarrow{AB} = (2, 2) - (-1, 0) = (3, 2) = 3i + 2j$

$\vec{v} = 3(3i + 2j) - 3i + 2j = 6i + 8j = (6, 8)$

Unit vector in direction of $\vec{v}$ is $\frac{\vec{v}}{|\vec{v}|} = \frac{(6,8)}{\sqrt{6^2+8^2}} = \left(\frac{3}{5}, \frac{4}{5}\right)$

Answer: A

102. Which of the following is a vector equation of the line tangent to the circle $x^2 + y^2 + 2x - 7 = 0$ at (1, 2) ….. UEE2004/12

A. $(x, y) = (0, 3) + \lambda(-1, 2)$ C. $(x, y) = (0, 3) + \lambda(1, -1)$

B. $(x, y) = (1, 2) + \lambda(2, -1)$ D. $(x, y) = (1, 2) + \lambda(-1, 2)$

Solution: $x^2 + y^2 + 2x - 7 = 0 \Rightarrow (x + 1)^2 + (y - 0)^2 = 8$

$\Rightarrow$ center, C(-1, 0), Let P(1, 2) and R(x, y) point on tangent line,

$\overrightarrow{CP} = (1, 2) - (-1, 0) = (2, 2)$

$\Rightarrow \overrightarrow{CP} \cdot \overrightarrow{PR} = 0 \Rightarrow (2, 2)\cdot(x-1, y-2) = 0$

$\Rightarrow 2x - 2 + 2y - 4 = 0 \Rightarrow x + y = 3$

- Direction vector $\vec{v} = (a, b)$ parallel to the line $x + y = 3$ is $x + y = 0 \Rightarrow y = -x$, $\vec{v} = (a, b) = (a, -a) = \lambda(1, -1)$
- take any arbitrary point on $x + y = 3$, say, (0, 3)

$\therefore$ vector equation of ℓ: $(x, y) = (0, 3) + \lambda(1, -1)$, Answer: C

103. Let ℓ be line given by $2x - y = 10$. what is the equation of the image of ℓ after a reflection in the line $y = 2x - 5$ followed by rotation through the angle of $\theta = 90^0$ about the origin ...UEE 2004/12

A. $x + 2y = 0$ C. $x + 2y = 5$

B. $2x + y = 0$ D. $x - 2y = 5$

Solution: ℓ: $y = 2x - 10$ is parallel to the reflecting axis.

Hence, the image ℓ': $y = 2x + b$ we need to find b.

Let (a, b) be any point on ℓ. say (0, -10)

So that its lies on ℓ': $y = 2x + b$

$$\Rightarrow M((0, -10)) = (a', b') \Rightarrow \frac{b'+10}{a'-0} = \frac{-1}{2} \Rightarrow a' = -2b' - 20$$

Also, mid-point of (0, -10) and (a', b')

$$\Rightarrow \left(\frac{0+a'}{2}, \frac{b'-10}{2}\right) \text{ lies on reflecting axis}$$

$$\Rightarrow \frac{b'-10}{2} = 2\left(\frac{a'}{2}\right) - 5 \Rightarrow a' = \frac{b'}{2}$$

$$\text{But } a' = -2b' - 20 \Rightarrow -2b' - 20 = \frac{b'}{2} \Rightarrow b' = -8$$

$a' = -4 \Rightarrow (-4, -8)$ lies on $y = 2x + b$

$\Rightarrow -8 = 2(-4) + b \Rightarrow b = 0 \Rightarrow \ell'$: $y = 2x$

$\Rightarrow$ Image of ℓ': $y = 2x$ when rotated by $\theta = 90^0$ about origin

$R(x, y) = (-y, x)$

$R(x, 2x) = (-2x, x)$

$\Rightarrow x = -2y \Rightarrow x + 2y = 0$ Answer: A

104. Let ℓ_1 and ℓ_2 be two lines in space intersecting at the origin (0,0,0). If ℓ_1 and ℓ_2 pass through point, A(1, 1, 0) and B(0, 1, 1) respectively, then the angle between ℓ_1 and ℓ_2 is equal to … UEE 2004/12.

A. 30° B. 45° C. 60° D. 90°

Solution: Let origin be 0 = (0, 0, 0)

$\Rightarrow \overrightarrow{OA} = (1, 1, 0)$ and $\overrightarrow{OB} = (0, 1, 1)$

Let θ be angle between ℓ_1 and ℓ_2

$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} = 60°$ **Answer: C**

105. Let $\vec{a} = -i + 3k$ and $\vec{b} = -i + j$ be vector in the spaced. What is the cosine of the angle between $\vec{a}$ and $\vec{a} - \vec{b}$ …UEE 2004/12.

A. $\frac{9}{10}$ B. $\frac{3}{5}$ C. $\frac{3}{\sqrt{10}}$ D. $\frac{-9}{10}$

Solution: Let θ be angle between $\vec{a}$ and $\vec{a} - \vec{b}$

$$\Rightarrow \cos\theta = \frac{\vec{a}\cdot(\vec{a}-\vec{b})}{|\vec{a}||\vec{a}-\vec{b}|} = \frac{(-i+0j+3k)\cdot(0i-j+3k)}{\sqrt{1+9}\sqrt{1+9}} = \frac{9}{10}$$

Answer: A

106. Let $\vec{v} = 3i - 4k$, i, j and k be standard unit vector in x, y, z axis and A = (0, 1, 2)

If $\overrightarrow{AB}$ is parallel to $\vec{v}$ and $\|\overrightarrow{AB}\| = 10$, then point B is at … UEE 2004/12

A. (-6, -1, 10) C. (6, 1, -6)
B. (-6, -1, -10) D. (-6,-1, 6)

Solution: Let B = (x, y, z)

Since $\overrightarrow{AB} \parallel \vec{v} \Rightarrow \overrightarrow{AB} = t\vec{v} \Rightarrow |\overrightarrow{AB}| = |t\vec{v}|$

$\Rightarrow t\vec{v} = t(3, 0, -4) \Rightarrow (3t, 0, -4t)s$

$\Rightarrow |t\vec{v}| = 10 \Rightarrow \sqrt{9t^2 + 0^2 + 16t^2} = \sqrt{25t^2} = |t|5$

$\Rightarrow 5|t| = 10 \Rightarrow t = 2$

$\overrightarrow{AB} = B - A = (x, y, z) - (0, 1, 2) = (x, y-1, z-2) = (2)(3, 0, -4)$

$\Rightarrow x = 6, y - 1 = 0 \Rightarrow y = 1, z - 2 = -8 \Rightarrow z = -6$

$\therefore B = (x, y, z) = (6, 1, -6)$ **Answer: C**

Supplementary exercise

1. The position vectors of four points A, B, C D and $\vec{a}, \vec{b}, 2\vec{a}+3\vec{b}$ and $\vec{a}-2\vec{b}$ respectively. Find $\overrightarrow{AC}$, $\overrightarrow{BC}$, $\overrightarrow{DA}$ and $\overrightarrow{DB}$ interms of $\vec{a}$ *and* $\vec{b}$
2. Verify whether the three points $\vec{a}-2\vec{b}+3\vec{c}$, $2\vec{a}+3\vec{b}-4\vec{c}$, $7\vec{b}+10\vec{c}$ are collinear
3. Show that the following vectors are coplanar $5\vec{a}+6\vec{b}+7\vec{c}$, $7\vec{a}-8\vec{b}+9\vec{c}$, $3\vec{a}+20\vec{b}+5\vec{c}$
4. For the vectors $\vec{a}$ *and* $\vec{b}$ where $\vec{a} = 2i + 3j + \vec{k}$, $\vec{b} = 3i - j -$ k, find the value of $\left|2\vec{a} + 3\vec{b}\right|$
5. ABCD is parallelogram. Find single vector of the following.

a)	$\overrightarrow{CB}-\overrightarrow{CA}$	d)	$\overrightarrow{AD}+\overrightarrow{DC}$
b)	$\overrightarrow{DC}-\overrightarrow{DA}$	e)	$\overrightarrow{DB}+\overrightarrow{BA}+\overrightarrow{DA}$
c)	$\overrightarrow{AC}+\overrightarrow{CB}$	f)	$\overrightarrow{CD}-\overrightarrow{CB}$

6. If the vectors $\vec{c}$ and $\vec{d}$ are two adjacent sides of regular hexagon, determine the other sides interms of $\vec{c}$ and $\vec{d}$
7. Find the magnitude and unit vector in the direction of the sum of the vectors $\vec{a} = i + 4j + 2\vec{k}$, $\vec{b} = 3i - 3j - 2k$, and $\vec{c} = -2i + 2j + 6k$
8. find the unit vector in the direction of the vector $\vec{r}_1 - \vec{r}_2$ where $\vec{r}_1 = i + 2j - k$ and $\vec{r}_2 = 3j + j - 5k$
9. Find the angle between the vector
   a) $\vec{a} = -3i + 2j$, $\vec{b} = 2i + 3j$ c) $\vec{a} = -2i$, $\vec{b} = \sqrt{3}i -$
   b) $\vec{a} = i + j$, $\vec{b} = 3i$
10. Find the value of t for which the vectors 3i + j +tk and 2i -2 j + are perpendicular to each other.
11. If $\vec{a} = 2i - 3j + k$, $\vec{b} = i + 2j + 5k$, $\vec{c} = -3i + j - 2k$ Find a) $\vec{a}\cdot\vec{b}$ b) $\vec{b}\cdot\vec{c}$

12. Given the vector $\vec{a} = i,\ \vec{b} = i + j$, find scalar k such that
a) the vector $\vec{a} + k\vec{b}$ is perpendicular to $\vec{a}$
b) the vector $\vec{a} + k\vec{b}$ is perpendicular to $\vec{b}$

13. Vectors $\vec{a}$ *and* $\vec{b}$ make an angle $\theta = \dfrac{\pi}{3}$. If $|\vec{a}| = 3$, $|\vec{b}| = 2$ then find a) $|2\vec{a} + \vec{b}|$ b) Angle between $\vec{b}$ and $(\vec{a} + \vec{b})$

14. Show that A(1, 2), B(3, 4) and C(5, 2) are vertices of a right triangle by considering the sides of the triangle as vectors.

15. Find the angle between the vectors
a) $\vec{a} = (1, 2)$ and $\vec{b} = \left(1, -\dfrac{1}{2}\right)$
b) $\vec{a} = i - \sqrt{3}\,j$ and $\vec{b} = i + \sqrt{3}\,j$

16. If $\vec{a} = 2i - 2j + k$, $\vec{b} = 2i + 3j + 6k$, $\vec{c} = -i + 2k$
Find the direction cosine of $\vec{a} - \vec{b} + 2c$

17. Find a vector equation and rectangular form of equation for the line that contains the given point and is parallel to a vector.
a) $\vec{r}_0 = (-3, 4)$; $\vec{a} = (2i + j)$ b) $r_0 = (5, 1)$; $\vec{a} = 3i - 2j$

18. Let $\vec{u} = i - 3j + 2k$, $\vec{v} = i + j$, $\vec{w} = 2i + 2j - 4k$
Find
a) $|u + v|$ c) $|3u - 5v + w|$
b) $|u| + |v|$ d) $\dfrac{1}{|w|}w$ f) $\left|\dfrac{1}{|w|}w\right|$

19. Find the component form of the vector $\vec{v}$ in 2-space that has the stated length and makes the stated angle θ with the positive x – axis
a) $|v| = 3,\ \theta = \dfrac{\pi}{4}$ c) $|v| = 5,\ \theta = 120^0$
b) $|v| = 2,\ \theta = 90^0$ d) $|v| = 1\ \theta = \pi$

20. Let $\vec{u} = (1, 3)$, $\vec{v} = (2, 1)$, $\vec{w} = (3, 4)$, find the vector $\vec{x}$ that satisfies $2\vec{u} - \vec{v} + \vec{x} = 7\vec{x} + \vec{w}$

21. Find parametric equations for the line through P_1 and P_2
a) $P_1(3, -2)$, $P_2(5, 1)$ b) $P_1(5, -2, 1)$, $P_2(2, 4, 2)$

22. Find thee equation of the line that passes through (3, 2) and
   a) Parallel to vector $2i - 4j$
   b) Perpendicular to the vector $-i + j$
23. Using vector methods, find the equation for circles whose ce[ntre] and radius are given below
   a) center (3, 4), $r = 2$ b) center (-1, 1), $r = \sqrt{3}$
24. Find the parametric equation of the line that is tangent to [the] circle $x^2 + y^2 = 25$ at the point (3, -4)
25. A translation T(-2, 3) mapped to the point (4, -1) then find [the] image of
   a) (0, 1) b) 1, 1)
   c) the circle with equation of $(x + 10^2 + (y - 2)^2 = 5$
26. Find the image of the following under the line of reflection x = ...
   a) $y = 2x + 1$, b) (1, 3) c) $y = 2$ d) $x = 1$
27. Find the image of the following under the line of reflectio[n] $\ell: x - y = 3$
   a) (1, 1) b) $(x - 1)^2 + (y - 1)^2 = 6$ c) x = ...
28. Find the coordinate of the image of the following after a rotation about the origin through each of i) 45^0 ii) 150^0 iii) 300^0
   a) (6, -2) b) (2, 1) c) $y = 1$ d) $y = 3x + 1$
   e) $x^2 + y^2 = 2$
29. Find the image of the following point after clockwise rotation through 90^0
   a) (4, -3) about i) (0, 0), ii) (1, 2)
   b) $y = 2x - 1$, about i) (0, 0) ii) (1, 1)
   c) $(x - 4)^2 + (y + 3)^2 = 4$ about i) (0, 0) ii) (1, 2)

Unit Eight

The set of complex number

$\frac{i}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

1. The concept of "complex Number".

Now consider the equation $x^2 + 1 = 0$. If we attempt to solve this equation, we have

$$x^2 + 1 = 0 \Leftrightarrow x^2 = -1 \Rightarrow x = \pm\sqrt{-1} = \pm i$$

A number whose square is -1 (Negative) is called imaginary number denoted by **i**, defined as $i = \sqrt{-1},\ i^2 = -1$

A complex number system (C) which contains both R (real number) and number whose square is Negative denoted by $\not\subset$ and given by

$\not\subset = \{Z/Z = x + yi\}$ where x and y are real number and

$$i = \sqrt{-1}$$

Illustrative Example

1. Solve each of the following

a) $x^2 + 4 = 0$ b) $x^2 + 9 = 0$ c) $x^2 + 64 = 0$

d) $x^2 + 2x + 10 = 0$

Solution:

a) $x^2 + 4 = 0 \Leftrightarrow x^2 = -4 \Rightarrow x = \pm\sqrt{-4} = \pm 2i \quad \therefore x=2i \text{ or } x=-2i$

b) $x^2 + 9 = 0 \Leftrightarrow x^2 = -9 \Rightarrow x = \pm\sqrt{-9} = \pm 3i \quad \therefore x=3i \text{ or } x=-3i$

c) $x^2 +64=0 \Leftrightarrow x^2 =-64 \Rightarrow x = \pm\sqrt{-64} = \pm 8i \quad \therefore x=8i \text{ or } x=-8i$

d) $x^2 +2x + 10 = 0 \Leftrightarrow x^2 + 2x+1 + 10- 1= 0 \Leftrightarrow (x + 1)^2 + 9 = 0$

$\Rightarrow (x + 1)^2 = -9 \Leftrightarrow x + 1 = \pm\sqrt{-9} = \pm 3i$

$\Rightarrow x + 1 = \pm 3i \qquad \therefore x = \{-1 + 3i, -1 -3i\}$

2. Write the expression in the form: $z = x + yi$

a) $z = 3 - \sqrt{-4}$ g) $z = \dfrac{\sqrt{-27}}{\sqrt{3}}$

b) $z = 1 + \sqrt{-25}$ h) $z = \sqrt{3}\sqrt{-4}$

c) $z = \sqrt[3]{-8} + \sqrt{-8}$ i) $z = \sqrt{(-4)(-3)}$

d) $z=\sqrt{-12}+\sqrt{-3}$

e) $z=\sqrt{-3}\sqrt{-4}$

f) $z=\sqrt{-18}$

j) $z=\left(2-\sqrt{-125}\right)-\left(3-\sqrt{-20}\right)$

k) $z=\left(-2+\sqrt{-4}\right)-\left(\sqrt{12}+\sqrt{-3}\right)$

Solution: Here

a) $z=3-\sqrt{-4} \Leftrightarrow z=3-2i$

b) $z=1+\sqrt{-25} \Leftrightarrow z=1+5i$

c) $z=\dfrac{\sqrt[3]{-8}+\sqrt{-8}}{2} \Leftrightarrow z=-1+\sqrt{2}i$

d) $z=\sqrt{-12}+\sqrt{-3} \Leftrightarrow 2\sqrt{3}i+\sqrt{3}i \qquad \therefore\ z=0+3\sqrt{3}i$

e) $z=\sqrt{-3}\cdot\sqrt{-4} \Leftrightarrow (\sqrt{3}i)\cdot(2i) \Rightarrow z=-2\sqrt{3}$

f) $z=\dfrac{\sqrt{-18}}{\sqrt{-2}} \Leftrightarrow z=\dfrac{3\sqrt{2}i}{\sqrt{2}i}=3+0i$

g) $z=\dfrac{\sqrt{-27}}{\sqrt{3}}=\dfrac{3\sqrt{3}i}{\sqrt{3}}=3i \qquad \therefore\ z=0+3i$

h) $z=\sqrt{3}\sqrt{-4}=2\sqrt{3}i \qquad \therefore\ z=0+2\sqrt{3}i$

i) $z=\sqrt{(-4)(-3)}=2\sqrt{3} \qquad \therefore\ z=2\sqrt{3}+0i$

j) $z=2-\sqrt{-125}-3+\sqrt{-20}=-1-5\sqrt{5}i+2\sqrt{5}i$

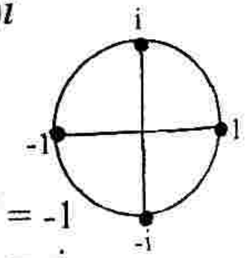
$\therefore\ z=-1-3\sqrt{5}i$

k) $z=-2+\sqrt{-4}-(\sqrt{12}+\sqrt{-3})$

$\therefore\ z=-2+2i-2\sqrt{3}-\sqrt{3}i$

$\therefore\ z=(-2-2\sqrt{3})+(2-\sqrt{3})i$

Power of i



*A few power of **i** are listed here*

- $i^1 = i$ • $i^5 = i$ • $i^9 = i$
- $i^2 = -1$ • $i^6 = -1$ • $i^{10} = -1$
- $i^3 = -i$ • $i^7 = -i$ • $i^{11} = -i$
- $i^4 = 1$ • $i^8 = 1$ • $i^{12} = 1$

The power of i rotate through the four numbers i, -1, -i, and 1.

Note: $i^n = i^r$, where $\Rightarrow n = 4m + r$
- n is positive integer
- r is the remainder when n is divided by 4
- $0 \le r \le 3$

Illustrative Example

3. Simplify and write as $z = x + yi$

a)	i^7	f)	i^{543}	k)	i^{205}
b)	i^{13}	g)	i^{176}	l)	i^{131}
c)	i^{15}	h)	$i^{298} + i^{723} + i^{10}$		
d)	i^{74}	i)	$i + i^2 + i^3 + i^4 + \ldots + i^{28} + i^{29} + i^{30}$		
e)	i^{102}	j)	$i^{4n+3} + (i^{2n+2})^2$		

Solution:

a) Divide 7 by 4, remainder, $r = 3$, $\therefore i^7 = i^3 = -i$

b) Divide 13 by 4, remainder, $r = 1$
$\therefore i^{13} = i \longleftarrow 13 = 4(3) + 1, r = 1$

c) Divide 15 by 4, Remainder, $r = 3$
$\therefore i^{15} = i^3 = -i$

d) $74 = 4(18) + 2$, $\therefore i^{74} = i^2 = -1 \longleftarrow$ Dividing 74 by 4 remainder 2

e) $i^{102} = i^{4(25)+2} \Rightarrow i^{102} = i^2 = -1 \leftarrow$ Dividing 102 by 4 remainder 2

f) Dividing 543 by 4, remainder, $r = 3$
$\therefore i^{543} = i^3 \longleftarrow 543 = 4(135) + 3$
$\therefore i^{543} = i^3 = -i$

g) $i^{176} = i^{4(44)+0} = i^0 = 1 \longleftarrow 176 = 4(44) + 0$, Dividing 176 by 4 remainder, $r = 0$

h) $i^{298} = i^2, \longleftarrow$ Dividing 298 by 4, remainder $r = 2$
$i^{723} = i^3, \longleftarrow$ Dividing 723 by 4, remainder $r = 3$
$i^{10} = i^2, \longleftarrow$ Dividing 10 by 4, remainder $r = 2$
$\therefore i^{298} + i^{723} + i^{10} = i^2 + i^3 + i^2 = -1 - i - i = -1 - 2i$

i) $i + i^2 + i^3 + i^4 + \ldots + i^{28} + i^{29} + i^{30}$
$= (i + -1 + -i + 1 + \ldots + i) + i - 1 = i = 1$

j) $i^{4n+3} = i^3$ and $(i^{2n+2})^2 = i^{4n+4} = i^4 = 1$
$\therefore i^{4n+3} + i^{4n+4} = i^3 + i^4 = -i + 1 = 1 - i$

k) $i^{205} = i^{4(51)+1} = i \longleftarrow$ Dividing 205 by 4, remainder $r = 1$

l) $i^{131} = i^{4(32)+3} = i^3 \longleftarrow$ Dividing 131 by 4, remainder $r = 3$
$\therefore i^{131} = i^3 = -i$

4. Simplify a) i^{2n} b) i^{2n+1}

Solution: a) $i^{2n} = \begin{cases} -1 \text{ if n is odd, } n = 1, 3, 5, 7..... \\ 1 \text{ if n is even, } n = 2,4,6,8,..... \end{cases}$

b) $i^{2n+1} = i^{2n} \cdot i = \begin{cases} -i \text{ if n is odd,} \\ i \text{ if n is even} \end{cases}$

Equality of complex Number

Let $z_1 = x + yi$ and $z^2 = a + bi$ then

$z_1 = z_2$ if and only if $x = a$ and $y = b$

Example: a) $x + yi = 3 + 4i$ iff $x = 3$ and $y = 4$

b) $(2x - 4) + 9i = 8 + 3yi$, iff $2x - 4 = 8$ and $3y = 9$

$\Rightarrow 2x = 12$ and $3y = 9$

$\therefore x = 6$ and $y = 3$

Operation on complex Number

Let $z_1 = x + yi$ and $z_2 = a + bi$ be complex number

i) Addition and subtraction:

$z_1 + z_2 = (x + yi) + (a + bi) = (x + a) + (y + b)i$

$z_1 - z_2 = (x + yi) - (a + bi) = (x - a) + (y - b)i$

ii) Multiplication of complex Number

$z_1 \cdot z_2 = (x + y_i)(a + bi) = x(a + bi) + yi(a + bi)$

$= ax + xbi + ayi - yb$

$= (ax - yb) + (xb + ay)i$

If $z_1 \cdot z_2 = 1 + 0i$ then

- z_1 and z_2 are multiplicative inverse of each other
- $1 + 0i$ is the identity element for multiplication of complex number

iii) Division of complex Number

- Let $z_1 = 1 + 0i = 1$ and $z_2 = a + bi$, then

$$\frac{z_1}{z_2} = \frac{1+0i}{a+bi} = \frac{1}{(a+bi)}\frac{a-bi}{(a-bi)} = \frac{a-bi}{a^2+b^2}$$

$$\therefore \frac{1}{z_2} = \frac{1}{a+bi} = \frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2}$$

Example: a) $\frac{1}{2+3i} = \frac{1}{2+3i} = \frac{2}{2^2+3^2} - \frac{3i}{2^2+3^2} = \frac{2}{13} - \frac{3i}{13}$

b) $\dfrac{1}{3-4i}=\dfrac{1}{3-4i}\cdot\dfrac{3+4i}{3+4i}=\dfrac{3}{3^2+4^2}+\dfrac{4i}{3^2+4^2}=\dfrac{3}{25}+\dfrac{4i}{25}$

Let $z_1 = a + bi$ and $z_2 = c + di$ then

- $\dfrac{z_1}{z_2}=\dfrac{a+bi}{c+di}=(a+bi)\left(\dfrac{1}{c+di}\right)$

$=(a+bi)\left(\dfrac{c}{c^2+d^2}-\dfrac{di}{c^2+d^2}\right)$

$=\dfrac{ac+bci-adi+bd}{c^2+d^2}$

$\therefore \dfrac{a+bi}{c+di}=\dfrac{(ac+bd)}{c^2+d^2}+\dfrac{(bc-ad)i}{c^2+d^2}$

Example C: $\dfrac{2+3i}{1+5i}=\dfrac{2+15}{1^2+5^2}+\dfrac{(3-10)i}{1^2+5^2}=\dfrac{17}{26}-\dfrac{7i}{26}$

Example d: $\dfrac{1-2i}{2+4i}=\dfrac{(2-8)}{2^2+4^2}-\dfrac{(4+4)i}{2^2+4^2}=\dfrac{-6}{20}-\dfrac{8i}{20}$

Multiplicative Inverse

For every $z \neq 0$ in $\not\subset$ there is multiplicative inverse $\dfrac{1}{z}$ such that

$$z\cdot\frac{1}{z}=1=\frac{1}{z}\cdot z$$

Example: Find the multiplicative inverse of each of the following

a) a + bi b) x + yi c) 2 + 3i d) $1-\sqrt{3}i$

Solution: a) multiplicative inverse of a + bi is $\dfrac{1}{a+bi}$

Thus, $\dfrac{1}{a+bi}=\dfrac{a}{a^2+b^2}-\dfrac{bi}{a^2+b^2}$ ← multiplicative inverse of a +bi

b) Multiplicative inverse of x + yi is $\dfrac{1}{x+yi}$

$$\therefore \frac{1}{x+yi}=\frac{x}{x^2+y^2}-\frac{yi}{x^2+y^2} \leftarrow \text{multiplicative inverse of } x+yi$$

c) Multiplicative inverse of 2 + 3i is $\dfrac{1}{2+3i}$

$$\therefore \frac{1}{2+3i}=\frac{2}{2^2+3^2}-\frac{3i}{2^2+3^2}=\frac{2}{13}-\frac{3i}{13} \leftarrow \text{multiplicative inverse of } 2+3i$$

d) Multiplicative inverse of $1-\sqrt{3}i$ is $\dfrac{1}{1-\sqrt{3}i}$

$$\therefore \frac{1}{1-\sqrt{3}i}=\frac{1}{1-\sqrt{3}i}\cdot\frac{1+\sqrt{3}i}{1+\sqrt{3}i}=\frac{1}{1^2+\sqrt{3}^2}=\frac{1}{4}+\frac{\sqrt{3}i}{4}$$

5. Find the multiplicative inverse and write as z = x + yi

a) $z=\dfrac{1+i}{2+i}$ b) $z=\dfrac{3-4i}{2+3i}$ c) $z=\dfrac{2-5i}{i}$

Solution: Multiplicative inverse of z is $\dfrac{1}{z}$

a) $z=\dfrac{1+i}{2+i} \Leftrightarrow \dfrac{1}{z}=\dfrac{2+i}{1+i}=(2+i)\left(\dfrac{1}{1+i}\right)=(2+i)\left(\dfrac{1}{2}-\dfrac{i}{2}\right)$

$$\Rightarrow \frac{1}{z}=\frac{2+i-i(2+i)}{2}=\frac{2+i-2i+1}{2}=\frac{3}{2}-\frac{i}{2}$$

$$\therefore \frac{1}{z}=\frac{3}{2}-\frac{i}{2} \longleftarrow \text{Multiplicative inverse of } z=\frac{1+i}{2+i}$$

b) $\dfrac{1}{z}=\dfrac{2+3i}{3-4i}=(2+3i)\left(\dfrac{1}{3-4i}\right)=(2+3i)\left(\dfrac{3}{3^2+4^2}+\dfrac{4i}{3^2+4^2}\right)$

$$\frac{1}{z}=\frac{3(2+3i)+4i(2+3i)}{25}=\frac{-6}{25}+\frac{17i}{25} \leftarrow \text{multiplicative}$$

inverse of $\dfrac{3-4i}{2+3i}$

c) $\dfrac{1}{z}=\dfrac{i}{2-5i}=(i)\left(\dfrac{1}{2-5i}\right)=i\left(\dfrac{2}{2^2+5^2}+\dfrac{5i}{2^2+5^2}\right)=\dfrac{-5}{29}+\dfrac{2i}{29}$

Illustrative Example

In each part solve for x and y

a) $(x + yi)(1 + 2i) = 4 + 7i$

b) $(x - yi)(4 + i) = 14 - 5i$

c) $\frac{x+yi}{3-i} = \frac{1}{5} + \frac{2}{5}i$

d) $(x + yi)i + 2 - 3x - y + 4i = 0$

e) $2x + yi = (3+5i)(2-4i)$

f) $\frac{3+i}{x+yi} = \frac{1}{2} + \frac{1}{2}i$

g) $\frac{i+2}{x-4yi} = 3i^{23}$

h) $\frac{2+3i}{3x-4yi} = i+1$

i) $(x + yi)^2 = i^5$

x+yi

Solution: Here

a) $(x+yi)(1+2i)=4+7i \Leftrightarrow x+yi$

$$= \frac{4+7i}{1+2i} \leftarrow \text{dividing both side by } (1 + 2i)$$

$$\therefore x + yi = \frac{4+7i}{1+2i} = (4 + 7i)\left(\frac{1}{1+2i}\right) = (4+7i)\left(\frac{1}{1^2+2^2} - \frac{2i}{1^2+2^2}\right)$$

$$\Rightarrow x + yi = \frac{4+7i-2i(4+7i)}{5} = \frac{4+14}{5} - \frac{i}{5}$$

$$\therefore x + yi = \frac{18}{5} - \frac{i}{5} \qquad \therefore x = \frac{18}{5}, \quad y = -\frac{1}{5}$$

b) $(x - yi)(4 + i) = 14 -5i \Leftrightarrow x - yi = \frac{14-5i}{4+i}$

$$\Rightarrow x - yi = (14-5i)\left(\frac{1}{4+i}\right) = (14+5i)\left(\frac{4}{4^2+1^2} - \frac{i}{4^2+1}\right)$$

$$\Rightarrow x - yi = \frac{4(14-5i) - i(14-5i)}{17} = \frac{51}{17} - \frac{34i}{17}$$

$$\therefore x = 3, \quad y = \frac{34}{17} = 2$$

c) $\frac{x+yi}{3-i} = \frac{1}{5} + \frac{2}{5}i \Leftrightarrow x + yi = (3-i)\left(\frac{1}{5} + \frac{2}{5}i\right)$

$$\Rightarrow x+yi=(3-i)\left(\frac{1+2}{5}i\right)=\frac{3(1+2i)-i(1+2i)}{5}$$

$$\Rightarrow x+yi=\frac{3+6i-i+2}{5}=\frac{5}{5}+\frac{5i}{5}=1+i$$

$\therefore$ x = 1 and y = 1

d) (x + yi)i + 2 – 3x – y + 4i = 0
$\Leftrightarrow$ (x + yi)i = -2 + 3x + y – 4i
$\Leftrightarrow$ ix – y = -2 + 3x +y – 4i
$\Rightarrow$ ix = -4i and –y = -2 + 3x + y
$\Rightarrow$ x = -4 and -2y = -2 +3x, put x = -4
$\Rightarrow$ -2y = -2 + 3(-4) = -14
$\Rightarrow$ x = -4 and -2y = -14, $\Rightarrow$ y = 7
$\therefore$x = -4 and y = 7

e) 2x + yi = (3 + 5i)(2 – 4i) = 6 – 12i + 10i + 20
$\Rightarrow$ 2x + yi = 26 -2i $\Rightarrow$ 2x = 26 and y = -2
$\therefore$ x = 13 and y = -2

f) $$\frac{3+i}{x+yi}=\frac{1}{2}+\frac{1}{2}i\Leftrightarrow 3+i=(x+yi)\left(\frac{1+i}{2}\right)$$

$$\Leftrightarrow x+yi=(3+i)\cdot\frac{2}{1+i}=(6+2i)\left(\frac{1}{1+i}\right)$$

$$\Leftrightarrow x+yi=(6+2i)\cdot\left(\frac{1}{1^2+1^2}-\frac{i}{1^2+1^2}\right)$$

$$\Rightarrow x+yi=\frac{6+2i}{2}-\frac{i(6+2i)}{2}=3+i-3i+1$$

$\Rightarrow$ x + yi = 4 – 2i $\quad\therefore$ x = 4, y = -2

g) $$\frac{i+2}{x-4yi}=3i^{23}\Leftrightarrow 2+i=3i^{23}(x-4yi)$$

$$\Rightarrow x-4yi=\frac{2+i}{3i^{23}}$$ Note, $i^{23}=i^3=-i$

$$\Rightarrow x-4yi=\frac{2+i}{-3i}=\left(\frac{2+i}{-3i}\right)\left(\frac{i}{i}\right)=\frac{2i-1}{3}$$

$$\Rightarrow x - 4yi = -\frac{1}{3} + \frac{2i}{3}$$

$$\Rightarrow x = \frac{-1}{3} \text{ and } -4y = \frac{2}{3} \Rightarrow y = -\frac{1}{6}$$

$$\therefore x = \frac{-1}{3} \text{ and } y = -\frac{1}{6}$$

h) $$\frac{2+3i}{3x-4yi} = i+1 \Leftrightarrow 2+3i = (i+1)(3x-4yi)$$

$$\Rightarrow 3x-4yi = \frac{2+3i}{1+i} = (2+3i)\left(\frac{1}{1+i}\right)$$

$$= 2+3i\left(\frac{1}{1^2+1^2} - \frac{i}{1^2+1^2}\right)$$

$$\Rightarrow 3x-4yi = (2+3i)\left(\frac{1}{2} - \frac{i}{2}\right) = \frac{2+3i-2i+3}{2}$$

$$\Rightarrow 3x-4yi = \frac{5}{2} + \frac{i}{2}$$

$$\Rightarrow 3x = \frac{5}{2} \text{ and } -4y = \frac{1}{2}$$

$$\therefore x = \frac{5}{6} \text{ and } y = -\frac{1}{8}$$

i) $(x + yi)^2 = i^5 \Leftrightarrow x^2 + 2xyi - y^2 = i$
$\Rightarrow x^2 - y^2 = 0$ and $2xy = 1$
$\Rightarrow x = y$ or $x = -y$ and $2xy = 1$
Substitute $y = x$ and $2xy = 1$
$\Rightarrow 2x \cdot x = 1 \Rightarrow 2x^2 = 1$

$$\Rightarrow x^2 = ½ \Rightarrow x = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

Since $x = y$, therefore, $y = \pm\frac{\sqrt{2}}{2}$, $\therefore x = \pm\frac{\sqrt{2}}{2}$ and $y = \pm\frac{\sqrt{2}}{2}$

Complex Conjugate

Definition: The complex conjugate of a complex number $z = x + yi$ denoted by $\bar{z}$ is given by $\bar{z} = x - yi$

Properties of complex conjugate

i) If $z = x + yi$, then

a) $\bar{z} = x - yi$

b) $z + \bar{z} = x + yi + x - yi = 2x$

c) $z - \bar{z} = x + yi - (x - yi) = 2yi$

d) $z \cdot \bar{z} = (x + yi)(x - yi) = x^2 + y^2$

ii) If z_1 and z_2 be complex number then

a) $\bar{\bar{z}}_1 = z_1$ b) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

c) $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$ d) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

e) $\overline{(z^n)} = (\bar{z})^n$

Definition: Modulus (the absolute value of complex number $z = x + yi$, denoted by $|z|$ is defined to be $|z| = \sqrt{x^2 + y^2}$

Note: If $z_1 = x + yi$ and $z_2 = a + bi$, then

$$|z_1 - z_2| = |(x - a) + (y - b)i| = \sqrt{(x-a)^2 + (y-b)^2}$$

Properties of Modulus

- $z \cdot \bar{z} = |z|^2$
- $|z^n| = |z|^n, \ n \in R$
- $|z| = |\bar{z}|$
- $|z_1 + z_2| \leq |z_1| + |z_2|$
- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
- $|z_1 - z_2| \geq |z_1| - |z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Illustrative Example

8. Find the conjugate of the expression

a) $z = -3 + 4i$ d) $z = \frac{2+3i}{4-2i}$

b) $z = 1 - 2i$ e) $z = \frac{1}{1+i}$

c) $z = x + yi$ f) $z = \dfrac{2i}{2+i} + \dfrac{3}{2-i}$

Solution: Conjugate of $z = x + yi$ is given by $\bar{z} = x - yi$, thus

a) $z = -3 + 4i$ then $\bar{z} = -3 - 4i$

b) $z = 1 - 2i$, then $\bar{z} = 1 + 2i$

c) $z = x + yi$, then $\bar{z} = x - yi$

d) we use $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$, thus

$$\bar{z} = \overline{\left(\frac{2+3i}{4-2i}\right)} = \frac{\overline{2+3i}}{\overline{4-2i}} = \frac{2-3i}{4+2i}$$

e) $\bar{z} = \overline{\left(\dfrac{1}{1+i}\right)} = \dfrac{\bar{1}}{\overline{1+i}} = \dfrac{1}{1-i} = \dfrac{1}{2} + \dfrac{i}{2}$

f) Exercise left for you (Answer $\bar{z} = \dfrac{9}{5} - \dfrac{6i}{5}$)

8. Find the modulus of the expression

a) $z = 2 + 3i$ d) $z = \dfrac{(1-i)^{20}}{(8+8\sqrt{3}i)(1-\sqrt{3}i)^6}$

b) $z = 2 - 6i$ e) $z = \sqrt{2+3i}$

c) $z = 3i^{497}$ f) $z = \sqrt[3]{8+8\sqrt{3}i}$

Solution: we know, if $z = x + yi$ then $|z| = \sqrt{x^2 + y^2}$, hence

a) $|z| = |2+3i| = \sqrt{2^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$

b) $|z| = |2-6i| = \sqrt{2^2 + 6^2} = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10}$

c) $|z| = |3i^{497}| = |3i| = \sqrt{0^2 + 3^2} = 3$

d) $|z| = \dfrac{|1-i|}{|8+8\sqrt{3}i||1-\sqrt{3}i|} = \dfrac{\left(\sqrt{1^2+(1)^2}\right)^{20}}{\sqrt{8^2+(8\sqrt{3})^2}\cdot\left(\sqrt{(1)^2+\left(\sqrt{-3}\right)^2}\right)^6}$

$$=\frac{\left(\sqrt{2}\right)^{20}}{16\cdot 2^{6}}=\frac{2^{10}}{2^{4}\cdot 2^{6}}=1$$

$$\therefore\ |z|=1$$

e) $|z|=|\sqrt{2+3i}|=|2+3i|^{\frac{1}{2}}=\left(\sqrt{2^2+3^2}\right)^{\frac{1}{2}}=\left(\sqrt{13}\right)^{\frac{1}{2}}=\sqrt[4]{13}$

f) $|z|=\left|\sqrt[3]{8+8\sqrt{3}i}\right|=\left|8+8\sqrt{3}i\right|^{\frac{1}{3}}=\left(\sqrt{8^2+(8\sqrt{3})^2}\right)^{\frac{1}{3}}$

$$=\sqrt[3]{16}=2\sqrt[3]{2}$$

9. Simplifying the indicated operation and write the result in the form x + yi

a) $\dfrac{2}{1+i}-\dfrac{3}{1-i}$

b) $\dfrac{(2-3i)(5i)}{2+3i}$

c) $\dfrac{2i}{2+i}+\dfrac{3}{2-i}$

d) $-6i^3+i^2$

e) $(1-2i)^2-(1+2i)^2$

f) $i^{44}+i^{150}-i^{74}+i^{109}+i^{61}$

g) $(3+\sqrt{-5})(7-\sqrt{-10})$

h) $-i(\sqrt{-4}-1)$

i) $4i^2-2i^3$

j) $\left(\sqrt{-2}\right)^6+\left(\sqrt{-5}\right)^3$

Solution:

a) $\dfrac{2}{1+i}-\dfrac{3}{1-i}=\dfrac{2(1-i)-3(1+i)}{(1+i)(1-i)}=\dfrac{2-2i-3-3i}{1^2+1^2}=\dfrac{-1}{2}-\dfrac{5i}{2}$

b) $\dfrac{(2-3i)(5i)}{2+3i}=\dfrac{10i+16}{2+3i}=\dfrac{16+10i}{2+3i}\cdot\left(\dfrac{2-3i}{2-3i}\right)$

$$=\frac{16(2-3i)+10i(2-3i)}{2^2+3^2}$$

$$=\frac{32-48i+20i-30}{13}=\frac{2}{13}-\frac{28i}{13}$$

c) $\dfrac{2i}{2+i}+\dfrac{5}{2-i}=\dfrac{2i(2-i)+5(2+i)}{(2+i)(2-i)}=\dfrac{4i+2+10+5i}{2^2+1^2}=\dfrac{12}{5}+\dfrac{9i}{5}$

d) $-6i^3 + i^2 = -6(-i) - 1 = -1 + 6i$

e) $(1-2i)^2 - (1+2i) = 1^2 - 4i + (2i)^2 - (1^2 + 4i + (2i)^2)$

$= 1 - 4i - 4 - 4i + 4 = 0 - 8i$

f) $i^{44} + i^{50} - i^{74} + i^{109} + i^{61} = i^4 + i^2 - i^2 + i + i = 1 + 2i$

g) $(3+\sqrt{-5})(7-\sqrt{-10}) = (3+\sqrt{5}i)(7-\sqrt{10}i)$

$$= 3(7-\sqrt{10}i) + \sqrt{5}i(7-\sqrt{10}i)$$
$$= 21 - 3\sqrt{10}i + 7\sqrt{5}i + 5\sqrt{2}$$
$$= 21 + 3\sqrt{10}i + 7\sqrt{5}i + 5\sqrt{2}$$

h) $-i(\sqrt{-4}-1) = -i(2i-1) = 2+i$

i) $4i^2 - 2i^3 = 4(-1) - 2(-i) = -4 + 2i$

j) $\left(\sqrt{-2}\right)^6 + \left(\sqrt{-5}\right)^3 = \left(\sqrt{2}i\right)^6 + \left(\sqrt{5}i\right)^3 = -8 - 5\sqrt{5}i$

10. Solve

a) $(3 + 4i)^2 - 2(x - iy) = x + iy$

b) $\left(\frac{1+i}{1-i}\right)^2 + \frac{1}{x+yi} = 1+i$

c) $(3 - 2i)(x + yi) = 2(x - 2iy) + 2i - 1$

Solution:

a) $(3 + 4i)^2 - 2(x - iy) = x + iy$

$\Leftrightarrow 9 + 24i - 16 - 2x + 2iy = x + iy$

$\Rightarrow 24i + 2iy - iy = x + 2x - 9 + 16$

$\Rightarrow 24i + iy = 3x + 7 + 0i$

$\Leftrightarrow (y + 24)i + 0 = (3x + 7) + 0i$

$\Leftrightarrow y + 24 = 0$ and $3x + 7 = 0$

$\therefore y = -24$ and $x = \frac{-7}{3}$

b) $\left(\frac{1+i}{1-i}\right)^2 + \frac{1}{x+yi} = 1+i \Leftrightarrow \frac{1}{x+yi} = 1+i-\left(\frac{1+i}{1-i}\right)^2$

$$\Leftrightarrow \frac{1}{x+yi} = 1+i-\left(\frac{1+i}{1-i}\cdot\frac{1+i}{1+i}\right)^2$$

$$\Leftrightarrow \frac{1}{x+yi} = 1+i-\left(\frac{2i}{2}\right)^2 \Rightarrow \frac{1}{x+yi} = 1+i+1$$

$$\Rightarrow \frac{1}{x+yi} = 2+i \Leftrightarrow x+yi = \frac{1}{2+i} = \frac{1}{2+i}\cdot\frac{2-i}{2-i}1 = \frac{2}{5}-\frac{i}{5}$$

$$\therefore\ x = \frac{2}{5} \text{ and } y = -\frac{1}{5}$$

c) $(3-2i)(x+yi) = (2x-4iy) + 2i - 1$

$\Rightarrow (3-2i)(x+yi) = (2x-1) + (2i-4iy)$

$\Rightarrow 3(x+yi) - 2i(x+yi) = (2x-1) + (2-4y)i$

$\Rightarrow 3x + 3iy - 2ix + 2y = (2x\ -1) + (2-4y)i$

$\Rightarrow (3x+2y) + (3y-2x)i = (2x-1) + (2-4y)i$

$\Rightarrow 3x + 2y = 2x - 1$ and $3y - 2x = 2 - 4y$

$\Rightarrow x + 2y = -1$ and $7y - 2x = 2$

Solving together $\begin{cases} -2x+7y=2 \\ x+2y=-1 \end{cases} \Rightarrow \dfrac{\begin{cases} -2x+7y=2 \\ 2x+4y=-2 \end{cases}}{11y=0} \Rightarrow y=0,\ x=-1$

11. Find a) $|4i^2-2i^3|$ b) $|-6i^3+i^2|$ c) $\left|\frac{-\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right|^{100}$

Solution:

a) $|4i^2-2i^3| = |-4+2i| = \sqrt{(-4)^2+(2)^2} = \sqrt{20} = 2\sqrt{5}$

b) $|-6i^3+i^2| = |6i-1| = \sqrt{6^2+(-1)^2} = \sqrt{36+1} = \sqrt{37}$

c) $\left|\frac{-\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right| = \left(\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2}\right)^{100} = \left(\sqrt{\frac{2}{4}+\frac{2}{4}}\right)^{100} = 1$

12. Solve each of the following: let $z = x + yi$

a) $x^2 + 4x + 20 = 0$

b) $z^2 - i = 0$

c) $z^2 = -5 + 12i$

d) $z = \sqrt{3-4i}$

e) $z = \sqrt{2i}$

Solution: Here

a) $x^2 + 4x + 20 = 0 \Leftrightarrow x^2 + 4x + 4 + 20 - 4 = 0$

$\Rightarrow (x + 2)^2 = -16$

$\Rightarrow x + 2 = \pm\sqrt{-16} \Rightarrow x + 2 = \pm 4i \Rightarrow x = -2 \pm 4i$

$\therefore$ S.S = {-2 -4i, -2 + 4i}

b) $z^2 - i = 0 \Leftrightarrow z^2 = i \Leftrightarrow z = \sqrt{i}$

$\Rightarrow (x + yi)^2 = i \Rightarrow x^2 + 2ixy - y^2 = i + 0$

$\Rightarrow x^2 - y^2 = 0$ and $2ix = i$

$\Rightarrow x = y$ and $2xy = 1$ put x in y

$$\Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

Since $x = y$, $\therefore x = \pm\frac{\sqrt{2}}{2}$ and $y = \pm\frac{\sqrt{2}}{2}$

$$\therefore x + iy = \pm\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$$

$$\therefore \text{S.S} = \left\{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right\}$$

c) $z^2 = -5+12i \Leftrightarrow (x+yi)^2 = -5 +12i$

$\Rightarrow x^2 + 2xyi - y^2 = -5 + 12i$

$$\Rightarrow x^2 - y^2 = -5 \text{ and } 2xy = 12 \Rightarrow y = \frac{12}{2x} = \frac{6}{x}$$

$$\Rightarrow x^2 - \left(\frac{6}{x}\right)^2 = -5 \Rightarrow x^2 - \frac{36}{x^2} = -5 \Rightarrow \frac{x^4 - 36}{x^2} = -5$$

$\Rightarrow x^4 + 5x^2 - 36 = 0$

$$\Rightarrow x^2 = \frac{-5 \pm \sqrt{25 - 4(36)}}{2} = \frac{-5 \pm \sqrt{169}}{2} = \frac{-5 \pm 13}{2}$$

$$\Rightarrow x^2 = \frac{-5 + 13}{2} = \frac{8}{2} = 4$$

$$\Rightarrow x = \pm\sqrt{4} = \pm 2 \text{ and } y = \frac{6}{\pm 2} = \pm 3$$

$\therefore$ the ***Solution*** set are z = {2 + 3i, -2-3i}

d) $z=\sqrt{3-4i} \Leftrightarrow z^2=3-4i \Leftrightarrow (x+yi)^2=3-4i$

$\Rightarrow x^2+2xyi-y^2=3-4i$

$\Rightarrow x^2-y^2=3$ and $2xy=-4 \Rightarrow y=\frac{-2}{x}$

$$\Rightarrow x^2\left(\frac{-2}{x}\right)^2=3 \Rightarrow x^2-\frac{4}{x^2}=3 \Rightarrow x^4-3x^2-4=0$$

$$\Rightarrow x^2=\frac{3\pm\sqrt{9-4(-4)}}{2}=\frac{3\pm 5}{2}=\frac{3+5}{2}=4$$

$\Rightarrow x^2=4 \Rightarrow x=2$ or -2 and $y=\frac{-2}{x}=\frac{-2}{2}=-1$

$y=\frac{-2}{-2}=1$

$\therefore z=2-i$ or $z=-2+i$

$\therefore$ S.S $=\{2-i, -2+i\} \longrightarrow \sqrt{3-4i}=2-i$ or $-2+i$

e) $z=\sqrt{2i} \Leftrightarrow z^2=2i \Leftrightarrow (x+yi)^2=2i$

$\Rightarrow x^2+2xyi-y^2=2i$

$\Rightarrow x^2-y^2=0$ and $2xy=2 \Rightarrow y=\frac{1}{x}$

$$\Rightarrow x^2-\left(\frac{1}{x}\right)^2=0 \Rightarrow x^2-\frac{1}{x^2}=0$$

$\Rightarrow x^4-1=0 \Rightarrow (x^2-1)(x^2+1)=0$

$\Rightarrow x^2-1=0$

$\Rightarrow x=1$ and $y=1$ and $x=-1$, $y=-1$

$\therefore z=1+i$ or $z=-1-i$

$\therefore$ S.S $=\{1+i, -1-i\}$

13. Find the cube root of each of the following.

a) 1 b) -1 c) -8 d) 27 e) 64

Solution: a) $z=\sqrt[3]{1} \Leftrightarrow z^3=1$

$\Rightarrow z^3-1=0 \Leftrightarrow (z-1)(z^2+z+1)=0$

$\Rightarrow z-1=0$ or $z^2+z+1=0$

$\Rightarrow z = 1$ or $z = \dfrac{-1 \pm \sqrt{1-4(1)}}{2} = \dfrac{-1 \pm \sqrt{3}i}{2}$

$\therefore$ S.S $= \left\{1, \dfrac{-1-\sqrt{3}i}{2}, \dfrac{-1+\sqrt{3}i}{2}\right\} \leftarrow$ cube root of 1

b) $z^3 + 1 = 0 \Leftrightarrow (z+1)(z^2 - z + 1) = 0$
$\Rightarrow z = -1$ or $z^2 - z + 1 = 0$

$\Rightarrow z = -1$ or $z = \dfrac{1 \pm \sqrt{1-4(1)}}{2} = \dfrac{1 \pm \sqrt{3}i}{2}$

$\therefore$ S.S $= \left\{-1, \dfrac{1-\sqrt{3}i}{2}, \dfrac{1+\sqrt{3}i}{2}\right\} \leftarrow$ cube root of -1

c) $z^3 + 8 = 0 \Leftrightarrow (z+2)(z^2 - 2z + 4) = 0$
$\Rightarrow z + 2 = 0$ and $z^2 - 2z + 4 = 0$

$\Rightarrow z = -2$ and $z = \dfrac{2 \pm \sqrt{4-4(4)}}{2} = \dfrac{2 \pm 2\sqrt{3}i}{2}$

$\therefore$ S.S $= \left\{-2, 1+\sqrt{3}i, 1-\sqrt{3}i\right\} \leftarrow$ cube root of -8

d) 27, 64 $\leftarrow$ exercise left for you

14. Find the fourth root of

a) 1 b) 16 c) 81 d) 4 e) 3

Solution: a) $z = \sqrt[4]{1} \Leftrightarrow z^4 = 1 \Leftrightarrow z^4 - 1 = 0$
$\Rightarrow z^4 - 1 = 0 \Leftrightarrow (z^2 - 1)(z^2 + 1) = 0$
$\Rightarrow (z-1)(z+1)(z+i)(z-i) = 0$
$\Rightarrow z = 1, -1, -i, i$
$\therefore$ S.S $= \{1, -1, -i, i\} \leftarrow$ cube root of 1

b) $z = \sqrt[4]{16} \Leftrightarrow z^4 = 16 \Leftrightarrow z^4 - 16 = 0$
$\Rightarrow (z^2 - 4)(z^2 + 4) = 0 \Rightarrow z^2 - 4 = 0$ and $z^2 + 4 = 0$
$\Rightarrow (z-2)(z+2)(z+2i)(z-2i) = 0$
$\Rightarrow z = 2, -2, -2i, 2i$
$\therefore$ S.S $= \{2, -2, -2i, 2i\} \leftarrow 4^{\text{th}}$ root of 16

c) $z = \sqrt[4]{81} \Leftrightarrow z^4 = 81 \Leftrightarrow z^4 - 81 = 0$
$\Rightarrow (z^2 - 9)(z^2 + 9) = 0 \Rightarrow (z - 3(z+3)(z+3i)(z -3i)$
$\Rightarrow z = 3, -3, -3i, 3i$

$\therefore$ S.S $= \{3,-3,-3i,3i\}$

d) $z=\sqrt[4]{4} \Leftrightarrow z^4=4 \Leftrightarrow z^4-4=0 \Leftrightarrow (z^2-2)(z^2+2)=0$

$\Rightarrow (z-\sqrt{2})(z+\sqrt{2})(z+\sqrt{2}i)(z-\sqrt{2}i)=0$

$\therefore$ S.S $= \{\sqrt{2},-\sqrt{2},-\sqrt{2}i,\sqrt{2}i\}$

e) $z=\sqrt[4]{3} \Leftrightarrow z^4=3 \Leftrightarrow z^4-3=0$

$\Rightarrow (z^2-\sqrt{3})(z^2+\sqrt{3})=0 \Rightarrow (z-\sqrt{3})(z+\sqrt{3})(z+\sqrt{3}i)(z-\sqrt{3}i)=$

$\therefore$ S.S $= \{\sqrt{3},-\sqrt{3},-\sqrt{3}i,\sqrt{3}i\} \longleftarrow$ fourth root of 3

15. Find the sixth root of:

a) 1 b) 64

Solution: Here

$(x-2)(x+2)$

a) $z=\sqrt[6]{1} \Leftrightarrow z^6=1 \Leftrightarrow z^6-1=0$

$\Rightarrow (z^3)^2-1^2=0 \Rightarrow (z^3-1)(z^3+1)=0$

$\Rightarrow (z-1)(z^2+z+1)(z+1)(z^2-z+1)=0$

$$\Rightarrow z=1 \text{ and } z=\frac{-1\pm\sqrt{1-4}}{2}, z=\frac{1\pm\sqrt{1-4}}{2}$$

$$\Rightarrow z=1,-1,\frac{-1-\sqrt{3}i}{2},\frac{-1+\sqrt{3}i}{2},\frac{1+\sqrt{3}i}{2},\frac{1-\sqrt{3}i}{2}$$

$$\therefore \text{S.S}=\left\{1,-1,\frac{-1-\sqrt{3}i}{2},\frac{-1+\sqrt{3}i}{2},\frac{1+\sqrt{3}i}{2},\frac{1-\sqrt{3}i}{2}\right\}$$

b) $z=\sqrt[6]{64} \Leftrightarrow z^6=64 \Leftrightarrow z^6-64=0$

$\Rightarrow (z^3)^2-8^2=0 \Rightarrow (z^3-8)(z^3+8)=0$

$\Rightarrow (z-2)(z+2)(z^2+2z+4)(z^2-2z+4)=0$

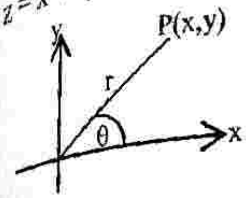
$$\therefore z=2,-2 \text{ and } z=\frac{-2\pm\sqrt{4-16}}{2} \text{ and } z=\frac{2\pm\sqrt{4-16}}{2}$$

$\therefore$ S.S $= \{2,-2,-1-\sqrt{3}i,-1+\sqrt{3}i,1-\sqrt{3}i,1+\sqrt{3}i\} \longleftarrow$ sixth root of 64

Argand diagram and polar representation of complex number

Argand Diagrams

There are two geometrical representation of the complex number $z = x + iy$



i) as the point p(x, y) in the xy plane

ii) as the vector from the origin to P(polar representation)

In each representation, the x-axis is called real axis and the y-axis is called the imaginary axis.

- A coordinate plane with a complex number assigned to each point is referred to as a complex (or Argand) plane instead of an xy-plane.
- Geometrical interpretation of a complex number $z = x+yi$ in complex plane:

The length of the line segment from the origin to the point P(x, y) in the complex plane is $|z| = \sqrt{x^2 + y^2}$

The distance between the points to z_1 and z_2 in the complex plane is $|z_1 - z_2|$

Example: The point corresponding to the complex numbers, z with $|z| = 1$ are on the unit circle.

16. Describe each of the following geometrically (graph, $z = x + yi$).

a) $|z - 2| = 3$ c) $|z + 2 - i| = 1$

b) $|z + i| < 2$ d) $|z + 2 - i| > 1$

Solution: a) $|z - 2| = 3 \Rightarrow |x+yi - 2| = 3 \Rightarrow |(x + 2) + yi| = 3$

$$\Rightarrow \sqrt{(x-2)^2 + y^2} = 3 \Rightarrow (x-2)^2 + y^2 = 9$$

Point on the circle $(x - 2)^2 + y^2 = 9$ with center (2, 0), r = 3

b) $|z + i| < 2 \Rightarrow |x + yi + i| < 2 \Rightarrow |x+(y+1)i| < 2$

$$\Rightarrow \sqrt{x^2 + (y+1)^2} < 2$$

$$\Rightarrow (x^2 + (y + 1)^2) < 4$$

$\therefore$ Point inside the circle $x^2 + (y + 1)^2 = 4$

c) $|z + 2- i| = 1 \Rightarrow |x + yi + 2 - i| = 1 \Rightarrow |(x+2)+(y-1)i| = 1$

$\Rightarrow (x + 2)^2 + (y - 1)^2$? 1

$\therefore$ Point outside the circle $(x + 2)^2 + (y - 1)^2 = 1$

17. Describe each of the following geometrically

a) $|z - 1| = |z + 2|$ c) $|z + 2i| = |z - 3|$

b) $|z + i| = |z - 1|$ d) $|z + 1| \geq |z|$

Solution: a) $|z - 1| = |z + 2| \Rightarrow |(x-1) + yi| = |(x+2) + yi|$

$\Rightarrow (x-1)^2 + y^2 = (x + 2)^2 + y^2$

$\Rightarrow x^2 - 2x + 1 + y^2 = x^2 + 4x + 4 + y^2$

$\Rightarrow -6x = 3 \Rightarrow x = \dfrac{-3}{6} = \dfrac{-1}{2}$

$\therefore$ Points on the line $x = \dfrac{-1}{2}$

b) $|z + i| = |z - 1| \Rightarrow |(x + (y + 1)i| = |(x - 1) + yi|$

$\Rightarrow x^2 + (y + 1)^2 = (x - 1)^2 + y^2 \Rightarrow 2y+1 = -2x+1$

$\Rightarrow y = -x \longrightarrow$ point on the line $y = -x$

c) $|z + 2i| = |z - 3| \Rightarrow |x + (y + 2)i| = |(x - 3) + yi|$

$\Rightarrow x^2 + (y+2)^2 = (x-3)^2+y^2 \Rightarrow 4y+4=-6x+9$

$\therefore$ point on the line $6x + 4y = 5$

18. How may following complex numbers, be obtained from $z = x + yi$ geometrically

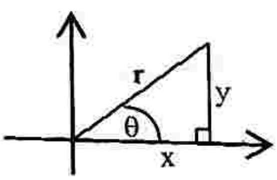
a) $\bar{z}$ b) $\overline{(-z)}$

Solution: $z = x + yi \Rightarrow \bar{z} = x - yi$

$\therefore$ a) $\bar{z} = x - yi$, by reflecting z on real x – axis

b) $\overline{(-z)} = -x + yi$ by reflecting z on imaginary (y) axis.

ii. Polar representation of complex Number (Trigonometric form of a complex number)



- A complex number $z = x + yi$ can be write interms of trigonometric

$\Rightarrow x = r\cos\theta,\ |z| = r = \sqrt{x^2 + y^2}$

$\Rightarrow y = r\sin\theta$

$\therefore z = x + yi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

This is called polar representation of z

- The polar angle θ is called the argument of z and is written

$$\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

- r is the modulus of a complex number z.

- $r=\sqrt{x^2+y^2}$
- $r(\cos\theta + i\sin\theta) = r(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi))$
  Where K is integer

Principal argument of z = x + yi

The principal argument is polar angle θ which is denoted by arg(z) is an angle between $-\pi < \arg(z) \le \pi$

- The principal argument of z is **calculated by:**

$$Arg(z)=\theta=\begin{cases}\tan^{-1}\left(\frac{y}{x}\right) & \text{if } z=x+yi \text{ lies on } 1^{st} \text{ quadrant}\\ \pi+\tan^{-1}\left(\frac{y}{x}\right) & \text{if } z=x+yi \text{ lies } 2^{nd} \text{ quadrant } (x<0, y>0)\\ \tan^{-1}\left(\frac{y}{x}\right)-\pi & \text{if } z=x+yi \text{ lies on } 3^{rd} \text{ quadrant } (x<0, y<0)\\ \tan^{-1}\left(\frac{y}{x}\right) & \text{if } z=x+yi \text{ lies on } 4^{th} \text{ quadrant}\end{cases}$$

Note: $\tan^{-1}(-x) = -\tan^{-1}(x)$

Note: If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then $z_1 = z_2 \Leftrightarrow r_1 = r_2$ and $\theta_1 = \theta_2 + 2\pi k,\ k \in z$.

Example:

a) $2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=2\left(\cos\frac{13\pi}{6}+i\sin\frac{13\pi}{6}\right)=2\left(\cos\frac{11\pi}{6}-i\sin\frac{11\pi}{6}\right)$

b) $3\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=3\left(\cos\frac{9\pi}{4}+i\sin\frac{9\pi}{4}\right)=3\left(\cos\frac{7\pi}{4}-i\sin\frac{7\pi}{4}\right)$

c) $6\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)=6\left(\cos\frac{11\pi}{5}+i\sin\frac{11\pi}{5}\right)=6\left(\cos\frac{9\pi}{5}-i\sin\frac{9\pi}{5}\right)$

d) $4\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)=4\left(\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}\right)=z=-2-2\sqrt{3}i$

19. Find the principal argument, and polar form of each of the following.

a) $z = 1 + i$ c) $z = 3\sqrt{3}+3i$

b) $z = 1+\sqrt{3}i$ d) $z = 4 + 3i$

Solution: since all of them are in first quadrant thus, the principal argument is given by.

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$, therefore

a) $\text{Arg}(1+i) = \theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$ ← principal argument

Modulus of z, r = $\sqrt{1^2+1^2} = \sqrt{2}$

$\therefore \left(\sqrt{2}, \frac{\pi}{4}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

c) $\text{Arg}(1+\sqrt{3}i) = \theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$ ← principal argument

$r = \sqrt{1^2+\sqrt{3}^2} = \sqrt{1+3} = 2$

$\therefore (r,\theta) = \left(2, \frac{\pi}{3}\right) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

c) $\text{Arg}\left(3\sqrt{3}+3i\right) = \theta = \tan^{-1}\left(\frac{3}{3\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ ← principal argument

$r = \sqrt{\left(3\sqrt{3}\right)^2 + 3^2} = 6$

$\therefore (r,\theta) = \left(6, \frac{\pi}{6}\right) = 6\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

d) $\text{Arg}(4+3i) = \theta = \tan^{-1}\left(\frac{3}{4}\right)$

$r = \sqrt{3^2+4^2} = 5$

$\therefore (r,\theta) = \left(5, \tan^{-1}\left(\frac{3}{4}\right)\right) = 5\left(\cos\left(\tan^{-1}\left(\frac{3}{4}\right)\right) + i\sin\left(\tan^{-1}\left(\frac{3}{4}\right)\right)\right)$

20. Find the principal argument and express in polar form

a) $z = -2+2i$ b) $z = -1+\sqrt{3}i$ c) $-2\sqrt{3}+2i$

Solution: z = -2 + 2i, z = -1 + $\sqrt{3}i$ and z = $-2\sqrt{3}+2i$ are found in **quadrant two.**

Therefore, the principal argument is given by

$\text{Arg}(z) = \pi + \tan^{-1}\left(\frac{y}{x}\right)$, hence

a) $\text{Arg}(-2+2i) = \theta = \pi + (\tan^{-1}\left(\frac{2}{-2}\right) = \pi + -\tan^{-1}(1)$

$= \pi - \frac{\pi}{4} = \frac{3\pi}{4} \longleftarrow$ principal argument

$r = \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

$\therefore (r, \theta) = \left(2\sqrt{2}, \frac{3\pi}{4}\right) = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

b) $\text{Arg}\,(1+\sqrt{3}i) = \theta = \pi + \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \pi - \tan^{-1}\left(\sqrt{3}\right) = \pi - \frac{\pi}{3} = \frac{2}{3}\pi$

$r = \sqrt{(-1)^2 + \sqrt{3}^2} = \sqrt{1+3} = 2$

$\therefore (r, \theta) = \left(2, \frac{2}{3}\pi\right) = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$\text{Arg}(-2\sqrt{3}+2i) = \theta = \pi + \tan^{-1}\left(\frac{2}{-2\sqrt{3}}\right) = \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi - \frac{\pi}{6} = \frac{5}{6}\pi$

$r = \sqrt{\left(-2\sqrt{3}\right)^2 + 2^2} = \sqrt{12+4} = \sqrt{16} = 4$

$\therefore (r, \theta) = \left(4, \frac{5}{6}\pi\right) = 4\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{6}\right)$

Find the principal argument and express in polar form:

a) z = -2 – 2i b) $z = -2\sqrt{3} - 2i$ c) $z = -4 - 4\sqrt{3}i$

Solution: z = -2 – 2i, z = $-2\sqrt{3} - 2i)$ *and* $z = -4 - 4\sqrt{3}i$ are all in 3rd quadrant, therefore the principal argument θ is given by

$\text{Arg}(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right) - \pi$, Hence

a) $\text{Arg}(-2-2i) = \theta = \tan^{-1}\left(\frac{-2}{-2}\right) - \pi = \tan^{-1}(1) - \pi = \frac{\pi}{4} - \pi = \frac{-3\pi}{4}$

$r = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$

$\therefore (r, \theta) = \left(2\sqrt{2}, \frac{-3\pi}{4}\right) = 2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) - i\sin\frac{3\pi}{4}\right)$

It is also possible, as: $\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

But $\frac{5\pi}{4}$ is not principal argument.

b) $\text{Arg}(-2\sqrt{3} - 2i) = \theta = \tan^{-1}\left(\frac{-2}{-2\sqrt{3}}\right) - \pi = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \pi$

$\Rightarrow \frac{\pi}{6} - \pi = \frac{5\pi}{6} \leftarrow$ principal argument

$r = \sqrt{\left(-2\sqrt{3}\right)^2 + (-2)^2} = \sqrt{16} = 4$

$\therefore (r, \theta) = \left(4, \frac{-5\pi}{6}\right) = 4\left(\cos\left(\frac{5\pi}{6}\right) - i\sin\left(\frac{5\pi}{6}\right)\right)$

c) $\text{Arg}(-4 - 4\sqrt{3}i) = \theta = \tan^{-1}\left(\frac{-4\sqrt{3}}{-4}\right) - \pi = \tan^{-1}\left(\sqrt{3}\right) - \pi$

$\Rightarrow \frac{\pi}{3} - \pi = \frac{-2}{3}\pi \longleftarrow$ principle argument

$r = \sqrt{(-4)^2 + (-4\sqrt{3})^2} = \sqrt{64} = 8$

$\therefore (r, \theta) = \left(8, \frac{-2}{3}\pi\right) = 8\left(\cos\frac{2}{3}\pi - i\sin\frac{2\pi}{3}\right)$

22. Find the principal argument and express in polar form.

a) $z = 4 - 4i$ b) $z = 1 - \sqrt{3}i$ c) $z = 2\sqrt{3} - 2i$

Solution: $z = 4 - 4i$, $z = 1 - \sqrt{3}i$ and $z = 2\sqrt{3} - 2i$ are all in 4^{th} quadrant, therefore, the principal argument θ is calculated by $\arg(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$. Hence

a) $\text{Arg}(4-4i) = \theta = \tan^{-1}\left(\frac{-4}{4}\right) = -\tan^{-1}(1) = -\left(\frac{\pi}{4}\right) \leftarrow$ principal argument

$r = \sqrt{4^2 + (-4)^2} = 8$

$\therefore (r, \theta) = \left(8, \frac{-\pi}{4}\right) = 4\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

It is also possible as: $8\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

b) $\text{Arg}(1-\sqrt{3}i) = \theta = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = \tan^{-1}\left(-\sqrt{3}\right)$

$= -\tan^{-1}\left(\sqrt{3}\right) = \frac{-\pi}{3}$

$r = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

$\therefore (r, \theta) = \left(2, \frac{-\pi}{3}\right) = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

Possible like to express:

$\frac{-\pi}{3} + 2\pi = \frac{5\pi}{3} \Rightarrow 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

c) $\text{Arg}(2\sqrt{3} - 2i) = \theta = \tan^{-1}\left(\frac{-2}{2\sqrt{3}}\right)$

$= \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{-\pi}{6}$

$r = \sqrt{\left(2\sqrt{3}\right)^2 + (-2)^2} = \sqrt{12+4} = 4$

$$\therefore (r, \theta) = \left(4, \frac{-\pi}{6}\right) = 4\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$$

Possible like to express: $\frac{-\pi}{6} + 2\pi = \frac{11\pi}{6}$

$$\Rightarrow 4\left(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right)$$

23. Find the principal argument θ

a) $z = \frac{\sqrt{3}}{2} - \frac{3}{2}i$ b) z = -4i c) z = -5

d) z = 2i e) $z = -2 + 2\sqrt{3}i$ f) z = -3 – 3i

Solution: a) $z = \frac{\sqrt{3}}{2} - \frac{3}{2}i$ it is on 4[th] quadrant.

$$\therefore \text{principal, } \arg(z) = \theta = \tan^{-1}\left(\frac{\frac{3}{-2}}{\frac{\sqrt{3}}{2}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\therefore \arg(z) = -\frac{\pi}{6}$$

b) since z = -4i concide with negative y axis

$$\therefore \text{principal, } \arg(-4i) = \theta = \tan^{-1}\left(\frac{-4}{0}\right) = -\frac{\pi}{2}$$

c) since z = -5 + 0i concide with negative x-axis

∴ principal, arg(-5) = θ = π

d) since z = 2i, concide wit positive y-axis

$$\therefore \arg(z) = \arg(2i) = \frac{\pi}{2}$$

e) since $z = -2 + 2\sqrt{3}i$ lies on 2[nd] quadrant

$$\therefore \arg(-2 + 2\sqrt{3}i) = \theta = \pi + \left(\tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = \pi - \tan^{-1}(\sqrt{3})\right)$$

$$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

f) since $z = -3 - 3i$ lies on 3^{rd} quadrant

$$\therefore \arg(-3-3i) = \theta = \tan^{-1}\left(\frac{-3}{-3}\right) - \pi = \tan^{-1}(1) - \pi = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$$

24. Find the corresponding complex number in rectangular form

a) $\left(4, \frac{2\pi}{3}\right)$ b) $\left(3, \frac{4\pi}{3}\right)$ c) $\left(2, \frac{-\pi}{3}\right)$ d) $(4, \pi)$

e) $\left(7, \frac{3}{2}\pi\right)$ f) $\left(5, \frac{-\pi}{4}\right)$ g) $\left(1 \frac{5\pi}{6}\right)$

Solution:

a) $\left(4, \frac{2\pi}{3}\right) = 4\left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right) = 4\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -2 + 2\sqrt{3}i$

b) $\left(3, \frac{4\pi}{3}\right) = 3\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = 3\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \frac{-3}{2} - \frac{\sqrt{3}}{2}i$

c) $\left(2, \frac{-\pi}{3}\right) = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1 - \sqrt{3}i$

d) $(4, \pi) = 4(\cos\pi + i\sin\pi) = (-1 + 0i) = -4$

e) $\left(7, \frac{3\pi}{2}\right) = 7\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = 7(0 - i) = -7i$

f) $\left(5, \frac{-\pi}{4}\right) = 5\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) = 5\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = \frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}i$

g) $\left(1, \frac{5\pi}{6}\right) = 1\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = \frac{\sqrt{3}}{2} + i\frac{1}{2} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

Product and quotient of complex number

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be complex number, then

i) $z_1 \cdot z_2 = r_1 \cdot r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

$$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

f) since z = -3 – 3i lies on 3rd quadrant

$$\therefore \arg(-3-3i) = \theta = \tan^{-1}\left(\frac{-3}{-3}\right) - \pi = \tan^{-1}(1) - \pi = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$$

24. Find the corresponding complex number in rectangular form

a) $\left(4, \frac{2\pi}{3}\right)$ b) $\left(3, \frac{4\pi}{3}\right)$ c) $\left(2, \frac{-\pi}{3}\right)$ d) $(4, \pi)$

e) $\left(7, \frac{3}{2}\pi\right)$ f) $\left(5, \frac{-\pi}{4}\right)$ g) $\left(1\frac{5\pi}{6}\right)$

Solution:

a) $\left(4, \frac{2\pi}{3}\right) = 4\left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right) = 4\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -2 + 2\sqrt{3}i$

b) $\left(3, \frac{4\pi}{3}\right) = 3\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = 3\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \frac{-3}{2} - \frac{\sqrt{3}}{2}i$

c) $\left(2, \frac{-\pi}{3}\right) = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1 - \sqrt{3}i$

d) $(4, \pi) = 4(\cos\pi + i\sin\pi) = (-1 + 0i) = -4$

e) $\left(7, \frac{3\pi}{2}\right) = 7\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = 7(0 - i) = -7i$

f) $\left(5, \frac{-\pi}{4}\right) = 5\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) = 5\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = \frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}i$

g) $\left(1, \frac{5\pi}{6}\right) = 1\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = \frac{\sqrt{3}}{2} + i\frac{1}{2} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

Product and quotient of complex number

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be complex number, then

i) $z_1 \cdot z_2 = r_1 \cdot r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

ii) $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$

- **Demoivies theorem**

iii) $z^n = r^n(\cos(n\theta) + i\sin\theta)$ for any integer

iv) $z^{-1} = r^{-1}(\cos\theta - i\sin\theta)$. If $n = -1$

$$\frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta)$$

Note: • $\text{Arg}(z_1 \cdot z_2) = \text{Arg}z_1 + \text{Arg}z_2 + 2\pi k$

• $\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2) + 2\pi k$

• $\text{Arg}(z^n) = n\text{Agr}(z) + 2\pi k$, where k, is integer

25. Find the indicated operation and express the results in rectangular form.

i) Let $z_1 = 4(\cos 15 + i\sin 15)$
$z_2 = 3(\cos 45 + i\sin 45)$ then find

a) $z_1 \cdot z_2$ b) $\frac{z_2}{z_1}$ c) $\frac{z_1}{z_2}$ d) $\frac{1}{z_2}$

e) $\text{Arg}(z_1 \cdot z_2)$ f) $\text{Arg}\left(\frac{z_1}{z_2}\right)$ g) $\text{Arg}\left(\frac{1}{z_2}\right)$

Solution: Here

a) $z_1 \cdot z_2 = (4 \cdot 3)(\cos(15^0 + 45^0) + i\sin(15^0 + 45^0))$

$$= 12(\cos 60^0 + i\sin 60^0) = 12\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 6 + 6\sqrt{3}i$$

b) $\frac{z_2}{z_1} = \frac{3}{4}(\cos(45^0 - 15^0) + i\sin(45^0 - 15^0)$

$$= \frac{3}{4}(\cos 30^0 - i\sin 30^0) = \frac{3}{4}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \frac{3\sqrt{3}}{8} + \frac{3}{8}i$$

c) $\frac{z_1}{z_2} = \frac{4}{3}(\cos(15^0 - 45^0) + i\sin(15^0 - 45^0)$

$$= \frac{4}{3}(\cos30^0 - i\sin30^0) = \frac{4}{3}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \frac{2\sqrt{3}}{3} - \frac{2}{3}i$$

d) $$\frac{1}{z_2} = \frac{1}{3}(\cos45^0 - i\sin45^0) = \frac{1}{3}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = \frac{\sqrt{2}}{6} - \frac{\sqrt{2}}{6}i$$

e) $\text{Arg}(z_1 \cdot z_2) = \text{Arg}z_1 + \text{Arg}z_2 = 15^0 + 45^0 = 60^0 + 2\pi k$,
k is integer

f) $$\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2) = 15^0 - 45^0 = -30^0 + 2\pi k,$$

k is integer

g) $$\text{Arg}\left(\frac{1}{z_2}\right) = -1\ \arg(z_2) = -(45^0) = -45^0$$

26. Let $z_1 = \sqrt{2}\,(\cos 84^0 + i\sin84^0)$
$z_2 = \sqrt{3}\,(\cos 126^0 + i\sin126^0)$ then find rectangular from of $z_1 \cdot z_2$:

Solution: $z_1 \cdot z_2 = \sqrt{2}\cdot\sqrt{3}\,(\cos(84^0+126^0) + i\sin(84^0+126^0))$

$$= \sqrt{6}\,(\cos210^0 + i\sin210^0)$$

$$= \sqrt{6}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \frac{-3\sqrt{2}}{2} - \frac{\sqrt{6}i}{2}$$

27. Let $z = 1(\cos20^0 + i\sin20^0)$ then find rectangular form of $z = x + yi$ and principal argument of each.

a) z^{30} b) z^{-15} c) z^9 d) z^{-12}

Solution: a) $z^{30} = 1^{30}(\cos20^0 + i\sin20^0)^{30}$

$$= 1(\cos600^0 + i\sin600^0)$$

$$= \cos240^0 + i\sin240^0) \longrightarrow 600^0 - 360^0 = 240^0$$

$$= \cos120^0 - i\sin120^0) \longrightarrow 240^0 - 360^0 = -120^0$$

$$= -\frac{1}{2} - \frac{\sqrt{3}}{2} \longrightarrow \sin 120^0 = \frac{\sqrt{3}}{2}$$

$$\cos 120^0 = -\frac{1}{2}$$

- $\text{Arg}(z^{30}) = 30(\text{Arg}z) = 30 \times 20^0 = 600^0$

$\therefore$ Principal argument is $600^0 - 720^0 = -120^0$

b) $z^{-15} = (\cos 20^0 + i\sin 20^0)^{-15} = \cos 300^0 - i\sin 300^0$

$= \cos 60^0 + i\sin 60^0 \longrightarrow 360^0 - 300^0 = 60^0$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$\therefore$ Principal $\arg(z^{-15}) = -15\arg(z) + 2\pi$

$= -15(20^0) + 2\pi = -300^0 + 360^0 = 60^0$

c) $z^9 = 1^9(\cos 20^0 + i\sin 20^0)^9 = 1(\cos 180^0 + i\sin 180^0)$

$= -1 + 0 = -1$

$\therefore$ Principal $\arg(z^9) = 9\arg(z) = (9)(20^0) = 180^0$

d) $z^{-12} = 1(\cos 240^0 - i\sin 240^0) = 1(\cos 120^0 + i\sin 120^0)$

$= \frac{-1}{2} + \frac{\sqrt{3}}{2}i$

$\therefore \text{Arg}(z^{-12}) = -12\text{Arg}(z) = -12 \times 20^0 = -240$

$\therefore$ Principal $\arg(z^{-12}) = -12\arg(z) + 360^0 = -240^0 + 360^0$

$= 120^0$

28. Let $z_1 = 2\sqrt{3} - 2i$ and $z_2 = -1 + \sqrt{3}i$, then find

a) polar form of $(z_1 \cdot z_2)$ c) $\text{Arg}(z_1 \cdot z_2)$

b) polar form of $\left(\frac{z_1}{z_2}\right)$ d) $\text{Arg}\left(\frac{z}{z_2}\right)$

Solution: • $|z_1| = r_1 = \sqrt{(2\sqrt{3})^2 + (-2)^2} = \sqrt{12+4} = 4 \leftarrow r_1$

• $Arg(z_1) = \theta_1 = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-2}{2\sqrt{3}}\right) = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

• $|z_2| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2 \leftarrow r_2$

• since $z_2 = -1 + \sqrt{\sqrt{3}^2}\, i$ lies on 2^{nd} quadrant principal

$\text{Arg}(z_2) = \pi + \tan^{-1}\left(\frac{y}{x}\right) = \pi + \tan^{-1}(-\sqrt{3}) = \pi - \tan^{-1}(\sqrt{3})$

$= \pi - \frac{\pi}{3} = \frac{2}{3}\pi$

Thus, a) $z_1 \cdot z_2 = r_1 r_2((\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

$$= 2.4\left(\cos\left(\frac{-\pi}{6}+\frac{2\pi}{3}\right)+i\sin\left(\frac{-\pi}{6}+\frac{2}{3}\pi\right)\right)$$

$$= 8\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right) \leftarrow \textit{polar form of } z_1\cdot z_2$$

$$= 8(0+i) - 8i \longleftarrow \text{rectangular form}$$

b) $$\frac{z_1}{z_2}=\frac{r_1}{r_2}(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2))$$

$$=\frac{4}{2}\left(\cos\left(-\frac{\pi}{6}-\frac{2\pi}{3}\right)+i\sin\left(\frac{-\pi}{6}-\frac{-2\pi}{3}\right)\right)$$

$$=2\left(\cos\frac{5\pi}{3}-i\sin\frac{5\pi}{3}\right) \longleftarrow \text{polar form of } \frac{z_1}{z_2}$$

$$=2\left[-\frac{\sqrt{3}}{2}+i\left(\frac{-1}{2}\right)\right]=-\sqrt{3}-i \leftarrow \text{rectangular form of } \frac{z_1}{z_2}$$

c) $\operatorname{Arg}(z_1\cdot z_2)=\operatorname{Arg}(z_1)+\operatorname{Arg}(z_2)=\dfrac{-\pi}{6}+\dfrac{2}{3}\pi=\dfrac{\pi}{2}$

d) $\operatorname{Arg}\left(\dfrac{z}{z_2}\right)=\operatorname{Arg}(z_1)-\operatorname{Arg}(z_2)=\dfrac{-\pi}{6}-\left(\dfrac{2}{3}\pi\right)=\dfrac{-5\pi}{6}$

29. Find, polar form, principal argument modulus of z^{12}

a) $z=\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i$ b) $z=\dfrac{-\sqrt{3}}{2}-\dfrac{1}{2}i$ c) $z=\sqrt{2}-\sqrt{6}i$

Solution: If $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos n\theta + i\sin n\theta)$ and substitute $n = 12$

a) $$Arg\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)=\tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)=\tan^{-1}(1)=\frac{\pi}{4}$$

$$|z|=\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2}=\sqrt{\frac{2}{4}+\frac{2}{4}}=1$$

$$z=\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\Rightarrow z^{12}=1^{12}\left(\cos\left(12\left(\frac{\pi}{4}\right)\right)+i\sin\left(12\left(\frac{\pi}{4}\right)\right)\right)$$

$= \cos 3\pi + i\sin 3\pi \longleftarrow$ polar form $= -1+0 = -1$

$$\therefore\ z^{12}=\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^{12} = \cos 3\pi + i\sin 3\pi = -1$$

$|z^{12}| = |-1|^{12} = 1 \longleftarrow$ modulus of $\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^{12}$

b) $$z=\left(\frac{-\sqrt{3}}{2}-\frac{1}{2}i\right)^{12}$$

$$Arg(z)=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)-\pi=\frac{\pi}{6}-\pi=\frac{-5\pi}{6}$$

$$z^{12}=\left(1\left(\cos\frac{5\pi}{6}-i\sin\frac{5\pi}{6}\right)\right)^{12} = \cos 10\pi - i\sin 10\pi)$$

$= \cos 0^0 - i\sin 0^0 \longleftarrow$ polar form

$\therefore$ Principal $\arg(z^{12}) = 0^0$

$$\therefore\ \left(\frac{-\sqrt{3}}{2}-\frac{1}{2}i\right)^{12} = 1+0i$$

c) $$z=\sqrt{2}-\sqrt{6}i, \Rightarrow z^{12}=\left(\sqrt{2}-\sqrt{6}i\right)^{12}$$

$$\text{Arg}(z)=\tan^{-1}\left(\frac{-\sqrt{6}}{\sqrt{2}}\right)=-\tan^{-1}\left(\sqrt{3}\right)=-\frac{\pi}{3}$$

$$|z|=\sqrt{(\sqrt{2})^2+(-\sqrt{6})^2}=\sqrt{2+6}=\sqrt{8}$$

379

$$z=\sqrt{8}\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)\Rightarrow z^{12}=\sqrt{8}^{12}\left(\cos\left(12\cdot\frac{\pi}{3}\right)-i\sin\left(12\cdot\frac{\pi}{3}\right)\right)$$

$$\Rightarrow z^{12}=\left(2^{\frac{3}{2}}\right)^{12}(\cos 4\pi - i\sin 4\pi) = 2^{18}(1-0i) = 2^{18}$$

$$\therefore z^{12}=\left(\sqrt{2}-\sqrt{6}i\right)^{12}=2^{18}(\cos 0^0 - i\sin 0^0) \longleftarrow \text{polar form}$$

- Modulus: $|z^{12}| = 2^{18}$
- Principal $\arg(z^{12}) = 0^0$

30. Find Modulus, principal argument and polar form:

a) $z=\dfrac{\sqrt{3}+\sqrt{3}i}{\left(1-\sqrt{3}i\right)\left(2-2i\right)}$ b) $z=\left(\dfrac{2\sqrt{3}+2i}{2+2i}\right)$

ⓧ NB

Solution: a) $z=\left|\dfrac{\left(\sqrt{3}+\sqrt{3}i\right)}{\left(1-\sqrt{3}i\right)\left(2-2i\right)}\right|=\dfrac{\left|\sqrt{3}+\sqrt{3}i\right|}{\left|1-\sqrt{3}i\right|\left|2-2i\right|}$

$$=\frac{\sqrt{\sqrt{3}^2+\sqrt{3}^2}}{\sqrt{1^2+\left(\sqrt{3}\right)^2}\cdot\sqrt{2^2+(-2)^2}}=\frac{\sqrt{6}}{2\sqrt{8}}=\frac{\sqrt{3}}{4}$$

- $Arg\left(\dfrac{\sqrt{3}+\sqrt{3}i}{\left(1-\sqrt{3}i\right)\left(2-2i\right)}\right)=Arg\left(\sqrt{3}+\sqrt{3}i\right)-[(Arg 1-\sqrt{3})+Arg(2-2i)]$

$$=\tan^{-1}\left(\frac{\sqrt{3}}{\sqrt{3}}\right)-\left(\tan^{-1}\left(\frac{-\sqrt{3}}{1}\right)+\tan^{-1}\left(\frac{-2}{2}\right)\right)$$

$$=\tan^{-1}(1)-\left(-\tan^{-1}\left(\sqrt{3}\right)-\tan^{-1}(1)\right)$$

$$=\frac{\pi}{4}-\left(\frac{-\pi}{3}-\frac{\pi}{4}\right)=\frac{10\pi}{12}=\frac{5}{6}\pi$$

$\therefore$ Principal arg(z) = $=\dfrac{5\pi}{6}$

$$\therefore z=\frac{\sqrt{3}+\sqrt{3}i}{\left(1-\sqrt{3}i\right)\left(2-2i\right)}=\frac{\sqrt{3}}{4}\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$$

$$= \frac{\sqrt{3}}{4}\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) = \frac{-3}{8}+\frac{\sqrt{3}}{8}i$$

b) $$z = \left|\frac{2\sqrt{3}+2i}{2+2i}\right| = \frac{\left|2\sqrt{3}+2i\right|^{16}}{\left|2+2i\right|^{16}}$$

$$= \frac{\left(\sqrt{\left(2\sqrt{3}\right)^2+2^2}\right)^{16}}{\left(\sqrt{2^2+2^2}\right)^{16}} = \frac{4^{16}}{\sqrt{8}^{16}} = \left(\sqrt{\frac{16}{8}}\right)^{16} = 2^8 = 256$$

- $\text{Arg}(z) = 16(\text{Arg}(z) = 16((\text{Arg}\left(2\sqrt{3}+2i\right) - \text{Arg}(2+2i))$

$$= 16\left(\frac{\pi}{6}-\frac{\pi}{4}\right) = \frac{-16\pi}{12} \longleftarrow \text{It is not principal argument of (z)}$$

$$\therefore \text{ Principal arg}(z) = \frac{-16\pi}{12}+2\pi = \frac{8\pi}{12} = \frac{2}{3}\pi$$

$$\therefore\ z = \left(\frac{2\sqrt{3}+2i}{2+2i}\right)^{16} = 256\left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right) \longleftarrow \text{polar form}$$

$$= 256\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right) = -128+128\sqrt{3}i \longleftarrow \text{rectangular form}$$

Solved Problems

31. The modulus of $\frac{1}{i^3}+\frac{-5+3i}{12-5i}$ is equal to .... UEE 2007

A. $\frac{\sqrt{15}}{13}$ B. $\frac{15}{13}$ C. $\frac{1+\sqrt{34}}{13}$ D. $\frac{20}{13}$

Solution:

$$\left|\frac{1}{i^3}+\frac{-5+3i}{12-5i}\right| = \left|\frac{12-5i+5i+3}{-i(12-5i)}\right| = \frac{|15|}{|-12i-5|} = \frac{15}{\sqrt{144+25}} = \frac{15}{13}$$

Answer: B

32. Simplified form of $i(1+\sqrt{3}i)^6 - 65i$ is equal to

A. 1 B. i C. –i D. i + 1

Solution:

$$i(1+\sqrt{3}i)^6 - 2^6\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^6 = 2^6(\cos 2\pi + i\sin 2\pi) = 64$$

$$\therefore i(1+\sqrt{3}i)^6 - 65i = i(64) - 65i = 64i - 65i = -i$$

Answer: C

33. If the complex number a + bi corresponds to the point (a, b) to what point does 2i - πi correspond ------2008.

A. (2, -π) B. (2-π, 0) C. (0, 2 -π) D. (2, π)

Solution: 2i - πi = 0 + (2-π)i = (0, 2-π), ∴ Answer C

34. What are the value of real number x and y that satisfies $\frac{x+3i}{4-2i} = \frac{2+yi}{20}$ ------------ UEE 2009

A. x = 2, y = 3 C. x = 2, y = 16

B. x = 6, y = 10 D. x = 4, y = 6

Solution: $\frac{x+3i}{4-2i} = \frac{2+yi}{20} \Rightarrow 20x + 60i = (4-2i)(2+yi)$

$= 8 + 4yi - 4i + 2y$

$\Rightarrow 20x + 60i = 8 + (4y-4)i + 2y$

$\Rightarrow 4y - 4 = 60 \Rightarrow 4y = 64 \Rightarrow y = 16$

$\Rightarrow 20x = 8 + 2y \Leftrightarrow 20x = 8+2(16) = 40$

$\Rightarrow 20x = 40 \Leftrightarrow x = 2$ **Answer: C**

35. For z = x + yi, which of the following represent the circle $(x-3)^2 + (y+1)^2 = 4$ …………… (UEE 2001)

A. $|z+3-i| = 4$ C. $|z+3-i| = 2$

B. $|z-3+i| = 4$ D. $|z-3+i| = 2$

Solution: $|z-3-i| = 2 \Rightarrow |x+yi-3+i| = 2$

$\Rightarrow |(x-3| + (y+1)i| = 2$

$= (x-3)^2 + (y+1)^2 = 4$

Answer: D

36. Let z be a complex number such that $\left|\frac{3z}{-4+3i}\right| = (2-i)\overline{(2-i)}$

Then the modulus of is equal to: ----------- UEE

A. $\frac{25}{3}$ B. $\frac{10}{3}$ C. $\frac{14}{3}$ D. $\frac{25}{9}$

Solution:

$$\left|\frac{3z}{-4+3i}\right| = (2-i)\overline{(2-i)} = \frac{3|z|}{\sqrt{4^2+3^2}} = (2-i)(2+i) = 5$$

$$\Rightarrow \frac{3|z|}{5} = 5 \Rightarrow 3|z| = 25 \Rightarrow |z| = \frac{25}{3}$$

Answer:

37. In the set of complex number, the ***Solution*** set of $z^3 - 8 = 0$ is –
UEE 2010

A. $\{2\}$

B. $\left\{2, 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right\}$

C. $\{2, 2+8i, 2-8i\}$

D. $\left\{2, 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)\right\}$

Solution: $z^2 - 8 = 0 \Rightarrow z^3 - 2^3 = 0 \Rightarrow (z-2)(z^2+2z+4) = 0$

$\Rightarrow z - 2 = 0$ and $z^2 + 2z + 4 = 0$

$$\therefore z = 2 \text{ and } \frac{-2 \pm \sqrt{4-4(4)}}{2} = \frac{-2 \pm 2\sqrt{3}i}{2} = -1 \pm \sqrt{3}i$$

$\therefore z = 2$ and $z = -1 + \sqrt{3}i$ or $z = -1 - \sqrt{3}i$

In polar form, $2, 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right), 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$\therefore$ Answer D

38. Polar form of $\left[\frac{2}{1+\sqrt{3}i}\right]^2$ equal -----

A. $\cos 120^0 + i\sin 120^0$ C. $\cos 60^0 + i\sin 60^0$

B. $\cos 240^0 + i\sin 240^0$ D. $\cos 120^0 + i\sin 120^0$

Solution: $\left[\dfrac{2}{1+\sqrt{3}i}\right]^2 = \dfrac{2^2(\cos 0^0 + i\sin 0^0)}{2^2(\cos 120^0 + i\sin 120^0)}$

$= \cos 120^0 - i\sin 120^0$

$\Rightarrow \cos 120^0 - i\sin 120^0 = \cos(240^0) + i\sin 240^0 \longrightarrow -120^0 + 360^0$

$= 240^0$

Answer: B

39. Solve each of the following expression over complex number

a. $x^3 + 2x^2 + 5x + 4 = 0$ c. $x^4 - x^2 - 20 = 0$

b. $x^2 + 2x + 6 = 0$ d. $x^3 + 8x = 0$

Solution: a) using rational root test $(-1)^3 + 2(-1)^2 + 5(-1) + 4 = 0$

$\therefore x^3 + 2x^2 + 5x + 4 = 0 \Rightarrow (x+1)(x^2 + x + 4) = 0$

$\Rightarrow x + 1 = 0$ or $x^2 + x + 4 = 0$

$\Rightarrow x = -1$ or $x = \dfrac{-1 \pm \sqrt{1-4(4)}}{2} = \dfrac{-1 \pm \sqrt{15}i}{2}$

$\therefore S.S = \left\{-1, \dfrac{-1+\sqrt{15}i}{2}, \dfrac{-1-\sqrt{15}i}{2}\right\}$

b) $x^2 + 2x + 6 = 0 \Rightarrow x^2 + 2x + 1 + 6 - 1 = 0 \Rightarrow (x+1)^2 + 5 = 0$

$\Rightarrow (x+1)^2 = -5 \Rightarrow x + 1 = \pm\sqrt{-5} = \pm\sqrt{5}i$

$\Rightarrow x = -1 \pm \sqrt{-5i} \Rightarrow x = -1 + \sqrt{5}i$ or $x = -1 - \sqrt{5}i$

$\therefore S.S = \{-1+\sqrt{5}i, -1-\sqrt{5}i\}$

c) $x^4 - x^2 - 20 = 0 \Rightarrow x = \dfrac{-1 \pm \sqrt{1-4(-20)}}{2} = \dfrac{1 \pm 9}{2} = \dfrac{10}{2}$ or $-\dfrac{8}{2}$

$\Rightarrow x^2 = 5$ or $x^2 = -4$

$\Rightarrow x = \pm\sqrt{5}$, or $x = \pm\sqrt{-4} = \pm 2i$

$\therefore S.S = \{-\sqrt{5}, \sqrt{5}, 2i, -2i\}$

d) $x^3 + 8x = 0 \Rightarrow x(x^2 + 8) = 0$

$\Rightarrow x(x - \sqrt{8}i)(x + \sqrt{8}i) = 0$

$\Rightarrow x = 0, x = \sqrt{8}i = 2\sqrt{2}i$ *or* $x = -\sqrt{8}i = -2\sqrt{2}i$

$\therefore S.S = \{0, 2\sqrt{2}i, -2\sqrt{2}i\}$

40. The simplified form of $z = \frac{(1-3i)(5+i)}{1+i}$ is .... UEE 2005

A. $-3-11i$ B. $3\sqrt{2}$ C. $\frac{2}{i}-\frac{5}{4}i$ D. $2i+3$

Solution:

$$\frac{(1-3i)(5+i)}{1+i} = \frac{(5+i-15i+3)}{1+i} = \frac{8-14i}{1+i} = 8-14i\left(\frac{1}{2}-\frac{i}{2}\right)$$

$$= \frac{8-14i-8i-14}{2}$$

$$\Rightarrow \frac{-6}{2} - \frac{22i}{2} = -3-11i$$

Answer: A

41. If z = -1 – i, then what is the value of z^{-6}

A. $1+\frac{1}{i}$ B) 8 = i C. $\frac{i}{8}$ D. $\frac{1}{360}i$

Solution:

$$\frac{1}{z^6} = \frac{1}{(-1-i)^6} = \frac{1}{\left((-1-i)^2\right)^3} = \frac{1}{(1+2i-1)^3} = \frac{1}{(2i)^3} = \frac{1}{-8i} = \frac{i}{8}$$

Answer: C

42. A polar form of $z = \frac{1+\sqrt{3}i}{1+i}$ is .... UEE

A. $\sqrt{2}\,(\cos 60^0 + i\sin 60^0)$ C. $\frac{\sqrt{2}}{2}(\cos 105^0 + i\sin 105^0)$

B. $2(\cos 30^0 + i\sin 30^0)$ D. $\sqrt{2}\,(\cos 15^0 + i\sin 15^0)$

Solution: $|z| = \left|\frac{1+\sqrt{3}i}{1+i}\right| = \frac{|1+\sqrt{3}i|}{|1+i|} = \frac{\sqrt{1^2+\sqrt{3}^2}}{\sqrt{1^2+1^2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\text{Arg}(z) = Arg\left(\frac{1+\sqrt{3}i}{1+i}\right) = Arg(1+\sqrt{3}i) - Arg(1+i)$

$= 60^0 - 45^0 = 15^0$

$\therefore$ polar form of $z = \sqrt{2}\,(\cos 15^0 + i\sin 15^0)$

Answer: D

43. A polar form of $z = \dfrac{1-i}{(\sqrt{3}+i)^2}$ with $a = \dfrac{\sqrt{2}}{4}$ is ….. UEE

A. $a[\cos 15^0 + i\sin 15^0]$ C. $a[\cos 255^0 + i\sin 255^0]$

B. $a[\cos 165^0 + i\sin 165^0]$ D. $a[\cos 275^0 + i\sin 275^0]$

Solution: $|z| = \dfrac{\sqrt{2}}{4}$

$$\text{Arg}(z) = \arg\left(\frac{1-i}{\left(\sqrt{3}+i\right)^2}\right) = \arg(1-i) - 2\arg\left(\sqrt{3}+i\right)$$

$$= \tan^{-1}\left(-\frac{1}{1}\right) - 2\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= -45^0 - 2(30^0) = -105^0 + 360^0 = 255^0$$

$\therefore$ Polar form $= a(\cos 105^0 - i\sin 105^0) = a(\cos 255 + i\sin 255^0)$

Answer: C

44. If $z = x + yi$, then when the equation $z(1+3i) - 4i = z$ is solved what are the value of x and y, respectively …..UEE 2006.

a) 3, 1 b) 0, 4 c) $\dfrac{4}{3}, 0$ d) 4, 3

Solution: $z(1 + 3i) - 4i = z \Rightarrow z + 3zi - 4i = z$

$\Rightarrow x + yi + 3(x + yi)I - 4i = z \Rightarrow x + yi + 3ix - 3y - 4i = x + yi$

$\Rightarrow (3x - 4)i - 3y = 0 \Rightarrow 3y = 0$ and $3x - 4 = 0$

$\therefore y = 0$ and $x = \dfrac{4}{3}$,

Answer: C

45. Find x + yi form of

a) $\left(-\sqrt{3}+i\right)^9$ b) $\left(2-2\sqrt{3}i\right)^5$ c) $\left(1+\sqrt{3}i\right)^8$

Solution: $\left|-\sqrt{3}+i\right| = \sqrt{3+1} = 2$

$$\text{Arg}\left(-\sqrt{3}+i\right)^9 = 9(\text{Arg}\left(-\sqrt{3}+i\right) = 9\left(\pi + \left(-\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)\right)$$

$$= 9\left(\pi - \frac{\pi}{6}\right) = \frac{45\pi}{6} = \frac{15\pi}{2}$$

$$\therefore\left(-\sqrt{3}+i\right)^9=2^9\left(\cos\frac{15\pi}{2}+i\sin\frac{15\pi}{2}\right)=2^9\left(\cos\frac{3}{2}\pi+i\sin\frac{3}{2}\pi\right)$$

$$=2^9(-i)=-512i$$

b) $$\left(2-2\sqrt{3}i\right)^5=\left(4\left(\cos\left(\frac{\pi}{3}\right)-i\sin\frac{\pi}{3}\right)\right)^5$$

$$=2^{10}\left(\cos\frac{5\pi}{3}-i\sin\frac{5\pi}{3}\right)$$

$$=2^{10}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\rightarrow\frac{-5\pi}{3}+2\pi=\frac{\pi}{3}$$

$$=2^{10}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$

$$=512+512\sqrt{3}i$$

c) $$|1+\sqrt{3}i|=\sqrt{1^2+\sqrt{3}^2}=\sqrt{1+3}=2$$

$$Arg(1+\sqrt{3}i)=\tan^{-1}\left(\sqrt{3}\right)=\frac{\pi}{3}$$

$$\therefore\left[1+\sqrt{3}i\right]^8=\left[2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right]^8=2^8\left(\cos\frac{8\pi}{3}+i\sin\frac{8\pi}{3}\right)$$

$$=2^8\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)=-128+128\sqrt{3}i$$

46. If $i^n=-i$ then which of the following is not possible values of n

a) 75 b) 307 c) 371 d) 397

Solution: A) If n = 75, then divide 75 by 4 $\Rightarrow r=3$

$\therefore i^{75}=i^3=-i \Leftrightarrow 75=4(18)+3 \longleftarrow r=3$

B) If $n=307 \Rightarrow 307=4(76)+3 \leftarrow$ divide 307 by 4, r = 3

C) If $n=371 \Rightarrow 371=4(92)+3 \leftarrow$ divide 371 by 4, r = 3

D) If $n=397 \Rightarrow 397=4(99)+1 \leftarrow$ divide 397 by 4, r = 1

$\Rightarrow i^{75}=i^3=-i,\ i^{307}=i^3=-i,\ i^{371}=i^3=-i$

But $i^{397}=i^1=i$

$\therefore$ possible values of n = 75, 307, and 371

Answer: D

47. For given cartesion coordinate, find polar coordinate.

a) (-1, 1) b) (0, 3) c) (0, -2) d) $\left(-\sqrt{3},-1\right)$

Solution: a) $r=\sqrt{(-1)^2+(1)^2}=\sqrt{2}$, and P(-1, 1) = -1 + i lies on 2nd quadrant.

$\text{Arg}(-1+i)=\pi+\tan^{-1}\left(\frac{-1}{1}\right)=\pi-\left(\frac{\pi}{4}\right)=\frac{3\pi}{4}$

$\therefore (r,\theta)=\left(\sqrt{2},\frac{3\pi}{4}\right)$

b) $r=\sqrt{0^2+3^2}=3$, $\text{Arg}(0+3i)=\frac{\pi}{2}$

$\therefore (0,3)=\left(3,\frac{\pi}{2}\right)$

c) $r=\sqrt{0^2+(-2)^2}=2$, $\text{Arg}(0-2i)=\frac{-\pi}{2}$

$\therefore (0,-2)=\left(2,\frac{-\pi}{2}\right)$

d) $r=\sqrt{\left(\sqrt{3}\right)^2+(-1)^2}=2$,

$\text{Arg}\left(-\sqrt{3}-i\right)=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)-\pi \Rightarrow \frac{\pi}{6}-\pi=\frac{-5\pi}{6}$

$\therefore P(x,y)=P\left(-\sqrt{3},-1\right)=(r,\theta)=\left(2,\frac{-5\pi}{6}\right)$

Let z be a complex number. which of the following is the solution set of $z^3-iz=0$? ……UEE

A. $\left\{\pm\frac{1}{\sqrt{2}}(1+i)\right\}$ C. $\left\{\pm\sqrt{2}(1+i)\right\}$

B. $\left\{0,\pm\frac{1}{\sqrt{2}}(1-i)\right\}$ D. $\left\{0,\pm\frac{1}{\sqrt{2}}(1+i)\right\}$

Solution: $z^3-iz=0 \Leftrightarrow z(z^2-i)=0 \Rightarrow z=0$ or $z^2-i=0$

$\Rightarrow z^2 = i \Rightarrow (x + yi)^2 = i \Leftrightarrow x^2 - y^2 + 2xyi = 0 + i$

$\Rightarrow x^2 - y^2 = 0$ and $2xyi = i$

$\Rightarrow x = \pm y$ and $2xy = 1$

So that, $2x\cdot(x) = 1 \Rightarrow 2x^2 = 1$

$$\Rightarrow x = \pm\frac{1}{\sqrt{2}} \text{ and } y = \pm\frac{1}{\sqrt{2}}$$

$$\Rightarrow z = x + yi = \pm\frac{1}{\sqrt{2}}(1+i)$$

$$\therefore \text{ S.S} = \left\{0, \pm\frac{1}{\sqrt{2}}(1+i)\right\}$$

Supplementary problem

1. Simplify write the answer in the form a+bi

a) $\sqrt{-16}+\sqrt{-4}$ c) $\dfrac{\sqrt[3]{-27}+\sqrt{-27}}{3}$

b) $\sqrt{-12}+\sqrt{-8}$ d) $\sqrt{32}+\sqrt{-32}$

2. Represent each of the following complex numbers as ordered pairs of number

a) 2-3i b) $(\sqrt{2}+1)+(\sqrt{3}-4)i$ c) 4i d) 5

3. In each of the following find the product in the form x+yi

a) (2+3i)(3-8i) c) $(5-2i)^2(3+i)$

b) $(\sqrt{2}+i)+(\sqrt{3}-i)$ d) 2i(3-5i)

4. In each of the following solve for x and y

a) (x+yi)(x-2i)=4+5i c) (x+yi)(2+3i)(2-i)=5+3i

b) (x-yi)(8+i)=6-I d) (2+3i)(x+yi)=7-3i

5. Compute $\overline{Z_1+Z_2}, \overline{Z_1}+\overline{Z_2}, \overline{Z_1 Z_2}$ and $\overline{Z_1}\cdot\overline{Z_2}$ in each of the following

a) $Z_1 = 2,-i; Z_2 = 3+i$ c) $Z_1 = 2, Z_2 = \sqrt{-3}$

b) $Z_1 = -1-2i, Z_2 = i$ d) $Z_1 = \sqrt{-5}, Z_2 = 3-\sqrt{-4}$

6. Simplify each of the following

a) $\dfrac{1}{2-3i}+\dfrac{3+5i}{2-i}$ b) $\dfrac{2+3i}{3-2i}+\dfrac{3+4i}{2+4i}$ c) i^{47}

7. Compute each of the following

a) $|(4-2i)(4+3i)|$ b) $\left|\sqrt{\frac{1+i}{1-1}}\right|$

8. Express each of the following complex numbers in polar form

a) $\frac{1}{2}-\frac{\sqrt{3}}{2}i$ b) 1-4i c) 1- i d) $\frac{1+2i}{1-3i}$

9. Find a) $(1-i)^{10}$ b) $(2+2i)^{7}$ c) $\left(\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)^{6}$

10. Find all roots of the equation

a) $x^3+8=0$ b) $x^3+\left(1+i\sqrt{3}\right)=0$

11. Find $\sqrt{-16+16\sqrt{3}i}$

12. Find real values of x and y for which the following equations are satisfied

a) (x+yi)(2-3i)=4+i b) $\frac{x-y}{3+i}+\frac{y-i}{3-i}=i$

c) $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y-i}{3-i}=i$

13. Find the center and radius

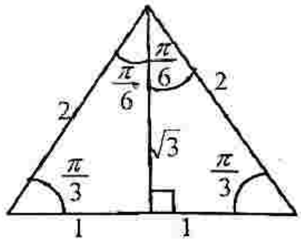
a) $Z\overline{Z}+(1+i)Z+(1-i)\overline{Z}=0$

b) $Z\overline{Z}-(2+3i)Z-(2-3i)\overline{Z}+9=0$

14. If $y+\frac{1}{y}=2\cos\theta$, *Find y*

Further on Trigonometric Function

- The function y = sec x, y = csc x and y = cot x
- $\sec x = \dfrac{1}{\cos x}$,
- $\csc x = \dfrac{1}{\sin x}$
- $\cot x = \dfrac{1}{\tan x}$



- Table of Trigonometric function of special angle

	X	Sinx	Cosx	Tanx	Cscx	Secx	Cotx
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

- The graph of y = secx, y = cscx and y = cot x
  To sketch the graph, each of the following elements are very important. These are:
- Domain, Range, period, asymptote.
- Determine for what x, the function is positive or negative
- Describe the behavior as x increase from:

$$0 \to \frac{\pi}{2}, \frac{\pi}{2} \to \pi, \pi \to \frac{3\pi}{2}, \frac{3\pi}{2} \to 2\pi$$

1. The graph of y = csc x

$$f(x) = \csc x = \frac{1}{\sin x}, \ \sin x \neq 0 \Rightarrow x \neq k\pi$$

- Domain = {x: $x \neq k\pi$, k is integer}
- Range = $R\setminus\{(-1,1)\} = (-\infty,-1] \cup [1, \infty)$
- Period = 2π i.e $f(x + 2\pi) = f(x)$ for all x.
- The asymptote = $\begin{cases} k\pi, \text{ where k is integer and} \\ \text{at which } \sin x = 0 \text{ and si} \\ \text{crosses the x - axis} \end{cases}$
- For what x, csc x is positive or negative

i) $\csc x > 0 \Rightarrow \dfrac{1}{\sin x} > 0 \Rightarrow \sin x > 0$

$\Rightarrow \csc x > 0$ when $-2\pi < x < -\pi$ or $0 < x < \pi$

$\therefore$ the graph lies above the x – axis

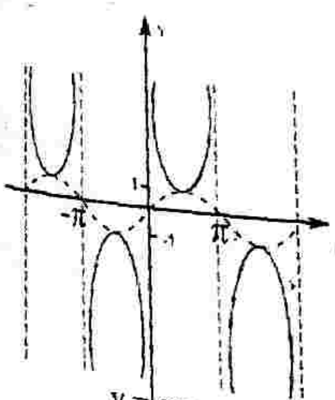
ii) $\csc x < 0 \Rightarrow \dfrac{1}{\sin x} < 0 \Rightarrow \sin x < 0$

$\therefore \csc x < 0$ when $-\pi < x < 0$ or $\pi < x < 2\pi$

That is the graph of $y = \csc x$, lies below the x – axis on this-interval

- Describing the behavior as x increase from

$0 \to \dfrac{\pi}{2}, \dfrac{\pi}{2} \to \pi, \pi \to \dfrac{3\pi}{2}, \dfrac{3\pi}{2} \to 2\pi$



y = cscx
- Domain: $x \neq k\pi$
- Range: $(-\infty, -1] \cup [1,\infty)$

i) **As x increase from 0 $\to \dfrac{\pi}{2}$**

- $\sin x$ increase from $0 \to 1$
$\Rightarrow$ csc x decease from $\infty \to 1$

ii) **As x increase from $\dfrac{\pi}{2} \to \pi$**

- $\sin x$ decrease from $1 \to 0$
- csc x increase from $1 \to \infty$

iii) **As x increase from $\pi \to \dfrac{3}{2}\pi$**

- $\sin x$ decrease from $0 \to -1$
- csc x increase from $-\infty \to -1$

iv) **As x increase from** $\frac{3}{2}\pi \rightarrow 2\pi$

- sin x increase from $-1 \rightarrow 0$
- csc x decrease from $-1 \rightarrow -\infty$

- **Symmetric property**

Csc (-x) = -csc x

$\Rightarrow$ it is an odd function

$\therefore$ The graph is symmetric with respect to the **origin.**

2. The graph of y = sec x

We have $g(x) = \sec x = \frac{1}{\cos x}, \cos x \neq 0 \Rightarrow x \neq \left(\frac{2k+1}{2}\right)\pi$

- Domain $= \left\{x \neq \frac{(2k+1)\pi}{2}, k \in Z\right\}$
- Range = $(-\infty\ -1] \cup [1\ \infty)$
- Period = 2π

Explanation: $-1 \leq \cos x \leq 1$

$\Rightarrow |\cos x| \leq 1 \Rightarrow \frac{1}{|\cos x|} \geq 1$

$\Rightarrow |\sec x| \geq 1$

$\therefore$ Rang: $(-\infty, -1] \cup [1, \infty)$

- Asymptote = $\begin{cases} \frac{(2k+1)\pi}{2}, k \in Z \text{ and at which } \cos x = 0 \\ \text{and cos x crosses the x - axis} \end{cases}$

- For what interval x, secx is positive or negative

i) $\sec x > 0 \Rightarrow \frac{1}{\cos x} > 0 \Rightarrow \cos x > 0$

$\therefore \sec x > 0$ when $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and $x \in (-2\pi\ \frac{-3\pi}{2})$

- That is the graph of g(x) = sec x lies above the x-axis ($g(x) \geq 1$)

ii) $\sec x < 0 \Rightarrow \frac{1}{\cos x} < 0 \Rightarrow \cos x < 0$

$\therefore$ $\sec x < 0$ when $x \in \left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right)$ and

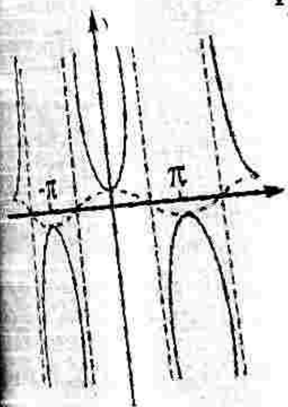
$x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

- When $x \in \left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right)$ and $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

  g(x) = sec x has maximum value -1, at x = -π and x = π but has no finite minimum value

- Describing the behavior as x increase from (0 0)

  $\to \frac{\pi}{2}, \frac{\pi}{2} \to \pi, \pi \to \frac{3\pi}{2}, \frac{3\pi}{2} \to 2\pi$



y = secx

• Domain: $x \neq \frac{\pi k}{2}$

• Range: $(-\infty, -1] \cup [1, \infty)$

i) As x increase from $0 \to \frac{\pi}{2}$

- cos x decrease from 1 →0
- sec x increase from 1→ ∞

(The graph goes up from 1→ ∞)

ii) As x increase from $\frac{\pi}{2} \to \pi$

- cos x decrease from 0 →-1
- sec x increase from -∞ → -1

(The graph goes up from -∞ → -1)

iii) **As x increase from** $\pi \to \frac{3\pi}{2}$

- cos x increase from -1 →0
- sec x decrease from 0 -1 → -∞

(The graph goes down from -1 → -∞)

iv) **As x increase from** $\frac{3}{2}\pi \to 2\pi$

- cos x increase from 0 →1
- sec x decrease from ∞ → 1
- The graph goes down from ∞→ 1)

Symmetry

Since cos(-x) = cos x ⇒ sec (-x) = sec x

⇒ g(...) ... function

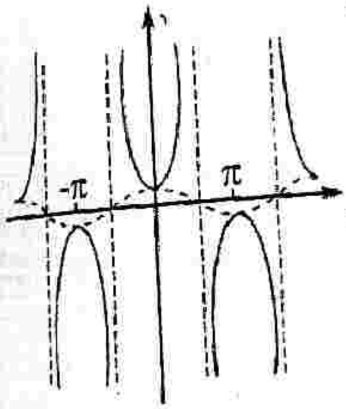
$x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

- When $x \in \left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right)$ and $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

  g(x) = sec x has maximum value -1, at x = -π and x = π but has no finite minimum value

- Describing the behavior as x increase from (0 0)

  $\rightarrow \frac{\pi}{2}, \frac{\pi}{2} \rightarrow \pi, \pi \rightarrow \frac{3\pi}{2}, \frac{3\pi}{2} \rightarrow 2\pi$



y = secx

- Domain: $x \neq \frac{\pi k}{2}$
- Range: $(-\infty, -1] \cup [1, \infty)$

i) **As x increase from** $0 \rightarrow \frac{\pi}{2}$

- cos x decrease from 1 →0
- sec x increase from 1→ ∞

(The graph goes up from 1→ ∞)

ii) **As x increase from** $\frac{\pi}{2} \rightarrow \pi$

- cos x decrease from 0 →-1
- sec x increase from -∞ → -1

(The graph goes up from -∞ → -1)

iii) **As x increase from** $\pi \rightarrow \frac{3\pi}{2}$

- cos x increase from -1 →0
- sec x decrease from 0 -1 → -∞

(The graph goes down from -1 → -∞)

iv) **As x increase from** $\frac{3}{2}\pi \rightarrow 2\pi$

- cos x increase from 0 →1
- sec x decrease from ∞ → 1
- The graph goes down from ∞→ 1)

Symmetry

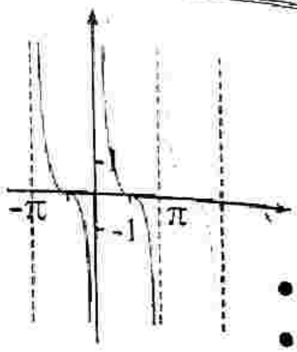
Since cos(-x) = cos x ⇒ sec (-x) = sec x

⇒ g(x) = sec x is an even function

∴ g(x) = sec x is symmetric w.r.t y-axis

3. The graph of y = cotx

We have $k(x) = \cot x = \dfrac{\cos x}{\sin x}$



- Range = ℝ
- Period = π
- Asymptote, = {x = kπ. x≠∈Z}
- x-intercept = $\left\{x = \left(\dfrac{2k+1}{2}\right)\pi\right.$
- for what interval x, cot x ≥ 0 cot x ≤ 0

  i) cot x ≥ 0 when $x \in \left(-2\pi, \dfrac{-3\pi}{2}\right] \cup \left(-\pi, \dfrac{-\pi}{2}\right]$

  $\cup \left(0, \dfrac{\pi}{2}\right] \cup \left(\pi, \dfrac{3\pi}{2}\right]$

  That is the graph decrease from ∞→0

  ii) cot x ≤ 0 when

  $x \in \left[\dfrac{-3\pi}{2}, -\pi\right) \cup \left[\dfrac{-\pi}{2}, 0\right) \cup \left[\dfrac{\pi}{2}, \pi\right) \cup \left[\dfrac{3\pi}{2}, 2\pi\right.$

  the graph lies below the x-axis

  The graph lies below the x-axis
- the graph k(x) = cot x is decreasing in its domain

Illustrative example

1. Find the interval between -2π ≤ x ≤ 2π on which:
   a) g(x) = secx is i) increasing ii) Decreasing
   b) f(x) = csc x is i) increasing ii) Decrease
   c) k(x) = cot x is i) increasing ii) decreasing.
   d) Both cosecant and secant function are
      i) increasing at the same time.
      ii) decreasing at the same time
   e) i) sec x ≥ 1 ii) sec x ≤ -1
   f) i) csc x ≥ 1 ii) csc x ≤ -1

Solution: Here by observing the graph.

a) i) g(x) = sec x is increasing on: [-2π, -π]∪[0,π].

$x \neq \frac{-3}{2}\pi$ and $x \neq \frac{\pi}{2}$

ii) g(x) = sec x is decreasing on: $[-\pi\ \ 0] \cup [\pi, 2\pi]$,

$x \neq \frac{-\pi}{2}$, and $x \neq \frac{3}{2}, \pi$

b) i) f(x) = csc x is increasing: $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right] \cup \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

for $x \neq -\pi$ and $x \neq \pi$.

ii) f(x) = csc x is decreasing on:

$\left(-2\pi, \frac{-3\pi}{2}\right] \cup \left[\frac{-\pi}{2}, \pi\right) \cup \left[\frac{3\pi}{2}, 2\pi\right)$ except

at x = 0,

c) i) k(x) = cot x has no interval on which cot x is increasing.

$\therefore$ s.s = ϕ

ii) k(x) = cot x is decreasing on interval $(-2\pi < x < 2\pi)$

except at $x = -\pi$, $x = 0$ $x = \pi$ and $x = 2\pi$.

d) i) f(x) = csc x and g(x) = sec x both are increasing at the same time on $\left(\frac{-3\pi}{2}, -\pi\right) \cup \left(\frac{\pi}{2}, \pi\right)$

ii) f(x) = csc x and g(x) = sec x both are decreasing at the same time on interval $\left(\frac{-\pi}{2}, 0\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$

e) i) sec x $\geq$ 1 on $\left[-2\pi, \frac{-3\pi}{2}\right) \cup \left(\frac{-\pi}{2}, \frac{\pi}{2}\right] \cup \left(\frac{3}{2}\pi\ \ 2\pi\right]$

on this interval sec x = 1 is the minimum value and No finite maximum value

ii) sec x $\leq$ -1 on $\left(\frac{-3}{2}\pi, \frac{-\pi}{2}\right) \cup \left(\frac{\pi}{2}\ \frac{3\pi}{2}\right)$

- on this interval sec x = -1 is the maximum value at $x = -\pi$ and $x = \pi$

f) i) csc x $\geq$ 1 on $(-2\pi, -\pi) \cup (0\ \ \pi)$

- on this interval csc x = 1 is the minimum value at $x = \frac{-3}{2}\pi$ and $x = \frac{\pi}{2}$ but has no finite maximum value

ii) csc x ≤ - 1 on (-π 0) ∪ (π 2π)

- on this interval csc x = -1 is the maximum value at $x = \frac{-\pi}{1}$ and $x = \frac{3}{2}\pi$
- But has no finite minimum value

2. Find the exact value

a) $\csc\left(\frac{2\pi}{3}\right)$

b) $\sec\left(\frac{2\pi}{3}\right)$

c) $\sec\left(\frac{-\pi}{6}\right)$

d) $\cot\left(\frac{33\pi}{4}\right)$

e) $\sec\left(\frac{23\pi}{3}\right)$

f) $\csc\left(\frac{-23\pi}{3}\right)$

g) $\csc\left(\frac{-5\pi}{4}\right)$

Solution:

a) $\csc\left(\frac{2\pi}{3}\right) = \frac{1}{\sin\left(\frac{2}{3}\pi\right)} = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$

b) $\sec\left(\frac{2\pi}{3}\right) = \frac{1}{\cos\left(\frac{2}{3}\pi\right)} = \frac{1}{-\cos\left(\frac{\pi}{3}\right)} = \frac{1}{\frac{-1}{2}} = -2$

c) $\sec\left(\frac{-\pi}{6}\right) = \frac{1}{\cos\left(\frac{\pi}{6}\right)} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$

d) $\cot\left(\frac{33\pi}{4}\right) = \frac{\cos\left(\frac{33\pi}{4}\right)}{\sin\left(\frac{33\pi}{3}\right)} = \frac{\cos\left(\frac{33\pi}{4} - 8\pi\right)}{\cos\left(\frac{33\pi}{4} - 8\pi\right)} = \frac{\cos\left(\frac{\pi}{4}\right)}{\sin\frac{\pi}{4}} = 1$

e) $\sec\left(\frac{23\pi}{3}\right) =$

$$\frac{1}{\cos\left(\frac{23}{3}\pi\right)} = \frac{1}{\cos\left(\frac{23}{3} - 8\pi\right)} = \frac{1}{\cos\left(\frac{-\pi}{3}\right)} = \frac{1}{\frac{1}{2}} = 2$$

f) $\csc\left(\frac{-23\pi}{3}\right) =$

$$\frac{1}{\sin\left(\frac{-23}{3}\pi + 8\pi\right)} = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

g) $\csc\left(\frac{-5\pi}{4}\right) = \frac{1}{\sin\left(\frac{-5\pi}{4} + 2\pi\right)} = \frac{1}{\sin\left(\frac{3\pi}{4}\right)} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$

Point about Inverse function Relationships between f^{-1} and f

- y= f^{-1}(x) iff x = f(y), x is in the domain of f^{-1}} and y is in the domain of f.
- Domain of f^{-1}= Range of f
- Range of f^{-1}= Domain of f.
- $f(f^{-1}(x))$ = x, for all x in the domain of f^{-1}
- $f^{-1}(f(y))$ = y, for all y in the domain of f.
- If (a,b) $\in$ f then (b, a) $\in$ f^{-1}
- The graph of f^{-1} and f are reflection of each other through the line y = x (i.e. M(a,b) $\xrightarrow{y=x}$ (b,a)

Inverse trigonometric function

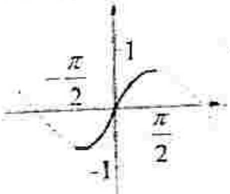
All the six basic trigonometric function are not one – to – one because they all repeated periodically and hence the horizontal line test crosses many time. Thus, to define inverse trigonometric function we must first restrict the domains of trigonometric function to make them **one – to – one**

Example: $\sin\left(\frac{5\pi}{6}\right) = \sin\left(\frac{-7\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$\therefore$ sine function is not one to one

A. Inverse sine function

The graph of y = sin x is one to one function on restricted domain:



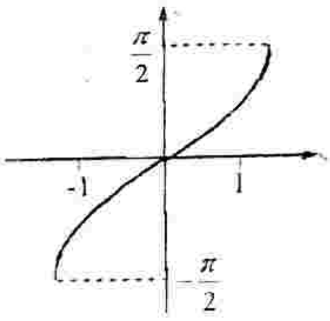
y = sinx

$$-\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

Thus, y = sin x is invertible on $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

> **Definition:** the inverse sine function (arc sine function) denoted by $\sin^{-1}$ or arc sin is defined by $y = \sin^{-1} x$ or y = arc sin x iff x = sin y for
>
> $$\frac{-\pi}{2} \le y \le \frac{\pi}{2} \text{ and } -1 \le x \le 1.$$

In function notation: Define $f^{-1} : [-1,1] \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is



$y = \sin^{-1}x$

$f^{-1}(x) = \sin^{-1} x$.

- Domain of $\sin^{-1}x$ is $[-1,1]$
- Range of $\sin^{-1}x$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

- The graph of y = sin x on $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is increasing implies graph of $y = \sin^{-1} x$ is also increasing on $[-1, 1]$.

Note a) • $\sin^{-1}(-x) = -\sin^{-1}x$.

- $\sin\frac{\pi}{2} = 1 \Rightarrow \sin^{-1}(1) = \frac{\pi}{2}$

- $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \Rightarrow \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$
- $\sin\frac{\pi}{6} = \frac{1}{2} \Rightarrow \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Illustrative example

3. If possible find the exact value

a) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ d) $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

b) $\sin^{-1}\left(\frac{-\sqrt{2}}{2}\right)$ e) $\sin^{-1}(1)$

c) $\sin^{-1}\left(-\frac{1}{2}\right)$ f) $\sin^{-1}(0)$

Solution: Let $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = y \Rightarrow \sin y = \frac{\sqrt{2}}{2} \Rightarrow y = \pi/4$

a) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

b) $\sin^{-1}\left(\frac{-\sqrt{2}}{2}\right) = -\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

c) $\sin^{-1}\left(-\frac{1}{2}\right) = -\sin^{-1}\left(\frac{1}{2}\right) = -\frac{\pi}{6}$

d) $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = -\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

e) $\sin^{-1}(-1) = -\sin^{-1}(1) = \frac{-\pi}{2}$

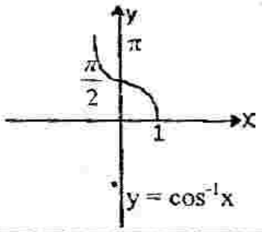
f) $\sin^{-1}(0) = 0$

B. Inverse cosine function

- The graph of y = cos x is one to one (decreasing) function on restricted domain: $0 \leq x \leq \pi$.
- Thus y = cos x is invertible on $o \leq x \leq \pi$ and $-1 \leq y \leq 1$

Definition: The inverse cosine function (arc cosine denoted by $\cos^{-1}$ or arc cos is defined by $y = \cos^{-1}x$ or y = arc cos iff $x = \cos y$ for $-1 \leq x \leq 1$ and $o \leq y \leq \pi$

Function notation: $f^{-1}: [-1,1] \longrightarrow [0, \pi]$ is $f^{-1}(x) = \cos^{-1}x$



- Domain of $\cos^{-1} x$ is $[-1, 1]$
- Range of (x) is $[0, \pi]$

The graph of y = cos x is decreasing on $[0,\pi]$, thus the graph of $y = \cos^{-1} x$ is also decreasing on [-1, 1].

Note. 1) $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

Example: $\cos^{-1}\left(\frac{-1}{2}\right) = \pi - \cos^{-1}\left(\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2}{3}\pi$

$$\therefore \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2}{3}\pi$$

4) If possible find the exact value of y

a) $y = \cos^{-1}\left(\frac{1}{2}\right)$ d) $y = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

b) $y = \cos^{-1}\left(-\frac{1}{2}\right)$ e) $y = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

c) $y = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ f) $y = \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

Solution: Here, the value of y in the range $0 \leq y \leq \pi$ of $\cos^{-1}$

- $\text{Cos}^{-1}(-x) = \pi - \cos^{-1}(x)$

a) $y = \cos^{-1}\left(\frac{1}{2}\right) \Leftrightarrow \cos y = \frac{1}{2} \Rightarrow y = \frac{\pi}{3}$

$$\therefore \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

b) $y = \cos^{-1}\left(-\frac{1}{2}\right) \Rightarrow y = \pi - \cos^{-1}\left(\frac{1}{2}\right) \Rightarrow y = \pi - \frac{\pi}{3} = \frac{2}{3}\pi$

$$\therefore y = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2}{3}\pi$$

c) $y = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \Leftrightarrow \cos y = \frac{1}{\sqrt{2}} \Rightarrow y = \frac{\pi}{4}$

$$\therefore \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

d) $y = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) \Rightarrow y = \pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$$\therefore y = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$$

e) $y = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \Leftrightarrow \cos y = \frac{\sqrt{3}}{2} \Rightarrow y = \frac{\pi}{6}$

f) $y = \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) \Rightarrow y = \pi - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$$\therefore y = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$

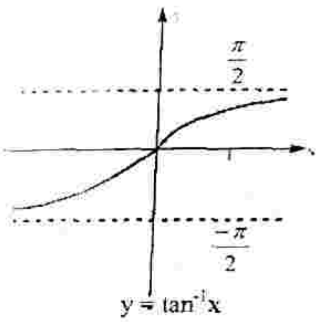
The inverse tangent function

- The inverse tangent function (arc tangent function is denoted $\tan^{-1}$ (arc tan) is defined by
  $y = \tan^{-1} x =$ arc tan if and only if

$x = \tan y$ for $-\infty < x < \infty$ and for $\frac{-\pi}{2} < y < \frac{\pi}{2}$

- Function notation

Define: $f^{-1} : \mathbb{R} \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is $f^{-1}(x) = \tan^{-1}x$



- Horizontal asymptote of are

Tangent: $y = \frac{-\pi}{2}$ *and* $y = \frac{\pi}{2}$

- Domain of $\tan^{-1} x$ is $\mathbb{R}$
- Range of $\tan^{-1}x$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
- $\tan^{-1}(-x) = -\tan^{-1} x$

5) If possible find the exact value of y

a) $y = \tan^{-1}(1)$ d) $y = \tan^{-1}(-\sqrt{3})$

b) $y = \tan^{-1}(-1)$ e) $y = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

c) $y = \tan^{-1}(\sqrt{3})$ f) $y = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

Solution: Here the value of y in the range $\frac{-\pi}{2} < y < \frac{\pi}{2}$

a) $y = \tan^{-1}(1) \Leftrightarrow \tan y = 1 \Rightarrow y = \frac{\pi}{4}$, $\therefore \tan^{-1}(1) = \frac{\pi}{4}$

b) $y = \tan^{-1}(-1) = -\tan^{-1}(1) = \frac{-\pi}{4}$ $\therefore \tan^{-1}(-\sqrt{3}) = \frac{\pi}{4}$

c) $y = \tan^{-1}(\sqrt{3}) \Rightarrow \tan y = \sqrt{3} \Rightarrow y = \frac{\pi}{3}$,

$\therefore \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

d) $y = \tan^{-1}(-\sqrt{3}) \Rightarrow y = -\tan^{-1}(\sqrt{3}) = \dfrac{-\pi}{3}$,

$\therefore \tan^{-1}(-\sqrt{3}) = \dfrac{-\pi}{3}$

e) $y = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) \Rightarrow \tan y = \dfrac{1}{\sqrt{3}} \Rightarrow y = \dfrac{\pi}{6}$,

$\therefore \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$

f) $y = \tan^{-1}\left(\dfrac{-1}{\sqrt{3}}\right) \Rightarrow y = -\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{-\pi}{6}$,

$\therefore \tan^{-1}\left(\dfrac{-1}{\sqrt{3}}\right) = \dfrac{-\pi}{6}$

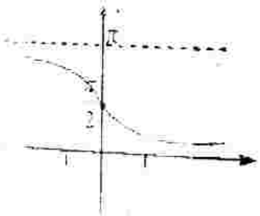
D. Inverse cotangent function

Definition: inverse cotangent function is denoted by $\cot^{-1}$ is defined by

$y = \cot^{-1} x$ if and only if $x = \cot y$

... number x and $0 < y < \pi$

Function notation



$y = \cot^{-1} x$

$f^{-1}: IR \longrightarrow (0, \pi)$ is $f^{-1}(x) = \cot^{-1}(x)$

- Domain of $\cot^{-1} x$ is IR
- Range of $\cot^{-1} x$ is $(0,\pi)$

$$\boxed{\mathrm{Cot}^{-1}x = \frac{\pi}{2} - \tan^{-1} x.}$$

...nd the exact value of y.

a) $y = \cot^{-1}(\sqrt{3})$ d) $y = \cot^{-1}(1)$

b) $y = \cot^{-1}(-\sqrt{3})$ e) $y = \cot^{-1}\left(\dfrac{\sqrt{3}}{3}\right)$

c) $y = \cot^{-1}(-1)$ f) $y = \cot^{-1}\left(\dfrac{-\sqrt{3}}{3}\right)$

Solution: use $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1}x$

a) $y = \cot^{-1}(\sqrt{3}) = \frac{\pi}{2} - \tan^{-1}(\sqrt{3}) = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$

b) $y = \cot^{-1}(-\sqrt{3}) = \frac{\pi}{2} - \tan^{-1}(\sqrt{3}) = \frac{\pi}{2} + \tan^{-1}(\sqrt{3})$

$= \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$

c) $y = \cot^{-1}(-1) = \frac{\pi}{2} - \tan^{-1}(-1) = \frac{\pi}{2} - (-\tan^{-1}(1))$

$= \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$

d) $y = \cot^{-1}(1) = \frac{\pi}{2} - \tan^{-1}(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

e) $y = \cot^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

f) $y = \cot^{-1}\left(\frac{-\sqrt{3}}{3}\right) = \frac{\pi}{2} - (-\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$

E) The inverse cosecant function $\csc^{-1} x$ or arc csc x is defined by

$y = \csc^{-1} x$ if and only if $x = \csc y$ for $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$, $y \neq 0$, $|x| \ge 1$

Note: If $|x| \ge 1$ then

- $\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$, let $y = \csc^{-1} x$

$\Rightarrow \csc y = x \Rightarrow \frac{1}{\sin y} = x \Rightarrow \sin y = \frac{1}{x} \Rightarrow y = \sin^{-1}\left(\frac{1}{x}\right)$

$$\boxed{\therefore \csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)}$$

7) Find the exact value of y:

a) $y = \csc^{-1}(2)$ c) $y = \csc^{-1}\left(\frac{-\sqrt{2}}{3}\right)$

b) $y = \csc^{-1}(-\sqrt{2})$ d) $y = \csc^{-1}\left(\frac{1}{3}\right)$

Solution: we have $\csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$

a) $y = \csc^{-1}(2) \Rightarrow y = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}, \therefore \csc^{-1}(2) = \frac{\pi}{6}$

b) $y = \csc^{-1}(-\sqrt{2}) \Rightarrow y = \sin^{-1}\left(\frac{-1}{\sqrt{2}}\right) = -\left(\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right) = -\frac{\pi}{4}$

c) $y = \csc^{-1}\left(\frac{-2}{\sqrt{3}}\right) \Rightarrow y = \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = -\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{-\pi}{3}$

d) $y = \csc^{-1}\left(\frac{1}{3}\right) \Rightarrow y = \sin^{-1}(3)$ not defined:

F) The inverse secant function

The inverse secant function is denoted by $\sec^{-1}$ (arc sec) is defined $y = \sec^{-1}x$, if and only if $x = \sec y$ where $0 \le y \le \pi$, $y \ne \pi$, $|x| \ge 1$

Note: i) $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$ for $|x| \ge 1$

Let $y = \sec^{-1}x \Rightarrow \sec y = x$

$$\Rightarrow \frac{1}{\cos y} = x \Rightarrow \cos y = \frac{1}{x} \Rightarrow y = \cos^{-1}\left(\frac{1}{x}\right)$$

$$\therefore \sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right) \text{ and}$$

- $\sec^{-1}(-x) = \pi - \cos^{-1}\left(\frac{1}{x}\right) = \pi - \sec^{-1}x$

8) Find the exact value of y:

a) $y = \sec^{-1}(\sqrt{2})$

b) $y = \sec^{-1}(-\sqrt{2})$

c) $y = \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

d) $y = \sec^{-1}\left(\frac{-2}{\sqrt{3}}\right)$

e) $y = \sec^{-1}(-1)$

f) $y = \sec^{-1}\left(\frac{-1}{2}\right)$

Solution:

a) $y = \sec^{-1}(\sqrt{2}) \Rightarrow y = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$, $\therefore \sec^{-1}(\sqrt{2}) = \frac{\pi}{4}$

b) $y = \sec^{-1}(\sqrt{2}) \Rightarrow y = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

c) $y = \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) \Rightarrow y = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$ $\therefore \text{-}\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{6}$

d) $y = \sec^{-1}\left(\frac{-2}{\sqrt{3}}\right) \Rightarrow y = \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \pi - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$$= \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

e) $y = \sec\text{-}1(-1) \Rightarrow y = \cos\text{-}1(-1) = \pi - \cos\text{-}1(1) = \pi - \pi = 0$

$\therefore \text{-}\sec^{-1}(-1) = 0$

f) $y = \sec^{-1}\left(\frac{-1}{2}\right) \Rightarrow y = \cos^{-1}(2) \longleftarrow$ not defined

Composition of trigonometric function, and their inverse

- If $-1 \leq x \leq 1$, then

i) $\sin(\sin^{-1}x) = x = \sin^{-1}(\sin x)$, if $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$

ii) $\cos(\cos^{-1}x) = x = \cos^{-1}(\cos x)$, if $0 \le x \le \pi$

iii) $\tan(\tan^{-1}x) = x = \tan^{-1}(\tan x)$, if $\frac{-\pi}{2} < x < \frac{\pi}{2}$

Note: $\sin^2 x + \cos^2 x = 1 \Rightarrow \sin x = \pm\sqrt{1-\cos^2 x}$

$\cos^2 x = 1 - \sin^2 x \Rightarrow \cos x = \pm\sqrt{1-\sin^2 x}$

9. Find the exact value:

a) $\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

b) $\cos\left(\sin^{-1}\left(\frac{12}{13}\right)\right)$

c) $\sin\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

d) $\cos\left(\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right)$

e) $\sin\left(2\cos^{-1}\left(\frac{3}{4}\right)\right)$

f) $\sin\left(2\sin^{-1}\left(\frac{4}{5}\right)\right)$

g) $\sin\left(\tan^{-1}\left(\frac{3}{4}\right)\right)$

h) $\sin\left(2\tan^{-1}\left(\sqrt{3}\right)\right)$

i) $\cos\left(\sec^{-1}(-2)\right)$

j) $\sec\left(\tan^{-1}\left(\frac{2}{3}\right)\right)$

k) $\sin\left(2\cos^{-1}\left(\frac{-1}{2}\right)\right)$

Solution: Use $\sin\theta = \pm\sqrt{1-\cos^2\theta}$, $\cos\theta = \pm\sqrt{1-\sin^2\theta}$

a) $\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

b) $\cos\left(\sin^{-1}\left(\frac{12}{13}\right)\right)$, Let $\sin^{-1}\left(\frac{12}{13}\right) = \theta \Rightarrow \sin\theta = \frac{12}{13}$.

$$\cos\theta = \sqrt{1-\frac{144}{169}} = \frac{5}{13} \Rightarrow \cos\left(\sin^{-1}\left(\frac{12}{13}\right)\right) = \cos\theta = \frac{5}{13}$$

c) $$\sin\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \sin\left(\pi-\frac{\pi}{6}\right) = \sin\frac{5\pi}{6} = \frac{1}{2}$$

d) $$\cos\left(\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right) = \cos\left(\frac{-\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

e) $$\sin\left(2\cos^{-1}\left(\frac{3}{4}\right)\right), \text{ Let } \cos^{-1}\left(\frac{3}{4}\right) = \theta \Rightarrow \cos\theta = \frac{3}{4}$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{3}{4}\right)^2} = \frac{\sqrt{7}}{4}$$

$$\Rightarrow \sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{\sqrt{7}}{4}\right)\left(\frac{3}{4}\right) = \frac{6\sqrt{7}}{16}$$

f) $$\sin\left(2\sin^{-1}\left(\frac{4}{5}\right)\right), \text{ Let } \sin^{-1}\left(\frac{4}{5}\right) = \theta \Rightarrow \sin\theta = \frac{4}{5}$$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{16}{25}\right)} = \sqrt{\frac{25-16}{25}} = \frac{3}{5}$$

$$\Rightarrow \sin\left(2\sin^{-1}\left(\frac{4}{5}\right)\right) = \sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{24}{25}$$

g) $$\sin\left(\tan^{-1}\left(\frac{3}{4}\right)\right), \text{ Let } \tan^{-1}\left(\frac{3}{4}\right) = \theta \Rightarrow \tan\theta = \frac{3}{4}$$

$$\Rightarrow \sin\theta = \frac{3}{5}, \Rightarrow \sin\left(\tan^{-1}\left(\frac{3}{4}\right)\right) = \sin\theta = \frac{3}{5}$$

h) $$\sin\left(2\tan^{-1}\left(\sqrt{3}\right)\right) = \sin\left(2\left(\frac{-\pi}{3}\right)\right) = \sin\left(\frac{-2\pi}{3}\right) = -\sin\frac{2\pi}{3} = \frac{-\sqrt{3}}{2}$$

i) $\cos\left(\sec^{-1}(-2)\right) \Rightarrow \cos\left(\cos^{-1}\left(\frac{-1}{2}\right)\right) = \cos\left(\frac{2\pi}{3}\right) = \frac{-1}{2}$

j) $\sec\left(\tan^{-1}\left(\frac{2}{3}\right)\right)$. Let $\tan^{-1}\left(\frac{2}{3}\right) = \theta \Rightarrow \tan\theta = \frac{2}{3}$

hypotenuse: $\sqrt{3^2+2^2} = \sqrt{13}$. $\sin\theta = \frac{2}{\sqrt{13}}, \cos\theta = \frac{3}{\sqrt{13}}$

$\Rightarrow \sec\left(\tan^{-1}\left(\frac{2}{3}\right)\right) = \sec(\theta) = \frac{1}{\cos\theta} = \frac{\sqrt{13}}{3}$

k) $\sin\left(2\cos^{-1}\left(\frac{-1}{2}\right)\right)$

$\Rightarrow \cos^{-1}\left(-\frac{1}{2}\right) = \pi - \cos^{-1}\left(\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\Rightarrow \sin\left(2\cos^{-1}\left(-\frac{1}{2}\right)\right) = \sin\left(2\cdot\frac{2\pi}{3}\right) = \sin\frac{4\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

10. Find the exact value

a) $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

b) $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$

c) $\sin^{-1}\left(\sin\frac{5\pi}{4}\right)$

d) $\cos^{-1}\left(\sin\frac{11\pi}{6}\right)$

e) $\tan^{-1}\left(\tan\frac{4\pi}{3}\right)$

f) $\sin^{-1}\left(\sin\frac{5\pi}{3}\right)$

g) $\sin^{-1}\left(\cos\frac{5\pi}{6}\right)$

Solution:

a) $\sin^{-1}\left(\sin\frac{5\pi}{6}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

b) $\cos^{-1}\left(\cos\frac{4\pi}{3}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \pi - \cos^{-1}\left(\frac{1}{2}\right) = \pi - \frac{\pi}{3}$

c) $\sin^{-1}\left(\sin\frac{5\pi}{4}\right) = \sin^{-1}\left(\frac{-\sqrt{2}}{2}\right) = -\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

d) $\cos^{-1}\left(\sin\frac{11\pi}{6}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \pi - \cos^{-1}\left(\frac{1}{2}\right) = \pi - \frac{\pi}{3}$

e) $\tan^{-1}\left(\tan\frac{4\pi}{3}\right) = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$

f) $\sin^{-1}\left(\sin\frac{5\pi}{3}\right) = \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = -\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

g) $\sin^{-1}\left(\cos\frac{5\pi}{6}\right) = \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = -\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

11. Simplify each of the expression interms of x.

a) $\sin(\text{arc}\cos(3x))$
b) $\cos(\text{arc}\sin(2x))$
c) $\sin(2\cos^{-1}x)$
d) $\cos(\sin^{-1}x)$
e) $\sin(\tan^{-1}x)$
f) $\cos(2\tan^{-1}x)$

Solution: use $\sin x = \sqrt{1-\cos^2 x}$, $\cos x = \pm\sqrt{1-\sin^2 x}$

a) $\sin(\cos^{-1}(3x))$ · Let $\cos^{-1}(3x) = \theta \Rightarrow \cos\theta = 3x$

$\Rightarrow \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-(3x)^2} = \sqrt{1-9x^2}$ *if* $-\frac{1}{3} \le x \le \frac{1}{3}$

$\Rightarrow \sin(\cos^{-1}(3x)) = \sin\theta = \sqrt{1-9x^2}$ · *if* $-\frac{1}{3} \le x \le \frac{1}{3}$

b) $\cos(\sin^{-1}(2x))$·Let $\sin^{-1}(2x) = \theta \Rightarrow \sin\theta = 2x$. $\cos\theta = \sqrt{1-4x^2}$

$\Rightarrow \cos\theta = \cos(\sin^{-1}(2x)) = \sqrt{1-4x^2}$ · *if* $-\frac{1}{2} \le x \le \frac{1}{2}$

c) $\sin(2\cos^{-1}x)$ · Let $\cos^{-1}x = \theta \Rightarrow \cos\theta = x$, $\sin\theta = \sqrt{1-x^2}$

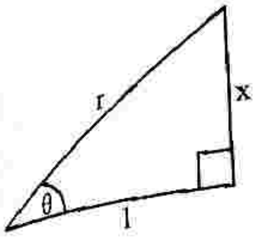
$\Rightarrow \sin(2\cos^{-1}x) = \sin(2\theta) = 2\sin\theta\cos\theta = 2x\sqrt{1-x^2}$

d) $\cos(\sin^{-1}x)$ · Let $\sin^{-1}x = \theta \Rightarrow \sin\theta = x$, $\cos\theta = \sqrt{1-x^2}$

$\Rightarrow \cos(\sin^{-1}x) = \cos\theta = \sqrt{1-x^2}$

e) $\sin(\tan^{-1}x)$ · Let $\tan^{-1}x = \theta \Rightarrow \tan\theta = x$

$r^2 = x^2 + 1 \Rightarrow r = \sqrt{x^2+1}$ · $\sin\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+1}}$



$$\Rightarrow \sin(\tan^{-1}x) = \sin\theta = \frac{x}{\sqrt{x^2+1}}$$

Example: $\sin(\tan^{-1}(2)) = \frac{2}{\sqrt{4+1}} = \frac{2}{\sqrt{5}}$

f) $\cos(2\tan^{-1}x)$ · Let $\tan^{-1}x = \theta \Rightarrow \tan\theta = x$

$$\Rightarrow \sin\theta = \frac{x}{\sqrt{x^2+1}}, \cos\theta = \frac{1}{r} = \frac{1}{\sqrt{x^2+1}}$$

$$\Rightarrow \cos(2\tan^{-1}x) = \cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{1}{x^2+1} - \frac{x^2}{x^2+1} = \frac{1-x^2}{x^2+1}$$

Example: $\cos(2\tan^{-1}(3)) = \frac{1-(3)^2}{3^2+1} = \frac{-8}{10} = \frac{-4}{5}$

12. Express x interms of y and determine the range of values x and y.

a) $y = \frac{1}{2}\sin^{-1}(x-3)$ c) $y = 2 + 3\cos^{-1}(5x-1)$

b) $y = 2 + 3 \text{ arc} \sin(5x-1)$

Solution: Here

a) $y = \frac{1}{2}\sin^{-1}(x-3) \Leftrightarrow 2y = \sin^{-1}(x-3)$

$$\Rightarrow \frac{-\pi}{2} \le 2y \le \frac{\pi}{2} \text{ and } -1 \le x-3 \le 1$$

$$\Rightarrow x-3 = \sin 2y, \frac{-\pi}{4} \le y \le \frac{\pi}{4} \text{ and } 2 \le x \le 4$$

$$\therefore\ x = 3 + \sin 2y, \frac{-\pi}{4} \le y \le \frac{\pi}{4} \text{ and } 2 \le x \le 4$$

b) $$\frac{y-2}{3} = \sin^{-1}(5x-1)$$

$$\Rightarrow \frac{-\pi}{2} \le \frac{y-2}{3} \le \frac{\pi}{2} \text{ and } -1 \le 5x-1 \le 1$$

$$\Rightarrow \sin\left(\frac{y-2}{3}\right) = 5x-1 \text{ if } 0 \le x \le \frac{2}{5}$$

$$\therefore\ x = \frac{1}{5} + \frac{1}{5}\sin\left(\frac{y-2}{3}\right) \text{ if } 0 \le x \le \frac{2}{5} \text{ and } 2-3\pi \le y \le 2+3\pi$$

c) $$\frac{y-2}{3} = \cos^{-1}(5x-1) \text{ if } 0 \le \frac{y-2}{3} \le \pi \text{ and } -1 \le 5x-1 \le 1$$

$$\Rightarrow \cos\left(\frac{y-2}{3}\right) = 5x-1 \text{ if } 0 \le y-2 \le 3\pi \text{ and } 0 \le 5x \le 2$$

$$\Rightarrow 5x = 1 + \cos\left(\frac{y-2}{3}\right) \text{ if } 2 \le y \le 2+3\pi \text{ and } 0 \le x \le \frac{2}{5}$$

$$\therefore\ x = \frac{1}{5} + \frac{1}{5}\cos\left(\frac{y-2}{3}\right) \text{ if } 2 \le y \le 2+3\pi \text{ and } 0 \le x \le \frac{2}{5}$$

13. Simplify each of the following

a) $\sin^{-1}x + \cos^{-1}x$

b) $\cos^{-1}(-x) + \cos^{-1}(x)$

c) $\sin(\sin^{-1}x + \cos^{-1}x)$

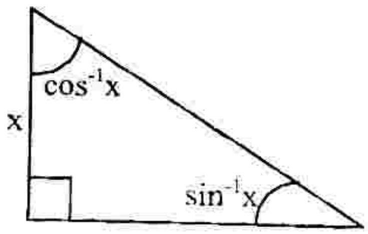
d) $\cos(\cos^{-1}(-x) + \cos^{-1}x)$

e) $\cos(\sin^{-1}x + \sin^{-1}(-x))$

Solution:

a) $\sin^{-1}x + \cos^{-1}x$, let $\sin^{-1}x = \alpha \Rightarrow \sin\alpha = x$

let $\cos^{-1}x = \beta \Rightarrow \cos\beta = x$



since it is right angle $\therefore \alpha + \beta = \frac{\pi}{2}$

$$\therefore \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}.$$

b) $\cos^{-1}(-x) = \pi - \cos^{-1}x \Rightarrow \pi - \cos^{-1}x + \cos^{-1}x = \pi$

$\therefore \cos^{-1}(-x) + \cos^{-1}x = \pi$

$$\Rightarrow \sin^{-1}\left(\frac{3}{5}\right) = \pi - \cos^{-1} x, \text{ Recall.} \cos^{-1}(x) + \cos^{-1}(-x) = \pi$$

$$\Rightarrow \sin^{-1}\left(\frac{3}{5}\right) = \cos^{-1}(-x)$$

$$\Rightarrow -x = \cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right), \; Let \; \sin^{-1}\left(\frac{3}{5}\right) = \theta$$

$$\Rightarrow \sin\theta = \frac{3}{5} \Rightarrow \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$$

$$\Rightarrow -x = \frac{4}{5}, \; therefore \quad x = \frac{-4}{5}$$

Graph of some trigonometric function

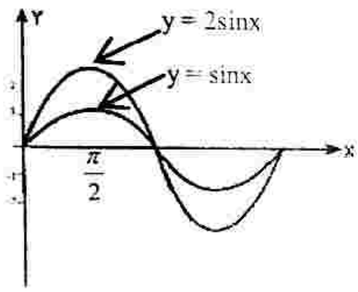
Let y = sinx and y = 2sinx. Then state:

a) Period
b) x-intercept
c) the amplitude
d) range
e) the maximum and minimum value and the value of x at which the maximum and minimum occur for y = 2sinx
f) sketch the graph on the same axes.

Solution:

a) The period: 2π
b) x-intercept: $x = \pi k$, k is an integer
c) Amplitude for, $y = \sin x$ is $|1| = 1$
   Amplitude for $y = 2\sin x$ is $|2| = 2$
d) Range for $y = \sin x$ is $[-1, 1]$
   Range for $y = 2\sin x$ is $[-2, 2]$
e) The maximum is 2 which occurs at $x = \frac{\pi}{2} + 2\pi k$, k is integer

   Minimum is -2, which occurs at $x = \frac{3\pi}{2} + 2\pi k$



Multiplying periodic function by a constant affects: the amplitude

ition: Amplitude $|a|$

c) $\sin(\sin^{-1}x + \cos^{-1}x) = \sin\left(\frac{\pi}{2}\right) = 1$

d) $\cos(\cos^{-1}(-x) + \cos^{-1}x = \cos\pi = -1$

e) $\cos(\sin^{-1}x + \sin^{-1}(-x)) = \cos(\sin^{-1}x + -\sin^{-1}x) = \cos 0 = 1$

14. Find the exact value of the expression

a) $\cos\left(\sin^{-1}\left(\frac{3}{4}\right) + \cos^{-1}\left(\frac{5}{13}\right)\right)$

b) $\sin\left(\sin^{-1}\left(\frac{2}{3}\right) + \cos^{-1}\left(\frac{1}{2}\right)\right)$

Solution:

a) Let $\sin^{-1}\left(\frac{3}{4}\right) = \alpha \Rightarrow \sin\alpha = \frac{3}{4} \Rightarrow \cos\alpha = \sqrt{1-\left(\frac{3}{4}\right)^2} = \frac{\sqrt{7}}{4}$

Let $\cos^{-1}\left(\frac{5}{13}\right) = \beta \Rightarrow \cos\beta = \frac{5}{13}$

$$\Rightarrow \sin\beta = \sqrt{1-\left(\frac{5}{13}\right)^2} = \frac{12}{13}$$

$$\Rightarrow \cos\left(\sin^{-1}\left(\frac{3}{4}\right) + \cos^{-1}\left(\frac{5}{13}\right)\right) = \cos(\alpha+\beta)$$

$$\Rightarrow \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \frac{\sqrt{7}}{4}\cdot\frac{5}{13} - \frac{3}{4}\cdot\frac{12}{13} = \frac{5\sqrt{7}-36}{52}$$

$$\therefore \cos\left(\sin^{-1}\left(\frac{3}{4}\right) + \cos^{-1}\left(\frac{5}{13}\right)\right) = \frac{5\sqrt{7}-36}{52}$$

b) Exercise left for you

15. Solve each of the following

a) $\sin^{-1}x = \cos^{-1}\left(\frac{5}{13}\right)$ c) $\cos^{-1}\left(x-\frac{1}{2}\right) = \frac{\pi}{3}$

b) $\sin^{-1}(x-1) = \frac{\pi}{2}$ d) $\tan^{-1}(x) = \sin^{-1}\left(\frac{3}{5}\right)$

e) $\sin^{-1}\left(\frac{3}{5}\right) + = \frac{\pi}{3}$

Solution:

a) $\sin^{-1}x = \cos^{-1}\left(\frac{5}{13}\right) \Leftrightarrow x = \sin\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$

$$Let\ \cos^{-1}\left(\frac{5}{13}\right) = \theta \Rightarrow \cos\theta = \frac{5}{13}$$

$$\Rightarrow \sin\theta = \sqrt{1-\left(\frac{5}{13}\right)^2} = \sqrt{1-\frac{25}{169}} = \frac{12}{13}$$

$$\therefore x = \sin\left(\cos^{-1}\left(\frac{5}{13}\right)\right) = \sin\theta = \frac{12}{13}$$

b) $\sin^{-1}(x-1) = \frac{\pi}{2} \Rightarrow x-1 = \sin\frac{\pi}{2} = 1 \Rightarrow x = 1+1 = 2$

c) $\cos^{-1}\left(x-\frac{1}{2}\right) = \frac{\pi}{3} \Rightarrow x-\frac{1}{2} = \cos\frac{\pi}{3} = \frac{1}{2}$

$$\Rightarrow x-\frac{1}{2} = \frac{1}{2}$$

$$\therefore\ x = \frac{1}{2}+\frac{1}{2} = 1$$

d) $\tan^{-1}x = \sin^{-1}\left(\frac{3}{5}\right) \Leftrightarrow x = \tan\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$

$$\Rightarrow \text{Let } \sin^{-1}\left(\frac{3}{5}\right) = \theta \Rightarrow \sin\theta = \frac{3}{5} \Rightarrow \cos\theta = \frac{4}{5}$$

$$\therefore\ \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{3}{4},\ \therefore x = \frac{3}{4}$$

e) $\sin^{-1}\left(\frac{3}{5}\right) + \cos^{-1}x = \pi$

$$\Rightarrow \sin^{-1}\left(\frac{3}{5}\right) = \pi - \cos^{-1} x, \text{ Recall.} \cos^{-1}(x) + \cos^{-1}(-x) = \pi$$

$$\Rightarrow \sin^{-1}\left(\frac{3}{5}\right) = \cos^{-1}(-x)$$

$$\Rightarrow -x = \cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right), \ Let \ \sin^{-1}\left(\frac{3}{5}\right) = \theta$$

$$\Rightarrow \sin\theta = \frac{3}{5} \Rightarrow \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$$

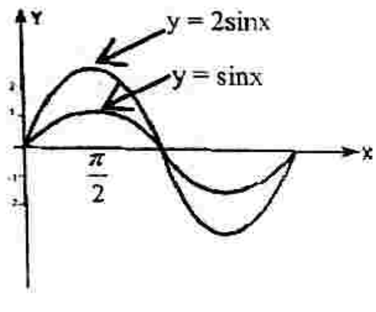
$$\Rightarrow -x = \frac{4}{5}, \ therefore \quad x = \frac{-4}{5}$$

Graph of some trigonometric function

16. Let y = sinx and y = 2sinx. Then state:

a) Period
b) x-intercept
c) the amplitude
d) range
e) the maximum and minimum value and the value of x at which the maximum and minimum occur for y = 2sinx
f) sketch the graph on the same axes.

Solution:



a) The period: 2π
b) x-intercept: $x = \pi k$, k is an integer
c) Amplitude for, y = sinx is $|1| = 1$
Amplitude for y = 2sinx is $|2| = 2$
d) Range for y = sinx is $[-1, 1]$
Range for y = 2 sinx is $[-2, 2]$
e) The maximum is 2 which occurs

at $x = \frac{\pi}{2} + 2\pi k$, k is integer

Minimum is -2, which occurs at $x = \frac{3\pi}{2} + 2\pi k$

Note: Multiplying periodic function by a constant affects: the amplitude

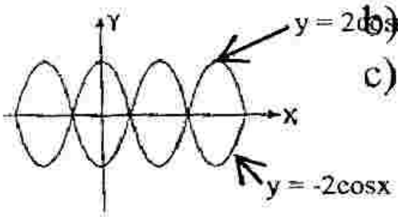
Definition: Amplitude $|a|$

The amplitude of periodic function with maximum M and minimum m is

$$|a| = \frac{M-m}{2}$$

17. Let y = -2cosx, then state
    a) Amplitude
    b) the range
    c) period
    d) how the graph differ from the graph of y = 2cosx

Solution: a) Amplitude is |-2| = 2



b) Range is [-2, 2]

c) period is 2π

d) It is the reflection of the graph of y = 2cosx about the x-axis.

Graph of y = asin(kx) and y = acos kx

- Amplitude = |a|
- Period = $\frac{2\pi}{|k|}$
- Effect of k to stretch or compress the cycle of the graph by factor of $\frac{1}{|k|}$

  i) if k > 1, the resulting graph is compressed (contract) horizontally (not vertically)

  ii) If 0 < k < 1, thee resulting graph is stretched horizontally.

18. Find, thee amplitude, period, x-intercept, the value of x which give minimum or maximum value sketch the graph

    a) f(x) = 3sin(2x)   d) f(x) = 3cos2x

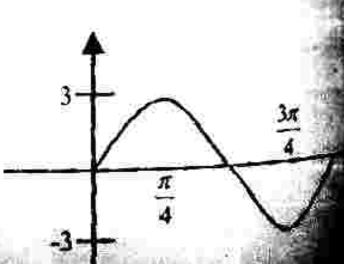
    b) f(x) = 4cos(πx)   e) $f(x) = 4\cos\left(\frac{-3x}{2}\right)$

    c) $f(x) = \cos\left(\frac{1}{4}x\right)$

Solution:

a) f(x) = 3sin(2x)

- Amplitude = |3| = 3,



- period, $P = \dfrac{2\pi}{2} = \pi$
- x-intercept (the graph crosses the x-axis:

  $3\sin 2x = 0 \Rightarrow 2x = k\pi$, k is integer $\Rightarrow x = \dfrac{\pi}{2}k$

  $\therefore$ the graph cross the x-axis at

  $$(-\pi,0),\left(\frac{-\pi}{2},0\right),(0,0),\left(\frac{\pi}{2},0\right),(\pi,0)$$

- maximum value is 3 which occurs:

  $\Rightarrow 3\sin 2x = \Rightarrow \sin 2x = 1$

  $\Rightarrow 2x = \dfrac{\pi}{2} + 2\pi k \Rightarrow x = \dfrac{\pi}{4} + \pi k, k \in z$

  $\therefore$ maximum attain, at $x = \dfrac{\pi}{4} + \pi k$

- $\sin 2x = -1 \Rightarrow 2x = \dfrac{3\pi}{2} + 2\pi k$, k is integer

  $\Rightarrow x = \dfrac{3\pi}{2} + \pi k$

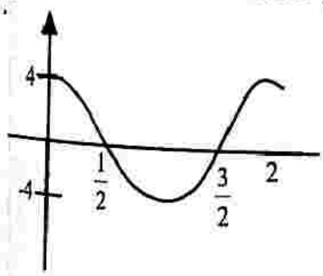
  $\therefore$ The function attains its minimum value at

  $x = \dfrac{3\pi}{4} + \pi k$

- One cycle of each graph is completed on the interval
  $0 \le 2x \le 2\pi \Rightarrow 0 \le x \le \pi$

b) $f(x) = 4\cos\pi x$

- Amplitude = 4, • Period = $\dfrac{2\pi}{\pi} = \pi$
- The graph crosses the x-axis: $\cos\pi x = 0$



$\Rightarrow \pi x = \dfrac{\pi}{2}, \dfrac{3}{2}\pi, \dfrac{5\pi}{2}, \ldots\ldots$

$\Rightarrow x = \dfrac{1}{2}, \dfrac{3}{2}, \dfrac{5}{2}, \ldots\ldots$

$\therefore$ x-intercept $\left(\frac{1}{2},0\right),\left(\frac{3}{2},0\right),\left(\frac{5}{2},0\right),\ldots$

- $3\cos\pi x = 3 \Rightarrow \cos\pi x = 1$

  $\cos\pi x = 1 \Rightarrow \pi x = 0 + 2\pi k \Rightarrow x = 2k$, k is integer

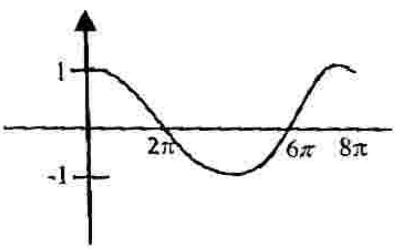
  $\therefore$ the function attain maximum at $x = 2k$
- $3\cos\pi x = -3 \Rightarrow \cos\pi x = -1$

  $\Rightarrow \pi x = \pi + 2\pi k \Rightarrow x = 1 + 2k,\ k \in z$

  $\therefore$ the function attain minimum value at $x = 1+2k$
- One cycle is completed on the interval:

  $0 \le \pi x \le 2\pi \Rightarrow 0 \le x \le 2$

c) $f(x) = \cos\left(\frac{1}{4}x\right)$



- Amplitude $= |1| = 1$, • Period, $P = \dfrac{2\pi}{\frac{1}{4}} = 8\pi$
- The graph crosses the x-axis: $\cos\left(\frac{1}{4}x\right) = 0$

$$\Rightarrow \frac{1}{4}x = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow x = 2\pi,\ 6\pi$$

$\therefore$ x-intercept $(-6\pi, 0)$, $(-2\pi, 0)$, $(2\pi, 0)$, $(6\pi, 0)$

- $\cos\left(\frac{1}{4}x\right) = 1 \Rightarrow \frac{1}{4}x = 0 + 2\pi k \Rightarrow x = 8\pi k$

  $\therefore$ the function attain maximum value at $x = 8\pi k$, k is integer
- $\cos\left(\frac{1}{4}x\right) = -1 \Rightarrow \frac{1}{4}x = \pi + 2\pi k \Rightarrow x = 4\pi + 8\pi k$

  $\therefore$ the function attain minimum value at $x = (1+2k)4\pi$, k is integer
- One cycle of each graph is completed on the interval:

$$0 \le \frac{1}{4}x \le 2\pi \Rightarrow 0 \le x \le 8\pi$$

d, e. exercise left for you

(18b) Find the amplitude and period fo the following graph.

a) $f(\mathrm{x}) = 6\sin\left(\frac{2\mathrm{x}}{5}\right)\cos\left(\frac{2\mathrm{x}}{5}\right)$

b) $f(\mathrm{x}) = 4\sin\left(\frac{\mathrm{x}}{3}\right)\cos\left(\frac{\mathrm{x}}{3}\right)$

Solution: Use $\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

a) $6\sin\left(\frac{2\mathrm{x}}{5}\right)\cos\left(\frac{2\mathrm{x}}{5}\right) = (6)\frac{1}{2}\left[\sin\left(\frac{2\mathrm{x}}{5}+\frac{2\mathrm{x}}{5}\right) + \sin\left[\frac{2\mathrm{x}}{5}-\frac{2\mathrm{x}}{5}\right]\right]$

$\therefore f(\mathrm{x}) = 6\sin\left(\frac{2\mathrm{x}}{5}\right)\cos\left(\frac{2\mathrm{x}}{5}\right) = 3\sin\left(\frac{4\mathrm{x}}{5}\right) + 0$

Then comparing $\mathrm{y} = 3\sin\left(\frac{4\mathrm{x}}{5}\right)$ to $\mathrm{y} = A\sin(\mathrm{wx})$

Amplitude $= |A| = 3$

Period: $P = \frac{2\pi}{\mathrm{w}} = \frac{2\pi}{\frac{4}{5}} = \frac{10\pi}{4} = \frac{5\pi}{2}$

b) Exercise left for you. (Ans. 2, 3π)

19. Find an equation for a sine function with the given information.

a) Amplitude a = 2, period, $P = 3\pi$

b) Amplitude a = 2, period, $P = \frac{2\pi}{3}$

c) Amplitude a = 4, period, $P = 2$

Solution: We have, $\mathrm{y} = a\sin(kx)$

a) $|a| = 2$, period, $P = \frac{2\pi}{k} = 3\pi \Rightarrow 3\pi k = 2\pi$

$\Rightarrow k = \frac{2}{3}$ $\qquad \therefore y = \pm 2\sin\left(\frac{2x}{3}\right)$

b) $|a| = 2$, Period, $P = \frac{2\pi}{k} = \frac{2\pi}{3} \Rightarrow k = 3,$

$\therefore \mathrm{y} = \pm 2\sin(3\mathrm{x})$

c) $|a| = 2$, Period, $P = \dfrac{2\pi}{k} = 2 \Rightarrow k = \pi$

$$\therefore\ y = \pm 2\sin\pi x$$

20. Find a function of the form $f(x) = a\cos kx$ satisfying the given properties

a) Amplitude, $a = \dfrac{2}{3}$, and $f(4) = 0$

b) Amplitude, $a = 3$, and $f(2) = 3$

c) Amplitude, $a = 2$, and the graph pass through $\left(\dfrac{\pi}{4}, 0\right)$

d) `Peak at $\left(\dfrac{\pi}{5}, 3\right)$

Solution: $f(x) = a\cos kx$

a) $|a| = \dfrac{2}{3}$, $f(4) = 0 \Rightarrow \cos(4k) = 0$

$$\Rightarrow 4k = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow k = \frac{\pi}{8} \text{ or } k = \frac{3\pi}{8}$$

$$\therefore\ f(x) = \pm\frac{2}{3}\cos\left(\frac{\pi}{8}x\right) \text{ or } f(x) = \pm\frac{2}{3}\cos\left(\frac{3\pi}{8}x\right)$$

b) $|a| = 3 \Rightarrow a = \pm 3$ and $f(2) = a\cos(2k) = 3$

$\Rightarrow 3\cos(2k) = 3 \Rightarrow \cos(2k) = 1$

$\Rightarrow 2k = 0, 2\pi, 4\pi \ldots \Rightarrow k = 0, \pi$

$$\therefore\ f(x) = 3\cos(\pi x)$$

c) $f\left(\dfrac{\pi}{4}\right) = \cos\left(\dfrac{\pi}{4}k\right) = 0 \Rightarrow \dfrac{\pi k}{4} = \dfrac{\pi}{2} \Rightarrow k = 2$

$\therefore\ f(x) = \pm 2\cos(2x)$

d) Peak at $x = \dfrac{\pi}{5} \Rightarrow f\left(\dfrac{\pi}{5}\right) = \cos\left(\dfrac{\pi}{5}k\right) = 3 \Rightarrow a = 3$

$$\Rightarrow \cos\left(\frac{\pi}{5}k\right) = 1 \Rightarrow \frac{\pi}{5}k = 2\pi \Rightarrow k = 10$$

$\therefore f(x) = 3\cos(10x)$

Graph of $f(x) = a\sin(kx + b) + c$ and $f(x) = a\cos(kx + b) + c$

- Amplitude = |a|
- Period, P = $\frac{2\pi}{k}, k > 0$
- Range = [c - |a|, c + |a|]
- Phase angle = -b
- Phase shift = $\frac{-b}{k} \longleftarrow$ the horizontal shift of the graph from y = a sinkx and y = acoskx along x –axis.
- One cycle of each graph is completed on the interval: $0 \le kx + b \le 2\pi \Rightarrow 0 - b \le kx \le 2\pi - b$

$$\Rightarrow \frac{-b}{k} \le x \le \frac{-b}{k} + \frac{2\pi}{k}$$

i) $x = \frac{-b}{k} \longleftarrow$ left end point of graph

ii) $x = \frac{-b}{k} + \frac{2\pi}{k} \longleftarrow$ Right end point of graph

- Vertical shift C – unit up in the positive y-direction if c > 0 and C unit down if C < 0 in negative y – direction

21. Find the amplitude, period, phase shift, phase angle, interval for one cycle, the range.

a) $f(x) = -\frac{1}{2}\sin(4x - 2)$ e) $f(x) = \frac{5}{4}\cos(3x + 2)$

b) $f(x) = \frac{3}{2}\sin\left(2x + \frac{\pi}{4}\right)$ f) $f(x) = 3\sin\left(\frac{1}{2}x + 3\right) - 2$

c) $f(x) = 3\cos\left(3x + \frac{3\pi}{4}\right)$ g) $f(x) = 2\cos\left(\frac{3}{2}x + \frac{\pi}{4}\right) + 4$

d) $f(x) = 4\sin\left(\frac{2}{3}x + \frac{\pi}{6}\right)$ h) $f(x) = -3\sin(\pi x + 1) - 1$

Solution:

a) $f(x) = -\frac{1}{2}\sin(4x - 2)$

c) $|a| = 2$, Period, $P = \dfrac{2\pi}{k} = 2 \Rightarrow k = \pi$

$$\therefore y = \pm 2\sin\pi x$$

20. Find a function of the form $f(x) = a\cos kx$ satisfying the given properties

a) Amplitude, $a = \dfrac{2}{3}$, and $f(4) = 0$

b) Amplitude, $a = 3$, and $f(2) = 3$

c) Amplitude, $a = 2$, and the graph pass through $\left(\dfrac{\pi}{4}, 0\right)$

d) `Peak at $\left(\dfrac{\pi}{5}, 3\right)$

Solution: $f(x) = a\cos kx$

a) $|a| = \dfrac{2}{3}$, $f(4) = 0 \Rightarrow \cos(4k) = 0$

$$\Rightarrow 4k = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow k = \frac{\pi}{8} \text{ or } k = \frac{3\pi}{8}$$

$$\therefore f(x) = \pm\frac{2}{3}\cos\left(\frac{\pi}{8}x\right) \text{ or } f(x) = \pm\frac{2}{3}\cos\left(\frac{3\pi}{8}x\right)$$

b) $|a| = 3 \Rightarrow a = \pm 3$ and $f(2) = a\cos(2k) = 3$

$\Rightarrow 3\cos(2k) = 3 \Rightarrow \cos(2k) = 1$

$\Rightarrow 2k = 0, 2\pi, 4\pi \ldots \Rightarrow k = 0, \pi$

$$\therefore f(x) = 3\cos(\pi x)$$

c) $f\left(\dfrac{\pi}{4}\right) = \cos\left(\dfrac{\pi}{4}k\right) = 0 \Rightarrow \dfrac{\pi k}{4} = \dfrac{\pi}{2} \Rightarrow k = 2$

$\therefore f(x) = \pm 2\cos(2x)$

d) Peak at $x = \dfrac{\pi}{5} \Rightarrow f\left(\dfrac{\pi}{5}\right) = \cos\left(\dfrac{\pi}{5}k\right) = 3 \Rightarrow a = 3$

$$\Rightarrow \cos\left(\frac{\pi}{5}k\right) = 1 \Rightarrow \frac{\pi}{5}k = 2\pi \Rightarrow k = 10$$

$\therefore f(x) = 3\cos(10x)$

Graph of $f(x) = a\sin(kx + b) + c$ and $f(x) = a\cos(kx + b) + c$

- Amplitude, $|a| = \left|\frac{-1}{2}\right| = \frac{1}{2}$, Period, $P = \frac{2\pi}{k} = \frac{2\pi}{4} = \frac{\pi}{2}$
- Phase shift, $4x - 2 = 0 \Rightarrow x = \frac{2}{4} = \frac{1}{2} \leftarrow$ Horizontal shift
- One cycle, $0 \leq 4x - 2 \leq 2\pi \Rightarrow \frac{1}{2} \leq x \leq \frac{1}{2} + \frac{\pi}{2}$
- Range $\left[\frac{-1}{2}, \frac{1}{2}\right]$

b) $f(x) = \frac{3}{2}\sin\left(2x + \frac{\pi}{4}\right)$, Amplitude, $a = \left|\frac{-3}{2}\right| = \frac{3}{2}$,

- Period, $P = \frac{2\pi}{k} = \frac{2\pi}{2} = \pi$
- Phase shift, $2x + \frac{\pi}{4} = 0 \Rightarrow 2x = \frac{-\pi}{4} \Rightarrow x = \frac{-\pi}{8}$
- Phase angle $\frac{\pi}{4}$
- One cycle, covers, $0 \leq 2x + \frac{\pi}{4} \leq 2\pi$

  $\Rightarrow \frac{-\pi}{8} \leq x \leq \frac{-\pi}{8} + \frac{2\pi}{2} \Rightarrow \frac{-\pi}{8} \leq x \leq \frac{7\pi}{8}$
- Range $\left[\frac{-3}{2}, \frac{3}{2}\right]$
- The graph has maximum at: $2x + \frac{\pi}{4} = \frac{\pi}{2}$

  $\Rightarrow x = \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8}$
- The graph has minimum, $\sin\left(2x + \frac{\pi}{4}\right) = -1$

$$\Rightarrow 2x+\frac{\pi}{4}=\frac{3\pi}{2}\Rightarrow x=\frac{5\pi}{8}$$

c) $f(x)=3\cos\left(3x+\frac{3\pi}{4}\right)$, Amplitude, $|a|=|3|\Rightarrow a=3$

- Period, $P=\frac{2\pi}{k}\Rightarrow P=\frac{2\pi}{3}$
- Phase shift, $\frac{-b}{k}=\frac{-3\pi}{(3)(4)}=\frac{-\pi}{4}$ *or* $3x+\frac{3\pi}{4}=0\Rightarrow x=\frac{-\pi}{4}$
- Phase angle $\frac{3\pi}{4}$
- One cycle, covers, $0\le 3x+\frac{3\pi}{4}\le 2\pi$

$$\Rightarrow \frac{-\pi}{4}\le x\le \frac{-\pi}{4}+\frac{2\pi}{3}$$

$$\Rightarrow \frac{-\pi}{4}\le x\le \frac{5\pi}{12}$$

- The graph has maximum when $\cos\left(3x+\frac{3\pi}{4}\right)=1$

$$\Rightarrow 3x+\frac{3\pi}{4}=2\pi\Rightarrow 3x=\frac{5\pi}{4}\Rightarrow x=\frac{5\pi}{12}$$

and at $x=\frac{-\pi}{4}$

- The graph has minimum when, $\cos\left(3x+\frac{3\pi}{4}\right)=-1$

$$\Rightarrow 3x+\frac{3\pi}{4}=\pi\Rightarrow 3x=\frac{\pi}{4}\Rightarrow x=\frac{\pi}{12}$$

d and e are exercise left for you

f) $f(x)=3\sin\left(\frac{1}{2}x+3\right)-2$, Amplitude, $a=3$

- Period, P = $\frac{2\pi}{k} = \frac{2\pi}{\frac{1}{2}} \Rightarrow P = 4\pi$
- Phase shift, $\frac{-b}{k} = \frac{-3}{\frac{1}{2}} = -6$
- Phase angle $= -3$
- One cycle, completed, $0 \le \frac{1}{2}x + 3 \le 2\pi$

$\Rightarrow -6 \le x \le -6 + 4\pi$

- The graph has maximum when $\sin\left(\frac{1}{2}x+3\right)=1$

$\Rightarrow \frac{1}{2}x + 3 = \frac{\pi}{2} \Rightarrow \frac{1}{2}x = \frac{\pi}{2} - 3 \Rightarrow x = \pi - 6$

- The graph has minimum when, $\sin\left(\frac{1}{2}x+3\right)=-1$

$\Rightarrow \frac{1}{2} + 3 = \frac{3\pi}{2} \Rightarrow \frac{1}{2}x = \frac{3\pi}{2} - 3 \Rightarrow x = 3\pi - 6$

- Range

$[c-|a|,\ c+|a|] = [-2-3, -2+|3|] = [-5, 1]$

g) f(x) = $2\cos\left(\frac{3}{2}x + \frac{\pi}{4}\right) + 4$, Amplitude, a = |2| = 2

- Period, P = $\frac{2\pi}{k} = \frac{2\pi}{\frac{3}{2}} = \frac{4\pi}{3}$
- Phase shift, $\frac{-b}{k} = \frac{\frac{-\pi}{4}}{\frac{1}{2}} = \frac{-\pi}{4} \cdot \frac{2}{3} = \frac{-\pi}{6}$
- Phase angle $\frac{\pi}{4}$

- One cycle of each graph is completed on interval

$$\le \frac{3}{2}x + \frac{\pi}{4} \le 2\pi \Rightarrow \frac{-\pi}{6} \le x \le \frac{-\pi}{6} + \frac{4\pi}{3}$$

- Range [-2+4, 2+4] = [2, 6]

- The graph has maximum when $\cos\left(\frac{3}{2}x + \frac{\pi}{4}\right) =$

$$\Rightarrow \frac{3}{2}x + \frac{\pi}{4} = 0 \ \ and \ \ \frac{3}{2}x + \frac{\pi}{4} = 2\pi$$

$$\Rightarrow x = \frac{-\pi}{6} \ \ and \ \ x = \frac{7\pi}{6}$$

h) exercise left for you

22. Find a function of the form f(x) = a sin(kx + b) + c satisfying the given properties.

a) Amplitude, a = 2, period, P = π, and phase shift $\frac{\pi}{3}$.

b) Amplitude, a = 3, period, P = 6, and phase shift $\frac{\pi}{4}$.

Solution: a) $|a| = 2 \Rightarrow a = \pm 2$, Period, $P = \frac{2\pi}{k} = \pi \Rightarrow k = 2$

- Phase shift, $\frac{-b}{k} = \frac{\pi}{3} \Rightarrow \frac{-b}{2} = \frac{\pi}{3} \Rightarrow b = -\frac{2\pi}{3}$

$$\therefore \ f(x) = \pm 2\sin\left(2x - \frac{2\pi}{3}\right)$$

b) exercise left for you

Trigonometric equation

Particular and General solution

Summary of features of trigonometric equation

Equation **General solution of equation**

- $\sin x = 0, \Rightarrow x = \pi k$, k is integer,
- $\sin x = 1, \Rightarrow x = \frac{\pi}{2} + 2\pi k$, k is integer,

- sinx = -1,$\Rightarrow x = \frac{3\pi}{2} + 2\pi k$, k is integer,
- cosx = 0,$\Rightarrow x = \frac{\pi}{2} + 2\pi k$, k is integer,
- cosx = 1,$\Rightarrow$x = 2πk, k is integer,
- cosx = -1,$\Rightarrow$x = π + 2πk $\Rightarrow$ x = (1 + 2k) π, k is integer
- tanx = 0,$\Rightarrow$x = πk, k is integer,
- tanx = 1,$\Rightarrow x = \frac{\pi}{4} + \pi k$, since the period of tangent is π,
- tanx = -1,$\Rightarrow x = \frac{3}{4}\pi + \pi k$, k is integer

Solving trigonometric Equation

There are two type of solution: These are

- **Particular Solution** is the Solution in its period.
- **General Solution** is particular Solution plus integer multiple of period of given trig. due to periodic nature of trigonometric function.

23. Find
    i) Particular Solution
    ii) General Solution for each of the following trigonometric equation.

a) $\sin x = \frac{1}{2}$

b) $\sin x = \frac{-1}{2}$

c) $\sin(3x) = \frac{1}{2}$

d) $\sin(3x) = \frac{-1}{2}$

e) $\sin\left(2x - \frac{\pi}{2}\right) = \frac{1}{2}$

f) $\sin(4x) = \frac{-\sqrt{3}}{2}$

g) $\sin x = \frac{-\sqrt{2}}{2}$

h) $\sin 6x = \frac{-\sqrt{2}}{2}$

i) $\cos x = \frac{\sqrt{3}}{2}$

j) $\cos(3x) = \frac{-\sqrt{3}}{2}$

k) $\tan x = \sqrt{3}$

Solution: a) $\sin x = \frac{1}{2}$

i) Particular Solution, $\left\{x : x = \frac{\pi}{6} \text{ and } x = \pi - \frac{\pi}{6}\right\} = \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$

in $0 \le x \le 2\pi$

ii) General Solution $= \left\{x : x = \frac{\pi}{6} + 2\pi k, \frac{5\pi}{6} + 2\pi k\right\}$

b) $\sin x = \frac{-1}{2}$

i) Particular Solution lies in 3rd and 4th quadrant

$\therefore S.S = \left\{\frac{\pi}{6} + \pi, \pi + \frac{5\pi}{6}\right\} = \left\{\frac{7\pi}{6}, \frac{11\pi}{6}\right\}$

ii) General Solution $= \left\{\frac{7\pi}{6} + 2\pi k, \frac{11\pi}{6} + 2\pi k\right\}$

c) $\sin(3x) = \frac{1}{2} \Rightarrow 3x = \frac{\pi}{6} \text{ and } 3x = \frac{5\pi}{6}$

$\Rightarrow x = \frac{\pi}{18} \text{ and } x = \frac{5\pi}{18}$

Period, $P = \frac{2\pi}{3}$

i) Particular Solution $= \left\{x : x = \frac{\pi}{18}, \frac{5\pi}{18}\right\}$

ii) General Solution $= \left\{x : x = \frac{\pi}{18} + \frac{2\pi k}{3}, \frac{5\pi}{18} + \frac{2\pi k}{3}\right\}$

d) $\sin(3x) = \frac{-1}{2} \Rightarrow -3x = \frac{\pi}{6} \text{ and } -3x = \frac{5\pi}{6}$

$\Rightarrow x = \frac{-\pi}{18} \text{ and } x = \frac{-5\pi}{18}$

$\Rightarrow x = \frac{-\pi}{18} + \frac{2\pi}{3} \text{ and } x = \frac{-5\pi}{18} + \frac{2\pi}{3}$

$$\Rightarrow x = \frac{11\pi}{18} \; and \; x = \frac{7\pi}{18}$$

i) Particular Solution in $\left[0, \frac{2\pi}{3}\right]$ are $\left\{x : x = \frac{11\pi}{18}, \frac{7\pi}{18}\right\}$

ii) General Solution = $\left\{x : x = \frac{11\pi}{18} + \frac{2}{3}\pi k, \frac{7\pi}{18} + \frac{2\pi k}{3}\right\}$

e) $\sin\left(2x - \frac{\pi}{2}\right) = \frac{1}{2} \Rightarrow 2x - \frac{\pi}{2} = \frac{\pi}{6} \; and \; 2x - \frac{\pi}{2} = \frac{5\pi}{6}$

$$\Rightarrow 2x = \frac{4\pi}{6} \Rightarrow x = \frac{\pi}{3} \; and \; x = \frac{2\pi}{3}$$

Period, $P = \frac{2\pi}{2} = \pi$

i) Particular Solution interval $[0, \pi]$ are $\left\{\frac{\pi}{3}, \frac{5\pi}{6}\right\}$

ii) General Solution = $\left\{\frac{\pi}{3} + \pi k, \frac{5\pi}{6} + \pi k\right\}$

f) $\sin(4x) = \frac{-\sqrt{3}}{2} \Rightarrow -4x = \frac{\pi}{3} \; and \; -4x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\Rightarrow x = \frac{-\pi}{12} \; and \; x = -\frac{\pi}{6}$ Period = $\frac{2\pi}{4} = \frac{\pi}{2}$

i) Particular Solution interval $\left[0, \frac{\pi}{2}\right]$ are $\left\{\frac{-\pi}{12} + \frac{\pi}{2}, \frac{-\pi}{6} + \frac{\pi}{2}\right\}$

$$= \left\{\frac{5\pi}{12}, \frac{\pi}{3}\right\}$$

ii) General Solution = $\left\{\frac{\pi}{3} + \frac{\pi}{2}k, \frac{5\pi}{12} + \frac{\pi}{2}k\right\}$

g) $\sin x = \frac{-\sqrt{2}}{2} \longleftarrow$ Exercise left for you

h) $\sin 6x = \frac{-\sqrt{2}}{2} \Rightarrow -6x = \frac{\pi}{4}$ *and* $-6x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\Rightarrow x = \frac{-\pi}{24}$ *and* $x = \frac{-\pi}{8}$

Period $= \frac{2\pi}{6} = \frac{\pi}{3}$

i) Particular solution on interval $\left[0, \frac{\pi}{3}\right]$ *are* $\left\{\frac{-\pi}{24} + \frac{\pi}{3}, \frac{-\pi}{8} + \frac{\pi}{3}\right\}$

$$= \left\{\frac{7\pi}{24}, \frac{5\pi}{24}\right\}$$

ii) General solution S.S $= \left\{\frac{5\pi}{24} + \frac{\pi k}{3}, \frac{7\pi}{24} + \frac{\pi k}{3}\right\}$.

i) $\cos x = \frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{6}$ *and* $x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

i) Particular solution $= \left\{\frac{\pi}{6}, \frac{11\pi}{6}\right\}$

ii) General solution $= \left\{\frac{\pi}{6} + 2\pi k, \frac{11\pi}{6} + 2\pi k\right\}$

j) $\cos(3x) = \frac{-\sqrt{3}}{2} \Rightarrow 3x = \pi - \frac{\pi}{3}$ *and* $3x = \pi + \frac{\pi}{3}$

$\Rightarrow x = \frac{2\pi}{9}$ *and* $x = \frac{4\pi}{9}$

Period $= \frac{2\pi}{3}$

i) Particular solution on interval $\left[0, \frac{2\pi}{3}\right]$ *are* $\left\{\frac{2\pi}{9}, \frac{4\pi}{9}\right\}$

ii) General solution S.S $= \left\{\frac{2\pi}{9} + \frac{2\pi k}{3}, \frac{4\pi}{9} + \frac{2\pi k}{3}\right\}$

k) $\tan x = \sqrt{3} \Rightarrow x = \frac{\pi}{3}$, Period, $P = \pi$

i) Particular solution = $\left\{\frac{\pi}{3}\right\}$

ii) General solution = $\left\{\frac{\pi}{3} + \pi k\right\}$

24. Find the general solution set for each of the following trigonometric equation.
   a) $2\sin^2 x + 3\cos x - 3 = 0$
   b) $2\cos x + \sin 2x = 0$
   c) $\sec^2(2x) - 2 = 0$

Solution: Here

a) $2\sin^2 x + 3\cos x - 3 = 0$

$\Rightarrow 2(1 - \cos^2 x) + 3\cos x - 3 = 0$

$\Rightarrow 2 - 2\cos^2 x + 3\cos x - 3 = 0$

$\Rightarrow -2\cos^2 x + 3\cos x - 1 = 0$

$\Rightarrow 2\cos^2 x - 3\cos x + 1 = 0$

Let $\cos x = u$

$\Rightarrow 2u^2 - 2u + 1 = 0$

$$\Rightarrow u = \frac{3 \pm \sqrt{9 - 4(2)}}{4} = \frac{3 \pm 1}{4}$$

$$\Rightarrow u = \frac{3+1}{4} = 1 \; and \; u = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$$

But $u = \cos x \Rightarrow \cos x = 1$ and $\cos x = \frac{1}{2}$

Equation **Solution**

$\cos x = 1 \Rightarrow x = 2\pi k$

$\cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3} + 2\pi k, \frac{5\pi}{3} + 2\pi k$

$$\therefore S.S = \left\{\frac{\pi}{3} + 2\pi k, \frac{5\pi}{3} + 2\pi k, 2\pi k\right\}$$

b) $2\cos x + \sin 2x = 0$

$\Rightarrow 2\cos x + 2\sin x \cos x = 0$

$\Rightarrow 2\cos x(1 + \sin x) = 0$

$\Rightarrow 2\cos x = 0$ or $\sin x + 1 = 0 \Rightarrow \cos x = 0$ or $\sin x = -1$

Equation	Solution
$\cos x = 0 \Rightarrow$	$x = \frac{\pi}{2} + 2\pi k$
$\sin x = -1 \Rightarrow$	$x = \frac{3}{2}\pi + 2\pi k$

$$\therefore S.S = \left\{\frac{\pi}{2} + 2\pi k, \frac{3}{2}\pi + 2\pi k\right\}$$

c) $\sec^2(2x) - 2 = 0 \Rightarrow (\sec(2x) - \sqrt{2})(\sec(2x) + \sqrt{2}) = 0$

$\Rightarrow \sec(2x) = \sqrt{2}$ or $\sec(2x) = -\sqrt{2}$

$\Rightarrow \cos 2x = \frac{1}{\sqrt{2}}$ or $\cos(2x) = \frac{-1}{\sqrt{2}}$

Equation	Solution for 2x	Solution for x
$\cos(2x) = \frac{1}{\sqrt{2}}$	$\Rightarrow 2x = \frac{\pi}{4} + 2\pi k \Rightarrow$	$x = \frac{\pi}{8} + \pi k$
	$\Rightarrow 2x = \frac{7\pi}{4} + 2\pi k \Rightarrow$	$x = \frac{7\pi}{8} + \pi k$
$\cos(2x) = -\frac{1}{\sqrt{2}}$	$\Rightarrow 2x = \frac{3\pi}{4} + 2\pi k \Rightarrow$	$x = \frac{3\pi}{8} + \pi k$
	$\Rightarrow 2x = \frac{5\pi}{4} + 2\pi k \Rightarrow$	$x = \frac{5\pi}{8} + \pi k$

$$\therefore S.S = \left\{\frac{\pi}{8} + \pi k, \frac{7\pi}{8} + \pi k, \frac{3\pi}{8} + \pi k, \frac{5\pi}{8} + \pi k\right\} = \left\{\frac{\pi}{8} + \frac{\pi k}{4}\right\}$$

25. Solve each of the following trigonometric equation

a) $\sin 2x = 2\sin x$ c) $\cos\left(\frac{\pi}{3}x - 2\right) = \frac{1}{2}$

b) $2\cos x + \sin 2x = 0$ d) $\sec\left(\frac{1}{3}x + \frac{\pi}{3}\right) = 2$ *and* $\tan x < 0$

e) $\tan\left(\frac{x}{2}\right) - 2\sin x = 0$

Solution: a) sin2x = 2sinx $\Rightarrow \frac{\sin 2x}{\sin x} = 2 \Rightarrow \frac{2\sin x\cos x}{\sin x} = 2$

$\Rightarrow$ 2cosx = 2 $\Rightarrow$ cosx = 1

$\therefore$ x = 2πk

b) 2cosx + sin2x = 0 $\Rightarrow$ 2cosx = -sin2x

$$\Rightarrow \frac{\sin 2x}{\cos x} = -2 \Rightarrow \frac{2\sin x\cos x}{\cos x} = -2$$

$$\Rightarrow \sin x = -1 \Rightarrow x = \frac{3}{2}\pi \;\therefore\; x = \frac{3}{2}\pi + 2\pi k$$

c) $\cos\left(\frac{\pi}{3}x - 2\right) = \frac{1}{2} \Rightarrow \frac{\pi}{3}x - 2 = \frac{\pi}{3} \Rightarrow \frac{\pi}{3}x = \frac{\pi}{3} + 2$

$\Rightarrow x = 1 + \frac{6}{\pi}$, since period, P = $\frac{2\pi}{\frac{\pi}{3}} = 6$

$\therefore$ General Solution = $\left\{1 + \frac{6}{\pi} + 6k\right\}$, k is integer

d) $\sec\left(\frac{1}{3}x + \frac{\pi}{3}\right) = 2 \; and \; \tan x < 0$. Period = $\frac{2\pi}{\frac{1}{3}} = 6\pi$

$\Rightarrow \cos\left\{\frac{1}{3}x + \frac{\pi}{3}\right\} = \frac{1}{2}$, since tan x < 0 and cosine value is positive, so lies on 4th quadrant

$$\Rightarrow \frac{1}{3}x + \frac{\pi}{3} = \frac{5\pi}{3} \Rightarrow \frac{1}{3}x = \frac{5\pi}{3} - \frac{\pi}{3} = \frac{4\pi}{3} \Rightarrow x = 4\pi$$

$\therefore$ General Solution = $\{4\pi + 6\pi k\}$

e) $\tan\left(\frac{x}{2}\right) - 2\sin x = 0 \Rightarrow \tan\left(\frac{x}{2}\right) - 2\left(\sin\left(\frac{x}{2} + \frac{x}{2}\right)\right) = 0$

$$\Rightarrow \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} - 2\left(2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\right) = 0$$

$$\Rightarrow \sin\left(\frac{x}{2}\right)\left(\frac{1}{\cos\left(\frac{x}{2}\right)} - 4\cos\left(\frac{x}{2}\right)\right) = 0 \Rightarrow \sin\left(\frac{x}{2}\right)$$

$$= 0,\ 1 - 4\cos^2\frac{x}{2} = 0$$

$\sin\left(\frac{x}{2}\right) = 0 \Rightarrow \frac{x}{2} = 0, \pi \ldots,$ x = 2π + 4πk, and $\cos\left(\frac{x}{2}\right) = \pm\frac{1}{2}$

$$\Rightarrow \frac{x}{2} = \frac{\pi}{3}, \frac{2\pi}{3}, \ldots x = \frac{2\pi}{3} + 4\pi k, \frac{4}{3}\pi + 4\pi k \ldots.$$

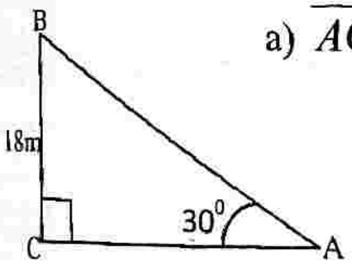
$$\therefore S.S = \left\{2\pi k, \frac{2}{3}\pi + 4\pi k, \frac{4}{3}\pi + 4\pi k\right\}$$

Solving triangles

I. Solving right triangle

26. For the right angled triangle given below. If $\overline{BC} = 18cm$ and $m(\angle A) = 30^0$ then how long is

a) $\overline{AC}$ b) $\overline{AB}$



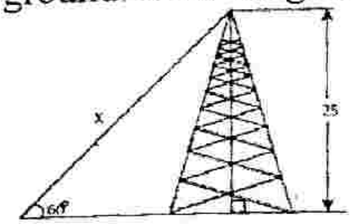
Solution: **To find** $\overline{AC}$: $\tan 30^0 = \frac{\overline{BC}}{\overline{AC}} = \frac{18}{\overline{AC}}$

$$\Rightarrow \overline{AC} = \frac{18cm}{\tan 30^0} = \frac{18}{\frac{1}{\sqrt{3}}} = 18\sqrt{3}cm$$

To find $\overline{AB}$: $\sin 30^0 = \dfrac{BC}{AB} = \dfrac{18}{AB}$

$$\Rightarrow \frac{1}{2} = \frac{18}{AB} \Rightarrow \overline{AB} = 36$$

27. A support cable for 25 meter tower is to make 60^0 angle with the ground. How long must the cable be.

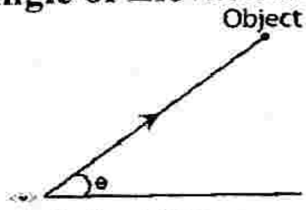


Solution:

$$\sin 60^0 = \frac{25}{x} \Rightarrow x = \frac{25}{\sin 60^0} = \frac{50}{\sqrt{3}} m.$$

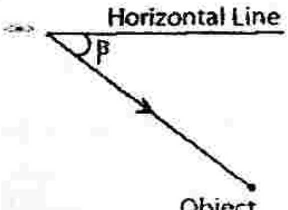
Angle of Elevation and Angle of Depression

Angle of Elevation: Looking up from a point.



The angle formed between the horizontal line and line of sight when we looking up is called **angle of elevation.**

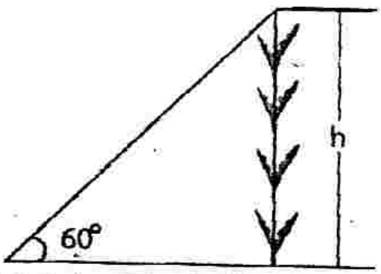
Angle of depression: Looking down from appoint.



The angle formed between the horizontal line and line of sight when we looking down is called angle of depression.

Using angle of elevation

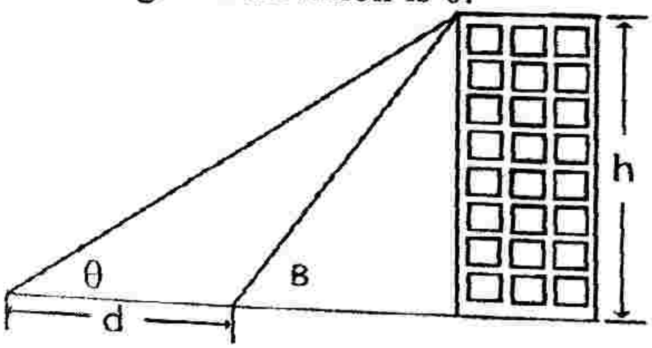
28. From a point 40meter from the base of kosso zaf tree, the angle of elevation to the top of tree is 60^0. Find the height of the tree.



Solution: $\tan 60^0 = \dfrac{h}{40m}$

$$\Rightarrow h = (40m)\tan 60^0 = (40)\sqrt{3} = 40\sqrt{3}m$$

Height of building When building top is looking from the point A, the angle of elevation is θ.



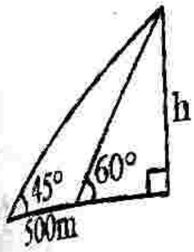
From a point B, which is closer to building, the angle of elevation increase to β and the distance between A to B is d, then the height of the building h given by

$$h = \frac{d}{\cot\theta - \cot\beta},$$

- h is height • d is distance between

29. The angle of elevation to an air plane from two point A and B on level ground are 45^0 and 60^0 respectively. The distance between A and B is 500 meter and the air plane is to the east of A to B in the same vertical plane. Find the altitude of the plane

Solution: $h = \dfrac{d}{\cot\theta - \cot\beta}$, $\theta = 45^0$, $\beta = 60^0$



$$\Rightarrow h = \frac{500m}{\cot 45^0 - \cot 60^0} = \frac{500m}{1 - \frac{1}{\sqrt{3}}} = \frac{500\sqrt{3}}{\sqrt{3}-1}m.$$

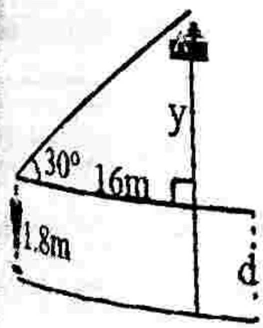
30. The angle of elevation from point A to the top of mountain is 30^0. From point B, 800m far there away, the angle of elevation is 45^0. Find the height of the mountain.

Solution: $h = \dfrac{d}{\cot\theta - \cot\beta}$, $\theta = 30^0$, $\beta = 45^0$

$$\Rightarrow h = \frac{800m}{\cot 30^0 - \cot 45^0} = \frac{800m}{\sqrt{3}-1} = \frac{800m}{\sqrt{3}-1} = 400(\sqrt{3}+1)m$$

31. A man 1.8m tall observes the angle of elevation of a tree to be 30^0. If he is standing 16m from the tree, find the height of the tree.

Solution: • $\tan 30^0 = \dfrac{y}{16} \Rightarrow y = 16\tan 30^0 = \dfrac{16}{\sqrt{3}}$



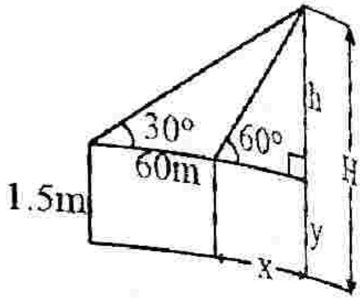
- height of tree, $h = y + d = \left(\dfrac{16}{\sqrt{3}} + 1.8\right)m$

d = 1.8 is height of man

32. Find the height of tower if the survey or measures the angle of elevation of its top changes from 30^0 to 60^0 at the survey or advances 60m toward its base, and that the instrument is 1.5m above ground level.

Solution:

$$\Rightarrow h = \frac{60m}{\cot 30^0 - \cot 60^0} = \frac{60m}{\sqrt{3} - \frac{1}{\sqrt{3}}}$$



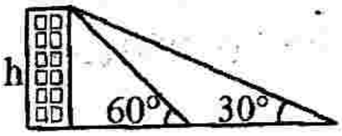
$$= \frac{60\sqrt{3}m}{2} = 30\sqrt{3}$$

$H = h + y = (30\sqrt{3} + 1.5)m \leftarrow$ Height of tower

41. Find the height of a building if the angle of elevation to the top of the building changes from 30^0 to 60^0 as the observer moves a distance of 100 meter toward the building.

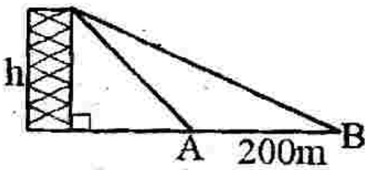
Solution:

$$\Rightarrow h = \frac{100m}{\cot 30^0 - \cot 60^0} = \frac{100m}{\sqrt{3} - \frac{1}{\sqrt{3}}} = \frac{100\sqrt{3}m}{3-1}$$



$\therefore h = 50\sqrt{3}m$

42. From the top of a tower T, the angle of depression to point B is 30^0 and the angle of depression to point A is 45^0. Point A and B are 200 meter a part.



By alternate interior angle

$\Rightarrow \angle B = 30^0$, and $\angle A = 45^0$

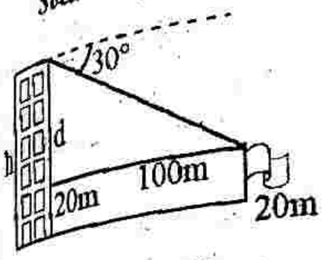
$$\Rightarrow h = \frac{200m}{\cot 30^0 - \cot 45^0}$$

43. The angle of depression of the top of the flag pole from the top of a building that is 100 meters away from the flag pole is 30^0. If the flag pole is 20 meters tall, then what is the height of the building ….. (UEE)

A. $20\left(\dfrac{3+5\sqrt{3}}{3}\right)meter$ C. $20(5\sqrt{3}-1)meter$

B. $20(1+5\sqrt{3})meter$ D. $100\sqrt{3}meter$

Solution: $\Rightarrow \tan 30^0 = \dfrac{d}{100} \Rightarrow d = 100(\tan 30^0)$



$$\Rightarrow d = (100)\left(\frac{1}{\sqrt{3}}\right) = \frac{100\sqrt{3}}{3}m$$

$$h = d + 20m$$

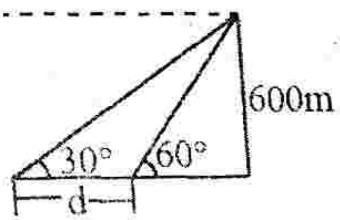
$$\Rightarrow h = \frac{100\sqrt{3}}{3} + 20m = 20\left(\frac{3+5\sqrt{3}}{3}\right)$$

Answer: A

44. If from the top of hill 600m above the sea level the angle of depressions of two boats in line with the observer were found to be 30^0 and 60^0 then the distance between the two boats was…

A. $400\sqrt{3}m$ B. 400m C. $300\sqrt{3}m$ D. $200\sqrt{3}m$

Solution: Use $h = \dfrac{d}{\cot\theta - \cot\beta}$



$h = 600m,\ \theta = 30^0,\ \beta = 60^0,\ d = ?$

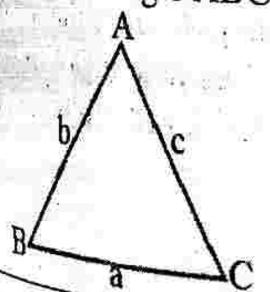
$$\Rightarrow 600m = \frac{d}{\cot 30^0 - \cot 60^0}$$

$$\Rightarrow d = 600m\ (\cot 30^0 - \cot 60^0)$$

$$\Rightarrow d = 600\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) = 600\left(\frac{3-1}{\sqrt{3}}\right) = \frac{1200}{\sqrt{3}} = \frac{1200\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}} = 400\sqrt{3}$$

Law of sines

* For any triangle ABC



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

- The length of the sides of a triangle are proportional to the sines of the measure of the opposite angle.

Note: The law of sine we apply when the following two cases satisfy

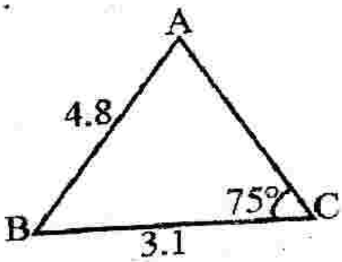
I. **SSA** ← two side and an angle opposite one of them.

II. **AAS or ASA** ← two angle and any sides

45. In ΔABC, m(c) = 75^0, $\overline{BC} = 3.1unit$ and $\overline{AB} = 4.8units$, If $\cos 15^0 = 0.96$ then what is the value of sin A? ... UEE.

A. 0.62 B. 0.48 C. 0.31 D. 0.71

Solution:



$$\frac{4.8}{\sin 75^0} = \frac{3.1}{\sin A} \Rightarrow \sin A = \left(\frac{\sin 75^0}{4.8}\right)(3.1)$$

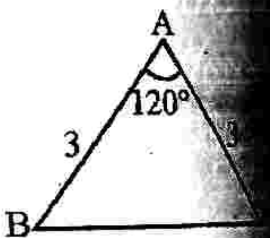
$\Rightarrow \sin 75^0 = \cos 15^0$ ← complementary relation

$$\Rightarrow \sin A = \left(\frac{\sin 15^0}{4.8}\right)(3.1) = \left(\frac{0.96}{4.8}\right)(3.1) = 0.62$$

46. In ΔABC, If $\overline{AB} = \overline{AC} = 3$ units and $m(\angle A) = 120^0$ then how long is $\overline{BC}$?

A. 3 unit C. $3\sqrt{3}\,units$

B. $3\sqrt{2}\,units$ D. $3\sqrt{3}\,units$



Solution:

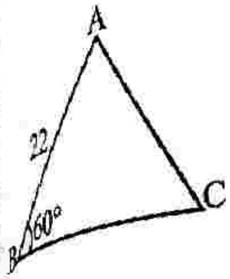
$$\frac{BC}{\sin 12^0} = \frac{3}{\sin 30^0} \Rightarrow BC = \frac{3}{\sin 30^0} \cdot \sin 120^0$$

$$\Rightarrow BC = \frac{3}{\frac{1}{2}} \cdot \frac{\sqrt{3}}{2} = \frac{6\sqrt{3}}{2} = 3\sqrt{3}$$ **Answer:**

47. In ΔABC, m(∠A) = 75^0, m(∠B) = 60^0, and AB = 22 units what is the length of $\overline{AC}$

A. $11\sqrt{3}\,units$ C. $\frac{11}{2}\sqrt{3}\,units$

B. $11\sqrt{6}$ *units* D. $11\sqrt{2}$ *units*



Solution: To find $(\angle C)$

$\Rightarrow 180^0 = (75^0 + 60^0 - m(C))$

$\Rightarrow m(C) = 180^0 - (75^0 + 60^0) = 45^0$

Thus,

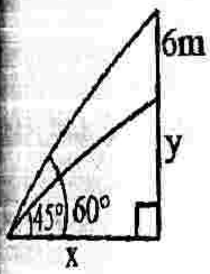
$$\frac{AC}{\sin 60^0} = \frac{22}{\sin 45^0} \Rightarrow \overline{AC} = \left(\frac{22}{\frac{1}{\sqrt{2}}}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$\therefore \overline{AC} = 11\sqrt{6}$ *units*

Answer: B

48. From the top of building, the angle of depression to the base of tree on the ground 60^0. If the angle of depression from a window 6 meter below the top of building is 45^0. Find the height of the building.

Solution: $\tan 60^0 = \dfrac{6+y}{x} \Rightarrow \sqrt{3} = \dfrac{6+y}{x} \Rightarrow \sqrt{3}x = 6 + y$



$$\tan 45^0 = \frac{y}{x} \Rightarrow y = x\tan 45^0 \Rightarrow y = x$$

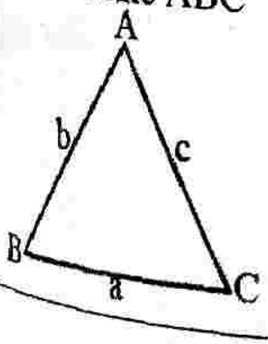
Solving together, $\begin{cases} \sqrt{3}x = 6 + y \\ y = x \end{cases}$

$$\Rightarrow \sqrt{3}x = 6 + x \Rightarrow \sqrt{3}x - x = 6 \Rightarrow y = \frac{6}{\sqrt{3}-1} = \frac{6(\sqrt{3}+1)}{2} = 3(\sqrt{3}+1)$$

$\Rightarrow$ Height, of the building, is $h = 6+y = 6 + 3\left(\sqrt{3}+1\right) = 9 + 3\sqrt{3}$

The law of cosines

For any triangle ABC



- $a^2 = b^2 + c^2 - 2bc\cos A$
- $b^2 = a^2 + c^2 - 2ac\cos B$
- $c^2 = a^2 + b^2 - 2ab\cos C$

The law of cosines can be apply to solve triangle for which these data are given

I. SSS ⟵ length of three side

II. SAS ⟵ two side and the included angle

48. Use the law of cosines to solve ΔABC if

a) $a = 5,\ b = 4\ \ m(\angle C) = 30^0$

b) $a = 2,\ c = 3,\ m(\angle B) = 60^0$

c) $b = 6,\ c = 8,\ m(\angle A) = 120^0$

Solution: a) $c^2 = a^2 + b^2 - 2ab(\cos c)$

$$\Rightarrow c^2 = 5^2 + 4^2 - 2(5)(4)\cos 30^0$$

$$\Rightarrow c = \sqrt{25 + 16 - 10\left(\frac{\sqrt{3}}{2}\right)} = \sqrt{41 - 5\sqrt{3}}$$

b) $b^2 = a^2 + c^2 - 2ac(\cos B)$

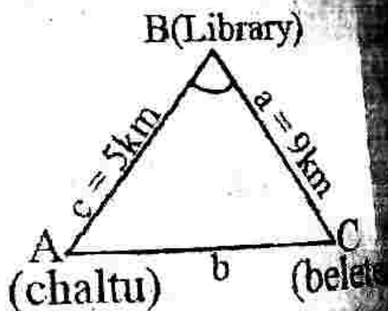
$$\Rightarrow b = \sqrt{4 + 9 - 2(2)(3)\left(\frac{1}{2}\right)} = \sqrt{7}$$

c) $$a = \sqrt{6^2 + 8^2 - 2(8)(6)\left(\frac{-1}{2}\right)} = 2\sqrt{37}$$

49. In the town where Chaltu and Beletech live, the public library is 5km away from Chaltu's home and 9km away from Beletech home as shown in the figure. What is the distance between Chaltu's home and Beletech home?

A. $\sqrt{61}km$ C. $106 + 45\sqrt{3}km$

B. $\sqrt{151}km$ D. $106 - 45\sqrt{3}km$



Solution:

$$\Rightarrow b^2 = a^2 + c^2 - 2ac\cos B$$

$$= 9^2 + 5^2 - 2(5)(9)\cos(120^0) = 81 + 25 + 45 = 151$$

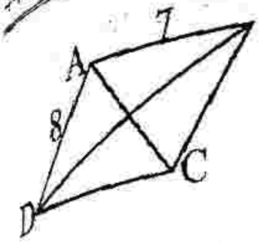
$$\therefore b = \sqrt{151}$$

Answer: B

50. ABCD is parallelogram with $\overline{AB} = 7cm$ and $\overline{AD} = 8cm$, $m(\angle DAB) = 60^0$ what is the length of the diagonal $\overline{AC}$

A. 13cm B. $\sqrt{113}cm$ C. 15cm D. $\sqrt{57}cm$

Solution: $m(\angle A) = 60^0 \Rightarrow m(\angle D) = 120^0$



$$\Rightarrow \overline{AC}^2 = (AD)^2 + (DC)^2 - 2(AD)(DC)\cos D$$
$$= 8^2 + 7^2 - 2(8)(7)\cos(120^0)$$
$$\Rightarrow \overline{AC}^2 = 64 + 49 + 59 = 169$$
$$\therefore AC = \sqrt{169} = 13$$

Answer: A

51. A rhombus has side of length 6cm and the angle at one vertex is 60^0. Find the length of the diagonal.

Solution: $D^2 = 6^2 + 6^2 - 2(6)(6)(\cos 60^0) = 72 - 36 = 36$

$$\Rightarrow D = \sqrt{36} = 6$$

52. In ΔDEF, $d = 4$, $e = 3$, $\cos F = \frac{5}{6}$. Find f

Solution: $f^2 = e^2 + d^2 - 2ed\cos F$

$$\Rightarrow f^2 = 3^2 + 4^2 - 2(3)(4)\left(\frac{5}{6}\right) = 5$$

53. In ΔPQR, $P = 5$, $q = 10$, $\cos R = 0$, Find r

Solution: $r^2 = p^2 + q^2 - 2pq\cos R \Rightarrow r^2 = 5^2 + 10^2 - 2(5)(10)(0)$

$$\Rightarrow r^2 = 125 \Rightarrow r = \sqrt{125} = 5\sqrt{5}$$

54. Find the cosine of each angle in the given triangle.

A. In ΔABC, $a = 2$, $b = 3$, $c = 4$

B. In ΔDEF, $a = 12$, $b = 5$, $c = 13$

Solution: a) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{3^2 + 4^2 - 2^2}{2(3)(4)} = \dfrac{21}{24}$

* $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac} = \dfrac{2^2 + 4^2 - 3^2}{2(2)(4)} = \dfrac{11}{16}$

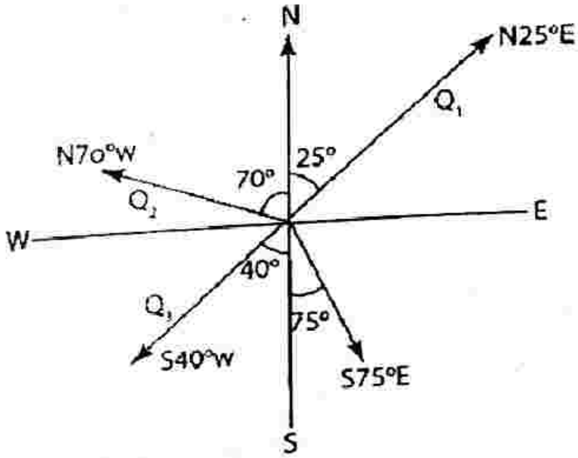
* $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab} = \dfrac{2^2 + 3^2 - 4^2}{2(2)(3)} = \dfrac{-3}{12} = \dfrac{-1}{4}$

$\therefore \Delta ABC$ is an obtuse angle triangle

c) Exercise left for you.

Navigation

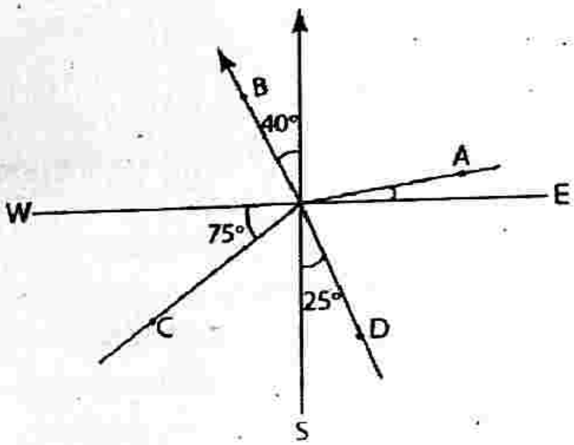
In certain navigation, surveying problems, **the direction**, or bearing.



* The angular direction used to locate one object in relation to another object is called bearing.
* **Bearing** is expressed interms of the acute angle formed by a **north-south line** and the line of direction.

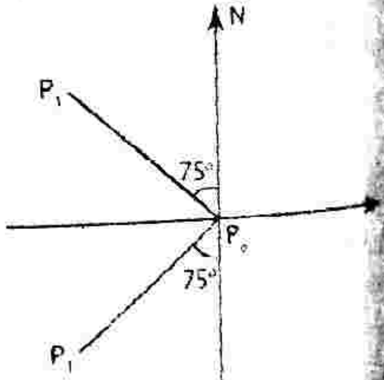
55. Find the bearing from P to each of the points A, B, C and D

Solution:



A. $N80^0E$
B. $N40^0W$
C. $S15^0W$
D. $S25^0E$

56. A car traveling from a parking lot P_0 travels 20kms in the direction N 75^0W to parking 10t P_1. Another car starting from P_0 travels 30kms in a direction S 75^0W to another lot P_2. which of the following is the distance between P_1 and P_2.



A. $10\sqrt{13-6\sqrt{3}}\,km$

B. $10\sqrt{10}km$

C. $10\sqrt{7}km$

D. $10\sqrt{19}km$

Solution:

The angle $(\angle P_1P_0P_2) = 180^0 - (75^0 + 75^0) = 30^0$

$\Delta P_1P_0P_2$, $\overline{P_1P_0} = 20km$, $\overline{P_0P_2} = 30km$

From cosine law,

$$(P_1P_1)^2 = (P_0P_1)^2 + (P_0P_2)^2 - 2(P_0P_1)(P_0P_2)\cos 30^0$$

$$= 20^2 + 30^2 - 2(20)(30)\left(\frac{\sqrt{3}}{2}\right)$$

$$\therefore \overline{P_1P_2} = \sqrt{1300 - 600\sqrt{3}} = 10\sqrt{13 - 6\sqrt{3}}$$

Answer: A

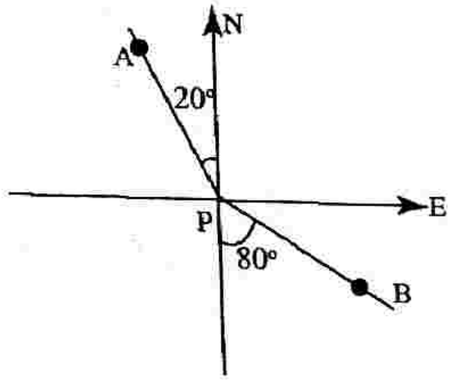
57. Two motor boats A and B leave a port P at the same time. A travels at constant speed of 20km/hr in the direction N20^0W and B travels speed of 15km/hr in the direction S 80^0E.
How far a part are the two boats after 2 hours …..UEE

A. $10\sqrt{31}km$

B. $5\sqrt{37}km$

C. $5\sqrt{13}k$

D. $10\sqrt{37}km$



Solution: In ΔAPB, angle,

$$m(\angle APB) = 20^0 + 90^0 + 10^0 = 120^0$$

$$\Rightarrow \overline{AP} = \left(20\frac{km}{hr}\right)(2hr) = 40km$$

$$\Rightarrow \overline{PB} = \left(15\frac{km}{hr}\right)(2hr) = 30km$$

$$\Rightarrow (\overline{AB})^2 = (\overline{AP})^2 + (\overline{PB})^2 - 2(\overline{AP})(\overline{BP})\cos P$$

$$= (40)^2 + (30)^2 - 2(40)(30)(\cos(120^0))$$

$$\Rightarrow \overline{AB} = \sqrt{1600 + 900 - (2400)\left(\frac{-1}{2}\right)}$$

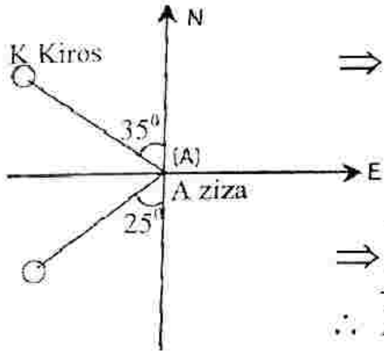
$$= \sqrt{2500 + 1200} = \sqrt{3700} = 10\sqrt{37}$$

Answer: D

58. A ziza located her two class mates, kiros at a distance of 20 meters N35^0W and Guta at 30 meters S 25^0 W away from her position. How far a part are kiros and Guta?

A. 40m B. $10\sqrt{19}m$ C. $10\sqrt{7}m$ D. 25m

Solution: In ΔAKG, the angle formed



$$m(\angle AKG) = 180^0 - (35^0 + 25^0) = 120^0$$

$$\Rightarrow \overline{KG}^2 = \overline{AG}^2 + \overline{AK}^2 - 2(\overline{AG})(\overline{AK})\cos 120$$

$$= 30^2 + 20^2 - 2(30)(20)\left(\frac{-1}{2}\right)$$

$$\Rightarrow \overline{KG} = \sqrt{900 + 400 + 600}$$

$$\therefore \overline{KG} = \sqrt{1900} = 10\sqrt{19},$$

Answer: B

Simple Harmonic Motion

The periodic nature of the trigonometric function is useful for describing the motion of a point on an object that vibrates, oscillates or moved by wave motion.

Examples of periodic quanties are

* Pendulum, spring, periodic fluctuation in the population of specias. Many of these quantities described by sine or cosine function, and is called simple harmonic motion.

A point that moves on a coordinate line is said to be in simple harmonic motion if its distance from the origin at time t is given by either.

d(t) = a sinwt or d(t) = acot wt

* amplitude, $|a|$
* period, $T = \dfrac{2\pi}{w}$
* frequency $f = \dfrac{1}{T} = \dfrac{w}{2\pi}$

59. Given the equation for simple harmonic motion

$$d = 6\cos\frac{3\pi t}{4}$$

Find

a) the maximum displacement
b) the frequency
c) the value of d when t = 4
d) the least positive value of t for which d = 0

Solution:

a) maximum displacement, $|a| = 6$

b) frequency, $f = \frac{w}{2\pi} = \frac{\frac{3\pi}{4}}{2\pi} = \frac{3}{8}$

c) $d(4) = 6\cos\left(\frac{3\pi}{4}(4)\right) = 6\cos 3\pi = 6(-1) = -6$

d) To find the least positive value of t for which d = 0, solve the equation $d = 6\cos\frac{3\pi}{4}t = 0$ to obtain

$$\frac{3\pi}{4}t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \Rightarrow t = \frac{2}{3}, 2, \frac{10}{3}$$

So the least positive value of t is $t = \frac{2}{3}$

60. The monthly sales (in thousands of unit) of a seasonal product in (ECX) is modeled by

$$S(t) = 56 + 9\sin\left(\frac{3\pi t}{5}\right); \ 0 \le t \le 12, \text{ t is time in month}$$

Determine:

a) the initial sale
b) the largest and smallest sale
c) the first time in which the sale reaches 51,500 Birr
d) the sale after one year

Solution:

a) The initial sale when t = 0, S(0) = 56
$\Rightarrow$ The initial sale, is 56,000 birr

b) The sale is largest if $\sin\left(\frac{3\pi t}{5}\right) = 1$ and

It is smallest if $\sin\left(\frac{3\pi t}{5}\right) = -1$

$\Rightarrow$ maximum sale = (56 + 9(1))1000 = 65,000 Birr
$\Rightarrow$ minimum sale = (56 + 9(-1))1000 = 47,000 Birr

c) $S(t) \times 1000 = 51{,}500 \Rightarrow 56 + 9\sin\left(\frac{3\pi t}{5}\right) = 51.5$

$$\Rightarrow \sin\left(\frac{3\pi t}{5}\right) = \frac{51.5 - 56}{9} = \frac{-4.5}{9} = -\frac{1}{2}$$

$$\Rightarrow \frac{3\pi t}{5} = \frac{7\pi}{6} \Rightarrow t = \frac{7\pi}{6} \cdot \frac{5}{3\pi} = \frac{35}{18} = 1\frac{17}{18}\ months$$

d) $s(12) = 56 + 9\sin\left(\frac{3\pi(12)}{5}\right) = 56 + 9\sin\left(\frac{-4\pi}{5}\right)$

$$= 56 + 9\sin\left(\frac{-4\pi}{5}\right) = 56 - 9(0.5878) = 50.7098$$

After one year the sale reaches 50,709 birr.

I. Sum and Difference formulas

- $\sin(A + B) = \sin A \cos B + \cos A \sin B.$
- $\sin(A - B) = \sin A \cos B - \cos A \sin B.$
- $\cos(A + B) = \cos A \cos B - \sin A \sin B.$
- $\cos(A - B) = \cos A \cos B + \sin A \sin B.$
- $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
- $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

II. Double angle formula

- $\text{Sin}2A = \sin(A + A) = 2\sin A \cos A$
- $\cos 2A = \cos(A + A) = \cos^2 A - \sin^2 A$
- $\tan 2A = \tan(A + A) = \dfrac{2\tan A}{1 - \tan^2 A}$

III. Half angle formula

- $\sin^2\left(\dfrac{A}{2}\right) = \dfrac{1 - \cos A}{2} \Rightarrow \sin\left(\dfrac{A}{2}\right) = \pm\sqrt{\dfrac{1 - \cos A}{2}}$
- $\cos^2\left(\dfrac{A}{2}\right) = \dfrac{1 + \cos A}{2} \Rightarrow \cos\left(\dfrac{A}{2}\right) = \pm\sqrt{\dfrac{1 + \cos A}{2}}$

- $\tan^2\left(\frac{A}{2}\right)=\frac{1-\cos A}{1+\cos A} \Rightarrow \tan\left(\frac{A}{2}\right)=\pm\sqrt{\frac{1-\cos A}{1+\cos A}}=\frac{\sin A}{1+\cos A}$

Find the exact value:

a) $\sin 75^0$ d) $\sin 105^0$ g) $\cos\frac{\pi}{12}$ j) $\sec\frac{11}{12}\pi$

b) $\sin 15^0$ e) $\tan (15^0)$ h) $\tan\frac{17\pi}{12}$

c) $\cos 165^0$ f) $\tan 195^0$ i) $\cot\frac{19\pi}{12}$

Solution:

a) $\sin75^0=\sin(45^0+30^0)=\sin45^0\cos 30^0+\cos 45^0\sin30^0$

$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}$$

b) $\sin15^0=\sin(45^0-30^0)=\sin45\cos 30^0-\cos 45^0\sin30^0$

$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}$$

c) $\cos165^0=\cos(180^0-150^0)=\cos180^0\cos15^0+\sin 180^0\sin 15^0$

$$=-\cos15^0$$

$\Rightarrow -\cos15^0=-\cos(45^0-30^0)$

$$=-(\cos 45\cos 30+\sin 45^0\sin 30^0)$$

$$=-\left(\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}\right)=\left(-\frac{\sqrt{6}+\sqrt{2}}{4}\right)$$

$$\Rightarrow \cos165^0=\cos(180^0-15^0)=-\cos15^0=\frac{-\sqrt{6}-\sqrt{2}}{4}$$

d) $\sin 105^0=\sin(60^0+45^0)=\sin 60^0\cos 45^0+\cos60^0\sin 45^0=\frac{\sqrt{6}+\sqrt{2}}{4}$

Note: $\cos105^0=\frac{\sqrt{2}-\sqrt{6}}{4}$

e) $\tan 150^0 = \tan(180^0 - 30^0) = \dfrac{\tan 180^0 - \tan 30^0}{1 + \tan 180^0 - \tan 30^0}$

f) $\tan 195^0 = \tan(150^0 + 45^0)$

$$= \frac{\tan 150^0 + \tan 45^0}{1 - \tan 150^0 \cdot \tan 45^0} = \frac{-\frac{\sqrt{3}}{3} + 1}{1 + \frac{\sqrt{3}}{3}} = 2 - \sqrt{3}$$

g) $\cos\left(\dfrac{\pi}{12}\right) = \cos\left(\dfrac{\pi}{3} - \dfrac{\pi}{4}\right) = \cos\dfrac{\pi}{3}\cos\dfrac{\pi}{4} + \sin\dfrac{\pi}{3}\sin\dfrac{\pi}{4}$

$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

h) $\tan\left(\dfrac{17\pi}{12}\right) = \tan\left(\dfrac{7\pi}{12} - 2\pi\right) = \tan\left(\dfrac{-7\pi}{12}\right) = -\tan\left(\dfrac{\pi}{3} + \dfrac{\pi}{4}\right)$

$$\Rightarrow \frac{-\left(\tan\frac{\pi}{3} + \tan\frac{\pi}{4}\right)}{1 - \tan\frac{\pi}{3}\tan\frac{\pi}{4}} = -\left(\frac{\sqrt{3} + 1}{1 - \sqrt{3}}\right) = -\sqrt{3} - 2$$

i) $\cot\left(\dfrac{19}{12}\pi\right) = \cot\left(\dfrac{19\pi}{12} - \pi\right) = \cos\left(\dfrac{7\pi}{12}\right) = \dfrac{1}{\tan\left(\dfrac{7\pi}{12}\right)} = 2 - \sqrt{3}$

j) $\sec\left(\dfrac{11\pi}{12}\right) = -\sec\left(\pi - \dfrac{11\pi}{12}\right) = -\sec\dfrac{\pi}{12} = -\sec\left(\dfrac{\pi}{3} - \dfrac{\pi}{4}\right)$

$$= \frac{-1}{\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)} = \frac{-1}{\cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}} = \frac{-4}{\sqrt{6} + \sqrt{2}} = \sqrt{2} - \sqrt{6}$$

Solved problem

63. Simplify each of the following expressions

a) $\sin(x + \pi)$ d) $\csc\left(x + \dfrac{\pi}{2}\right)$ g) $\cos\left(x - \dfrac{\pi}{2}\right)$

b) $\sin\left(x+\frac{\pi}{2}\right)$ e) $\cos\left(x+\frac{\pi}{2}\right)$ h) $\cos(x + k\pi)$, $k\in Z$

c) $\tan\left(x+\frac{\pi}{2}\right)$ f) $\sec\left(x-\frac{\pi}{2}\right)$ i) $\sin(x + k\pi)$

Solution:

a) $\sin(x+\pi) = \sin x \cos\pi + \cos x \sin\pi = -\sin x$

b) $\sin\left(x+\frac{\pi}{2}\right) = \sin x \cos\frac{\pi}{2} + \cos x \sin\frac{\pi}{2} = 0 + \cos x = \cos x$

c) $\tan\left(x+\frac{\pi}{2}\right) = \dfrac{\sin\left(x+\frac{\pi}{2}\right)}{\cos\left(x+\frac{\pi}{2}\right)} = \dfrac{\cos x}{\cos x\cos\frac{\pi}{2} - \sin x \sin\frac{\pi}{2}} = \dfrac{\cos x}{-\sin x} = -\cot x$

d) $\csc\left(x+\frac{\pi}{2}\right) = \dfrac{1}{\sin\left(x+\frac{\pi}{2}\right)} = \dfrac{1}{\cos x} = \sec x$

e) $\cos\left(x+\frac{\pi}{2}\right) = \cos x \cos\frac{\pi}{2} - \sin x \sin\frac{\pi}{2} = 0 - \sin x = -\sin x.$

f) $\sec\left(x-\frac{\pi}{2}\right) = \dfrac{1}{\cos\left(x-\frac{\pi}{2}\right)} = \dfrac{1}{\sin x} = \csc x$

g) $\cos\left(x-\frac{\pi}{2}\right) = \cos x \cos\frac{\pi}{2} + \sin x \sin\frac{\pi}{2} = \sin x.$

h) $\cos(x + k\pi) = \cos x \cos\pi k - \sin x \sin\pi k = (-1)k \cos x$, k is integer

i) $\sin(x + k\pi) = \sin x \cos\pi k + \cos x \sin\pi k = (-1)^k \sin x.$

63. Simplify

a) $\dfrac{\cos 48^0 \sin 12^0 + \sin 48^0 \cos 12^0}{\cos 72^0 \cos 12^0 + \sin 72^0 \sin 12^0}$

b) $\dfrac{\tan 189^0 - \tan 144^0}{1 + 189^0 \tan 144^0}$

c) $\cos 130^0 \cos 50^0 - \sin 130^0 \sin 50^0$

Solution:

a) $\Rightarrow \dfrac{\sin(48^0+12^0)}{\cos(72-12)} = \dfrac{\sin 60^0}{\cos 60} = \tan 60^0 = \sqrt{3}$

b) $\tan(189^0 - 144^0) = \tan 45^0 = 1$

c) $\cos(130^0 + 50^0) = \cos 180^0 = -1$

Sum – to – product formula

- $\sin x + \sin y = 2\sin\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$
- $\sin x - \sin y = 2\cos\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$
- $\cos x + \cos y = 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$
- $\cos x - \cos y = -2\sin\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$

64. Simplify: a) $\dfrac{\sin 2x + \sin 4x}{\cos 2x - \cos 4x} = \dfrac{2\sin\left(\dfrac{4x+2x}{2}\right)\cos\left(\dfrac{2x-4x}{2}\right)}{-2\sin\left(\dfrac{2x+4x}{2}\right)\sin\left(\dfrac{2x-4x}{2}\right)}$

$$\Rightarrow \frac{2\sin 3x\cos x}{-2\sin 3x\sin(-x)} = \frac{-\cos x}{\sin(-x)} = \frac{\cos x}{\sin x} = \cot x.$$

b) $\cos 195^0 + \cos 105^0 = 2\cos\left(\dfrac{195^0+105^0}{2}\right)\cos\left(\dfrac{195^0-105^0}{2}\right) = 2\cos 150^0\cos 45^0$

$$\Rightarrow 2\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{-\sqrt{6}}{2}$$

64) Find the exact value

a) $\tan^{-1}(-\sqrt{3})$ c) $\sin\left(\cos^{-1}\left(-\dfrac{1}{2}\right)\right)$

b) $\sec^{-1}(-2)$ d) $\sin^{-1}\left(\sin\dfrac{5\pi}{4}\right)$

Solution:

a) $\tan^{-1}\left(-\sqrt{3}\right) = -\tan^{-1}\left(\sqrt{3}\right) = \dfrac{-\pi}{3}$

b) $\sec^{-1}(-2) = \pi - \sec^{-1}(2) = \pi - \cos^{-1}\left(\dfrac{1}{2}\right) = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$

c) $\sin\left(\cos^{-1}\left(-\frac{1}{2}\right)\right) = \sin$

$\left(\pi - \cos^{-1}\left(\dfrac{1}{2}\right)\right) = \sin\left(\pi - \dfrac{\pi}{3}\right) = \sin\dfrac{2\pi}{3} = \dfrac{\sqrt{3}}{2}$

d) $\sin^{-1}\left(\sin\dfrac{5\pi}{4}\right) = \sin^{-1}$

$\left(-\sin\left(\dfrac{5\pi}{4} - \pi\right)\right) = \sin^{-1}\left(\dfrac{-\sqrt{2}}{2}\right) = \dfrac{-\pi}{4}$

65) If $\sin\theta - \cos\theta = ½$ then what is the value of $\sin 2\theta$ . . . UEE

A. $\dfrac{3}{8}$ B. $\dfrac{3}{4}$ C. $\dfrac{5}{8}$ D. $\dfrac{5}{4}$

Solution:

$\Rightarrow (\sin\theta - \cos\theta)^2 = ¼ \Rightarrow \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta = ¼$

$\Rightarrow 1 - 2\sin\theta\cos\theta = ¼ \Rightarrow 2\sin\theta\cos\theta = 3/8$

$\Rightarrow 2\sin\theta\cos\theta = \sin 2\theta = 3/8$

Answer: A

66) $\cos(\sin^{-1}(x))$ is equal to …UEE

A. $\sqrt{x^2+1}$ C. $\dfrac{1}{1-x^2}$

B. $\sqrt{1-x^2}$ D. $\dfrac{1}{\sqrt{x^2+1}}$

Solution: Let $\sin^{-1} x = \theta \Rightarrow \sin\theta = x$, $\cos\theta = \sqrt{1-\sin^2\theta}$

$\Rightarrow \cos(\sin^{-1} x) = \cos\theta = \sqrt{1-x^2}$,

Answer: B

67) If sin 37 = 0.6 and sin 53^0 = 0.8, then cos 16^0 is . . . UEE

A. 0.02 B. 0.90 C. 0.48 D. 0.96

Solution: $\cos 16^0 = \cos(53-37) = \cos(53)\cos(37) + \sin(53)\sin(37)$

$\Rightarrow \cos 16^0 = \sin 37 \sin 53 + \sin 53 \sin 37$

$= 2(\sin 37 \sin 53) = 2(0.6)(0.8) = 0.96$

68) If $\cos\theta = \frac{-5}{13}$ $for \frac{\pi}{2} < \theta < \pi$, then $\sin 2\theta$ is equal to...

EHEECE

A. $\frac{120}{169}$ B. $\frac{-120}{240}$ C. $\frac{169}{240}$ D. $\frac{10}{13}$

Solution:

$$\text{If } \cos\theta = \frac{-5}{13} \Rightarrow \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{25}{169}\right.} = \frac{12}{13}$$

$$\Rightarrow \sin 2\theta = \sin(\theta+\theta) = 2\sin\theta\cos\theta = 2\left(\frac{12}{13}\right)\left(\frac{-5}{13}\right) = \frac{-120}{169}$$

Answer: B

69) Given $\Delta ABC \overline{AB} = 30\ unit$, $\overline{AC} = 22$ units and $\overline{BC} = 18$ units then cos B is equal to :

A. $\frac{37}{54}$ B. $\frac{54}{37}$ C. $\frac{3}{5}$ D. $\frac{11}{15}$

Solution: Let, $\overline{BC} = a = 18$, $\overline{AC} = b = 22$, and $\overline{AB} = c = 30$

$$\Rightarrow \cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{18^2+30^2-22^2}{2(18)(30)} = \frac{324+900-484}{1080} =$$

70) In ΔABC, m(<A) = 30^0, m(<B) = 45^0 and BC = 12 units Then how long is $\overline{AC}$...

A. $12\sqrt{3}$ unit B. $4\sqrt{6}$ C. $6\sqrt{2}$ unit D. $12\sqrt{2}$ unit

Solution:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{12}{\sin 30^0} = \frac{AC}{\sin 45^0} \Rightarrow AC = \frac{12}{\frac{1}{2}} \cdot \frac{\sqrt{2}}{2} = 12\sqrt{2}$$

71) If $\theta = \arc\cos\left(\frac{1}{2}\right)$, what is the value of $\tan\theta$ . . . UEE 2002.

A. $\frac{1}{\sqrt{3}}$ B. $\sqrt{3}$ C. $\frac{\sqrt{3}}{2}$ D. 1

Solution: $\theta = \cos^{-1}\left(\frac{1}{2}\right) \Rightarrow \cos\theta = \frac{1}{2}$,

$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{4}}$

$$\Rightarrow \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3},$$

Answer: B

72) If $x = \arc\sin\left(\frac{3}{4}\right)$, the value of $\cos(2x)$. . . UEE

A. $\frac{3}{4}$ B. $\frac{-3}{4}$ C. $\frac{-1}{8}$ D. $\frac{1}{8}$

Solution: $x = \sin^{-1}\left(\frac{3}{4}\right) \Rightarrow \sin x = \frac{3}{4} \Rightarrow \cos x = \sqrt{1-\sin^2 x}$

$$= \sqrt{1-\frac{9}{16}} = \frac{\sqrt{7}}{4}$$

$$\Rightarrow \cos 2x = \cos(x+x) = \cos^2 x - \sin^2 x = \frac{7}{16} - \frac{9}{10} = \frac{-2}{16} = \frac{-1}{8}$$

Answer: C

73) $\text{Sec}\left(\tan^{-1}\left(-\sqrt{3}\right)\right)$ is equal to:

A. $\frac{-3\sqrt{3}}{2}$ B. -2 C. $\frac{3\sqrt{3}}{2}$ D. 2

Solution: $\sec\left(\tan^{-1}\left(-\sqrt{3}\right)\right) = \sec\left(\frac{-\pi}{3}\right) = 2.$

Answer: D

74) The Solution set of the equation: $2\cos^2 x - \sin^2$

A. $\left\{\frac{\pi}{2}+k\pi\right\}$ C. $\left\{\frac{\pi}{2}+k\pi\right\}$

B. $\left\{\frac{\pi}{4}+k\pi, \frac{\pi}{2}+k\pi\right\}$ D. $\left\{\frac{\pi}{4}+k\pi, \frac{\pi}{2}+2k\pi\right\}$

Solution: $2\cos^2x - \sin2x = 0 \Leftrightarrow \quad 2\cos^2x - 2\sin x \cos x = 0$
$\Rightarrow 2\cos x(\cos x - \sin x) = 0 \Rightarrow 2\cos x = 0$ or $\cos x = \sin x$

$\Rightarrow x = \frac{\pi}{2} + 2\pi k, x = \frac{\pi}{4} + k\pi$

Answer: D

75) If $\theta = \arcsin\left(\frac{1}{2}\right)$, then $\tan(2\theta)$ is equal to ... UEE 2003

A. $\frac{1}{\sqrt{3}}$ B. 1 C. $\sqrt{3}$ D. $\frac{\sqrt{3}}{2}$

Solution: $\sin^{-1}\left(\frac{1}{2}\right) = \theta \Rightarrow \sin\theta = \frac{1}{2}$,

$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$

$$\Rightarrow \tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \frac{2\left(\frac{1}{2}\right)\frac{\sqrt{3}}{2}}{\frac{3}{4}-\frac{1}{4}} = \frac{\sqrt{3}}{2}\cdot(2) = \sqrt{3}$$

76) What is the amplitude and period respectively of the graph
$f(x) = 2 - 3\sin\left(\frac{x+\pi}{4}\right)$...

A. $2, \frac{\pi}{4}$ B. $-3, 8\pi$ C. $3, \frac{\pi}{2}$ D. 3, 8

Solution: Amplitude, $a = |-3| = 3$, period, $p = \frac{2\pi}{k} = \frac{2\pi}{\frac{1}{4}} = 8$

Answ

77) Which of the following is the Solution set of $\sqrt{3}\sin 2x = \cos 2x$ on $[0,\pi]$…

A. $\left\{\frac{\pi}{6},\frac{\pi}{12}\right\}$ B. $\left\{\frac{\pi}{6},\frac{\pi}{3}\right\}$ C. $\left\{\frac{\pi}{12},\frac{7\pi}{12}\right\}$ D. $\left\{\frac{\pi}{6},\frac{2}{3}\pi\right\}$

Solution: $\sqrt{3}\sin 2x = \cos 2x \Rightarrow \tan 2x = \frac{\sin 2x}{\cos 2x} = \frac{1}{\sqrt{3}}$

$\Rightarrow 2x = \frac{\pi}{6},\frac{7\pi}{6} \Rightarrow x = \left\{\frac{\pi}{12},\frac{7\pi}{12}\right\}$ **Answer: C**

78) A 1 meter high pole is fixed vertically on the ground at a point $50\sqrt{3}$ m far from the foot of building. If the angle of elevation from the top of the pole to the top of building is 30^0. What is the height of the building… UEE 2003.

A. 50m B. 51m C. 150m D. 151m

Solution: $\tan 30^0 = \frac{d}{50\sqrt{3}} \Rightarrow d = 50\sqrt{3}.\frac{1}{\sqrt{3}} = 50m$

$\Rightarrow h = d + y = 50m + 1m = 51m$

Answer: B

79) Let $f(x) = -3\sin\left(\frac{1}{2}x\right)$. which of the following is true…UEE2002

A. The amplitude is -3 C. f is decreasing on $[-2\pi,0]$
B. The range is $[-1, 1]$ D. f is increasing on $[\pi, 3\pi]$

Solution: Amplitude, $a = |-3| = 3$, Range $= [-3,3]$

- f is increasing on $\frac{1}{2}x = \left[\frac{\pi}{2},\frac{3\pi}{2}\right] \Rightarrow [\pi, 3\pi]$

Answer: D

80) In the interval $[0,2\pi]$, what is the Solution set of the equation $\cos^2 x + \sin x\cos x = 1$ …UEE

Solution: $\cos^2 x + \sin x\cos x = 1 \Rightarrow \sin x\cos x = 1 - \cos^2 x = \sin^2 x$
$\Rightarrow \sin x\cos x = \sin^2 x \Rightarrow \sin^2 x - \sin x\cos x = 0 \Rightarrow \sin x(\sin x - \cos x) = 0$

$\Rightarrow \sin x = 0, \sin x = \cos x \Rightarrow x = \left\{0,\frac{\pi}{4},\frac{5\pi}{4},\pi\right\}$ **Answer: A**

81) Given that $\tan\theta = \frac{-4}{3}$, θ is in quadrant II, and $\tan\beta = \frac{-5}{12}$ for β in quadrant IV then find

A. $\sin(\theta+\beta)$ B. $\cos(\theta+\beta)$ C. quadrant of $(\theta-\beta)$

Solution:

$\tan\theta = \frac{-4}{3} \Rightarrow \sin\theta = \frac{4}{5}$ *and* $\cos\theta = \frac{-3}{5}$ $\tan\beta = \frac{-5}{12} \Rightarrow$

$\sin\beta = \frac{-5}{13}$ *and* $\cos\beta = \frac{12}{13}$

$\Rightarrow \sin(\theta+\beta) = \sin\theta\cos\beta + \cos\theta\sin\beta = \frac{4}{5}\cdot\frac{12}{13} + \frac{-3}{5}\cdot\frac{-5}{13} = \frac{63}{65}$

$\Rightarrow \cos(\theta+\beta) = \cos\theta\cos\beta - \sin\theta\sin\beta = \left(\frac{-3}{5}\right)\left(\frac{12}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{-5}{13}\right) = \frac{-16}{65}$

- $\sin(\theta+\beta) > 0$ and $\cos(\theta+\beta) < 0 \Rightarrow (\theta+\beta)$ lies on II quadrant

82) Exact value of $\cos\left(\sin^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{-4}{5}\right)\right)$

A. −1 B. 1 C. ½ D. −½

Solution:

Let $\sin^{-1}\left(\frac{3}{5}\right) = \theta \Rightarrow \sin\theta = \frac{3}{5}$, $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{-9}{25}} = \frac{4}{5}$

Let $\cos^{-1}\left(\frac{-4}{5}\right) = \beta \Rightarrow \cos\beta = \frac{-4}{5}, \sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\frac{16}{25}} = \frac{3}{5}$

$\Rightarrow \cos\left(\sin^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{-4}{5}\right)\right) \Rightarrow \cos(\theta+\beta) = \cos\theta\cos\beta - \sin\theta\sin\beta$

$\Rightarrow \left(\frac{4}{5}\right)\left(\frac{-4}{5}\right) - \left(\frac{3}{5}\right)\left(\frac{3}{5}\right) = \frac{-16-9}{25} = -1$ **Answer: A**

83) If $\cos\theta = \frac{-4}{5}$ *for*, $\pi < \beta < \frac{3\pi}{2}$ *then find*

A. $\sin\left(\frac{\theta}{2}\right)$ B. $\cos\left(\frac{\theta}{2}\right)$ C. $\tan\left(\frac{\theta}{2}\right)$ D. $\cos 2\theta$ E. $\sin 2\theta$

Solution: $\pi < \theta < \dfrac{3\pi}{2} \Rightarrow \dfrac{\pi}{2} < \dfrac{\theta}{2} < \dfrac{3\pi}{4}$

$\Rightarrow \sin\left(\dfrac{\theta}{2}\right) > 0$ and $\cos\left(\dfrac{\theta}{2}\right) < 0$

a) $\quad \sin\left(\dfrac{\theta}{2}\right) = \sqrt{\dfrac{1-\cos\theta}{2}} = -\sqrt{\dfrac{1+\frac{4}{5}}{2}} = \dfrac{3}{\sqrt{10}}$

b) $\quad \cos\left(\dfrac{\theta}{2}\right) =- \sqrt{\dfrac{1+\cos\theta}{2}} = \sqrt{\dfrac{1-\frac{4}{5}}{2}} = -\sqrt{\dfrac{3}{10}} = \dfrac{-1}{\sqrt{10}}$

c) $\quad \tan\left(\dfrac{\theta}{2}\right) =- \dfrac{\sin\left(\dfrac{\theta}{2}\right)}{\cos\left(\dfrac{\theta}{2}\right)} = \dfrac{\frac{3}{\sqrt{10}}}{\frac{-1}{\sqrt{10}}} = -3$

d) $\quad \cos 2\theta = \cos^2\theta - \sin^2 = \left(\dfrac{-4}{5}\right)^2 - \left(\dfrac{-3}{25}\right)^2 = \dfrac{16-9}{25} = \dfrac{7}{25}$

e) $\quad \sin 2\theta = \sin(\theta+\theta) = 2\sin\theta\cos\theta = 2\left(\dfrac{-3}{5}\right)\left(\dfrac{-4}{25}\right) = \dfrac{24}{25}$

84) Let $\sin\left(2x - \dfrac{\pi}{4}\right) = d$ *then solve for*

a) $\quad d = 0 \quad$ b) $\quad d = 1 \quad$ c) $\quad d = ½$

Solution: $\sin\left(2x - \dfrac{\pi}{4}\right) = 0 \Rightarrow 2x - \dfrac{\pi}{4} = 0, \pi \ldots \text{period} = \pi$

$\Rightarrow 2x = \dfrac{\pi}{4}, \dfrac{5\pi}{4} \ldots, x = \dfrac{\pi}{8} + \pi k, \dfrac{5\pi}{4} + \pi k$

$\therefore SS = \left\{\dfrac{\pi}{8} + \pi k; \dfrac{5\pi}{4} + \pi k\right\}$

b) $\sin\left(2x-\frac{\pi}{4}\right)=1 \Rightarrow 2x-\frac{\pi}{4}=\frac{\pi}{2} \Rightarrow 2x=\frac{3\pi}{4}+2\pi k$

$\Rightarrow x=\frac{3\pi}{8}+\pi k$

d) $\sin\left(2x-\frac{\pi}{4}\right)=\frac{1}{2} \Rightarrow 2x-\frac{\pi}{4}=\frac{\pi}{6},\frac{5\pi}{6}$

$\Rightarrow 2x=\frac{\pi}{6}+\frac{\pi}{4}+2\pi k,\frac{5\pi}{6}+\frac{\pi}{4}+2\pi k$

$\Rightarrow x=\frac{10\pi}{48}+\pi k,\frac{15\pi}{48}+\pi k=\left\{\frac{5\pi}{24}+\pi k,\frac{5\pi}{16}+\pi k\right\}$

Supplementary exercise

1. Find all Solutions of the equation in the interval, $[0,2\pi]$

a) $\sin\left(x+\frac{\pi}{3}\right)+\sin\left(x-\frac{\pi}{3}\right)=1$

b) $\cos\left(x+\frac{\pi}{4}\right)-\cos\left(x-\frac{\pi}{4}\right)=1$

c) $\tan(x+\pi)+2\sin(x+\pi)=0$

2. Find the amplitude, phase shift, period for the following functions

a) $y=2\sin\left(x-\frac{\pi}{2}\right)$ c) $y=-4\sin\left(\frac{2x}{3}+\frac{\pi}{6}\right)$

b) $y=\cos\left(2x-\frac{\pi}{4}\right)$ d) $y=\frac{5}{4}\cos(3x-2\pi)$

3. Find phase shift and period for the function

a) $y=2\tan\left(2x-\frac{\pi}{4}\right)$ b) $y=-3\csc\left(\frac{x}{3}+\pi\right)$

4. Graph one full period of each function.

a) $y=\sin\left(x-\frac{\pi}{2}\right)$ d) $y=-\sin(\pi x+1)-2$

b) $y=\cos\left(\frac{x}{2}+\frac{\pi}{3}\right)$ e) $y=4\cos(\pi x-2)+1$

c) $y = \tan\left(x - \frac{\pi}{4}\right)$

5. Find the equation of

a) the cosine function with amplitude 3 period 3π, and phase shift $\frac{-\pi}{4}$

b) the tangent function with period, 2π, phase shift $\frac{\pi}{2}\ \frac{\pi}{2}$

c) the secant function with period 4π and phase shift $\frac{3\pi}{4}$

Find the exact value

a) $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

e) $\cos(2\sin^{-1}\left(\frac{\sqrt{2}}{2}\right))$

b) $\tan^{-1}(-1)$

f) $\sin(2\sin^{-1}\left(\frac{4}{2}\right))$

c) $\sec^{-1}(2)$

g) $\sin^{-1}[\cos\left(\frac{-2\pi}{3}\right)]$

d) $\cos(\sec^{-1}2)$

h) $\sin^{-1}(½ \sin^{-1}x)$

Find an algebraic expression for each of the following

a) $\csc(\cos^{-1}(x))$

c) $\sin(\cos^{-1}x - \sin^{-1}3x)$

b) $\sin(2\cos^{-1}x)$

Find the exact value of the given expression.

a) $\sin\left(\sin^{-1}\left(\frac{2}{3}\right) + \cos^{-1}\left(\frac{1}{2}\right)\right)$ b) $\tan\left(\cos^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(\frac{3}{4}\right)\right)$

Solve the equation for x algebraically.

a) $\tan^{-1}x = \sin^{-1}\left(\frac{24}{25}\right)$

c) $\sin^{-1}x + \cos^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{6}$

b) $\cos^{-1}(x - ½) = \frac{\pi}{3}$

d) $\cos^{-1}x + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{2}$

10. Solve for y interms of x

a) $5x = \tan^{-1}3y$ b) $x - \frac{\pi}{3} = \cos^{-1}(y-3)$

11. solve the equation

a) $y = \sec\left(\sin^{-1}\left(\frac{12}{13}\right)\right)$ d) $y = \cos\left[\sin^{-1}\left(-\frac{3}{5}\right) + \cos^{-1}\left(\frac{5}{13}\right)\right]$

b) $2\sin\text{-}1(x-1) = \frac{\pi}{3}$ e) $y = \cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$

c) $\sin^{-1}x + \cos^{-1}\frac{4}{5} = \frac{\pi}{2}$

12. Given that $\tan A = \frac{12}{5}$ and A is in quadrant II and that $\cos B = \frac{9}{4}$, and B is in quadrant I.

Find the following

a) $\cos(A - B)$ C) $\sin(A - B)$

b) $\tan(A + B)$ D) quadrant of $A - B$

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